MATHEMATICAL MODELS, ANALYSIS AND SIMULATIONS OF THE HANDY MODEL WITH MIDDLE CLASS

by

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To all the beloved ones who passed without seeing me accomplish this.

To you with love.

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ABSTRACT

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This study presents three different mathematical versions of the HANDY (Human And Nature DYnamics) model for the socioeconomic dynamics of a large stratified society. The basic model was introduced in the ground breaking publications of Motesharrei (dissertation 2014) and Motesharrei et. al. (2016). The original model consists of a nonlinear system of four ordinary differential equations (ODEs) which describe the development, in time, of a 'very simple' society consisting of two populations: the Elite (rich) and Commoners (workers). Included also are the use of natural (renewable and nonrenewable) resources and the accumulation of human wealth. The model's solutions depict the dynamics of these variables. Motesharrei's main impetus and interest was to use the model as a tool for evaluating the conditions that contribute to the flourishing, sustainability, or collapse of complex societies.

This dissertation expands the basic HANDY model and studies its mathematical properties and those of its three extensions. It establishes the existence of solutions to the models, as well as their uniqueness, boundedness and

positivity. Furthermore, it investigates the stability of the systems' steady states, which describe the long-time behavior of the societies. It also presents a number of qualitatively different computer simulations, providing insights into potential behaviors of the societies described by these models. The main contributions of this work are the mathematical analysis of the basic HANDY model, its three expansions and their analysis, and computer simulations.

The first extension, the HANDY-SM model, includes social mobility. Rich individuals may go bankrupt and become workers, and some workers may become rich. It also allows for two different aspects of inequality, through variations in salaries and the wealth structure. The second extension, the HANDY-MC-I model, includes the Middle Class population, making the model more practical when applied to modern societies. It expands the system into five ODEs, and allows for social mobility among the three populations. Finally, in the third extension, HANDY-MC-II, two variables describe the natural resources: the renewable resources (wood, solar and wind energies), and nonrenewable resources (coal, oil, gas). This particular extension makes the model more realistic, but it also adds considerable complexity since it consists of six nonlinear coupled ODEs. The model simulations depict the consequences of having three different populations with different income status, two natural resources, and unequal contributions to wealth structure.

Analysis of the models' steady states shows that the state is stable when the populations and wealth die out but nature (the resources) is at its equilibrium. The model has also asymptotically stable, nonzero steady states to which the populations, the resources and the wealth converge in the long-time limit. The simulations also show the existence of periodic solutions in which the populations, the natural resources and wealth undergo large oscillations, indicating cycles of ‘boom and bust.’ Finally, the simulations demonstrate that the models may have chaotic solutions, pointing to a high level of unpredictability.

This dissertation describes three increasingly more complex HANDY models. It paves the way and raises mathematically interesting topics for their further study. In particular, the uniqueness of the solutions, and the questions of the existence of periodic solutions, limit cycles and chaos, remain unresolved, yet. Furthermore, it suggests the possibility of tailoring such models to existing societies, and then using them as tools for evaluation of the potential outcomes of various policy decision.

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CHAPTER ONE

INTRODUCTION

This dissertation presents a mathematical model, and three of its extensions, for the study of the dynamics of large societies. It focuses on the relationships among the populations of Elites (rich), Commoners (workers) and the Middle Class (professionals and managers), their use of Natural Resources (renewables and non-renewables), and the generation of human Wealth. The mathematical analysis and computer simulations provide insights into potential flourishing, long-term sustainability, or collapse of such societies.

The ground breaking HANDY– Human And Nature DYnamics – model was constructed in Motesharrei's dissertation [12]. The original model consists of four ordinary differential equations (ODE's), representing two populations, their use of natural resources and wealth. The model was simulated to study its behavior in [12] and in Motesharrei et. al. [13, 14]. In our research we also performed computer simulations that reinforced the findings in these publications, which showed that economic stratification and rapid depletion of natural resources were among the main reasons for societal collapse. The results indicated that an irreversible collapse could be avoided if the society first adopted sustainable behaviors to reduce the depletion of natural resources and, secondly, if it moved towards an equitable distribution of resources. In these publications, the authors provided many historical examples and studies of earlier civilizations, making a general comparison between the archaeological results of ancient collapsed civilizations. The main theme throughout the model for collapsed civilizations, (in addition to individual issues specific to each civilization), was that both over-depletion of resources and inequality led to collapse.

In Brendan and Taylor [2], the authors introduced a simpler model to study the collapse of the society in Easter Island. They used the Ricardo-Malthus model with two ODE's, one for renewable resources that included a logistic term with saturation, and a harvesting term. The population growth rate was described by three terms: the birth rate, the death rate, and a fertility factor using the Malthusian hypothesis [11]. This hypothesis claimed that higher income rate stimulates population growth, leading to the depletion of resources and eventually to famine. On the other hand, it was found that too low wages may cause a population collapse caused by conflict. Various civilizations have gone through such a collapse, including the western Roman empire, and the Mayan and the Mesopotamian civilizations. A similar model for the societal dynamics was considered in [6], consisting of three ODEs representing the population, resources, and reserves or surpluses. They used a factor r for the amount of reserve per capita, and proposed that the population over-grown when the accumulated reserve was abundant (above a threshold) and died out when the reserves were insufficient. Furthermore, they found in their simulations patterns, similar to those in the HANDY model, where Nature's slow growth and rapid decline, known as the Seneca effect. This observation led the authors to write: "Human civilizations collapse is not impossible and that even if it were to be followed by a new resurgence, such a collapse would still lead to a complete transfiguration of the human/nature interaction."

There are many reasons for civilizations' collapse, and the aim of using mathematical models is to study such processes as "thought experiments," or "mathematical experiments," since the models allow one to effectively investigate various scenarios and the dependence of the systems' dynamics on various inputs. This enables finding common causes for collapse that cannot be discerned by only studying the historical records. Furthermore, such systems may be sensitive to some of the inputs, and not so to other inputs. This implies that in an attempt to describe accurately a society's' dynamics, the inputs to which the models are sensitive must be found or estimated precisely, whereas those to which the system is not sensitive can be estimated with much less accuracy and still provide useful insights. A statistical sensitivity analysis to all the parameters used in the HANDY model was conducted in [16] to find those that impacted the behavior the most during short time periods. The analysis found the most affecting values and then showed the results in computer simulations. The sensitivity analysis in the thesis [16] focused on the inequitable society, since the scenario reached an equilibrium following some oscillations. The results showed that the Elite's birth rate and maximum death rate have a larger impact on the simulations than the depletion and the inequality factors. It was claimed in [16] that the inequality factor had a minor effect, whereas our findings indicated that that is not the case.

The power of the mathematical models lies in their ability to show clearly the similarities in historical collapses, and to provide guidance on how to prevent them. Although, it was customary to primarily blame a complex society's collapse on overpopulation, there wasn't enough historical record not support this claim. Wealth inequality and workers' low quality of life were also causes for a society to fail [19]. In addition, poor management of sustenance, draining renewable resources, inflation, and food shortages lead to populist rebellions against a government's poor resource management, weakening the political structure and causing it to fail.

This research expands the HANDY model in stages making it more realistic for the study of the conditions for civilization's flourishing, long-term sustainability, or collapse. First, the inequality factor is split into the consumption rate and wealth inequality factors. Then, we introduce the HANDY-SM model that allows for social mobility, so that some Elites may go bankrupt and some Commoners become rich.

The HANDY-MC-I model adds the middle class and its interactions with the other classes, resources and wealth. Finally, the HANDY-MC-II model separates the Natural resources into renewables (solar, wind energies, wood) and non-renewables (oil, gas). This expands the model into six ODEs, three for the populations, one for wealth, one for each of renewable and non-renewable energy and resources. Since each of the models consists of a non-linear coupled system of ODEs, from the mathematical point of view, the existence, positivity and boundedness of the solutions is a first step in their analysis. This is done for each one of the models, including the original HANDY model. Then, we obtain the steady states of the systems and investigate their stability. It is found that the steady state with zero populations, resources or wealth is unstable, while the one without populations or wealth, but with renewable resources, is stable. In the latter case nature regenerates to its carrying capacity in the absence of humans.

The main contributions of this dissertation are: three extensions of the original HANDY model two of which include the Middle Class; the mathematical analysis of the model and its extensions; and some computer simulations, which indicate the complexity of the potential scenarios the models predict. The analysis includes proofs of the existence, boundedness and positivity of the solutions. Then, the steady states are found, and their stability established.

The dissertation is structured as follows. Chapter 2 presents a summary of the necessary mathematical tools used in this work. Chapter 3 provides a short description of the original HANDY model. Then, it establishes the existence of its unique solution. It also proves that the solution is bounded and nonnegative, both qualities that are necessary for the interpretation of the solutions. However, the uniqueness of the solution depends on splitting of the domain $\mathbb{R}^4_+$ into subdomains where the function $\boldsymbol{F}$ is Lipschitz continuous, since it is not globally Lipschitz. This is an interesting observation that needs further study. The stability of the steady states of the system is established, and it is found that those that are steady represent the long-term behavior of the society. Those that are unstable seem to lead to periodic oscillations, limit cycles and possibly to non-periodic or chaotic behavior. Chapter 4 describes the computer algorithm that was used to simulate the model. Then, the results of a few typical, but interesting, solutions are depicted and briefly commented on. The system appears to have very sharp declines or raises in some of the variables, representing periods of very rapid change. Chapter 5 extends the model by allowing social mobility. The existence of the solutions, their boundedness and positivity are proven, but the uniqueness question remains unresolved. It is followed by computer simulations of the model, depicted in Chapter 6. Chapter 7 extends the original model into HANDY-MC-I by introducing the middle class. This adds another ODE into the system, and entails appropriate changes in the system parameters. The existence of bounded and non-negative solutions is proved in a similar manner to the one in Chapter 5. Then, the study of the steady states is provided. Computer simulations are shown in Chapter 8. Chapter 9 presents the HANDY-MC-II model, in which the resources are split into renewables and non-renewables. This adds another ODE to the system. Analysis of this model is left open and will be completed elsewhere. The simulations depict more complex solutions. Finally, Chapter 10 provides a summary of the results in this dissertation and lists some questions and directions for future work.

CHAPTER TWO

MATHEMATICAL PRELIMINARIES

This Chapter provides the mathematical background and tools used in this dissertation.

We begin with the standard existence, uniqueness and regularity results for nonlinear systems of Ordinary Differential Equations (ODEs) and Inequalities.

2.1 Existence, uniqueness and regularity of the solutions

We consider a general system of ODEs with n equations of the first order. We denote by $\boldsymbol{x} = \boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))$ the vector of the unknown functions, the solutions to the problem on the time interval $[t_0, T]$. We may consider the solutions as trajectories in $\mathbb{R}^n$, and once the initial condition, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is prescribed, $\boldsymbol{x}(t)$ describes a curve or trajectory that begins at $\boldsymbol{x}_0 \in \mathbb{R}^n$. We denote by prime over a variable the time derivative, thus, $d\boldsymbol{x}/dt = \boldsymbol{x}'(t) = (x_1'(t), \ldots, x_n'(t))$. For the sake of simplicity, we assume that $t_0 = 0$, unless specifically mentioned.

Next, we note that a function $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}) : [0, T] \to \Omega \subset \mathbb{R}^n$ is said to be *Lipschitz continuous* with *Lipschitz constant* K_L, if there is a positive constant K_L such that for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$,

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \le K_L \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Here and below, $\| \cdot \|$ denotes the standard norm in $\mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ is assumed to be a domain (open, connected and bounded set) in $\mathbb{R}^n$.

We are interested in the existence and properties of the solutions of the following problem, consisting of n ODEs.

Problem 2.1. *Find a trajectory or a solution* $\boldsymbol{x}(t)$ *in* $\mathbb{R}^n$*, such that*

$$\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2.1}$$

Here, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}) : [0, T] \to \mathbb{R}^n$ *is a given vector function and* $\boldsymbol{x}_0$ *is the initial condition.*

Next, we use the following terminology.

Definition 2.2. *(Global and local solutions) Let* $\boldsymbol{x}(t)$ *be a solution of system* (2.1)*.*

1. If the solution exists only for finite time $[t_0, T]$*, for* $T < \hat{T}$*, and some* $0 < \hat{T}$*, then* $\boldsymbol{x}$ *is called a local solution.*

2. If the solution exists on every time interval $[0, T]$*, then* $\boldsymbol{x}(t)$ *is called a global solution.*

We begin with a standard existence and uniqueness result for ODEs.

Theorem 2.3 ([17], Theorem 2.1.3, p. 102)**.** *[Fundamental Existence-Uniqueness Theorem for ODEs.] Consider the initial value problem 2.1. Suppose that* $\boldsymbol{F}$ *is Lipschitz continuous on* $E \subset \mathbb{R}^n$*. Then, there exists a unique local solution of this initial value problem. More precisely, there exists an* $l > 0$ *and a unique solution* $\boldsymbol{x} : (-l, l) \to \mathbb{R}^n$ *of (2.1). When* $\boldsymbol{F}$ *is Lipschitz continuous on* $\mathbb{R}^n$*, the solution exists and is unique on every time interval* $[0, T]$*, for* $0 < T$*. If, in addition* $\boldsymbol{x}(t)$ *is bounded, i.e.* $\|\boldsymbol{x}(t)\| \leq C$*, then it exists on* $[0, \infty)$*.*

The solution is local since it may become unbounded as $t \to \pm l$. However, if F is Lipschitz everywhere, the solution may become infinite only when $T \to \infty$. Hence, if it is bounded, it exists on $[0, \infty)$. We use this theorem in Chapter 3.
The following is another well known existence theorem for ODEs which is based on the Shaeffer fixed point theorem, (see, e.g., [10] or [4]).

Theorem 2.4. *Let* $\boldsymbol{F} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and suppose that there exists* $L > 0$ *such that for all* $\widehat{\lambda} \in (0, 1)$*, if*

$$\boldsymbol{x}' = \widehat{\lambda} \boldsymbol{F}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2.2}$$

for all $t \in [0, T]$*, then* $\|\boldsymbol{x}\| < L$*. Then, there exists a solution to the system of ODEs,*

$$\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{2.3}$$

for $t \in [0, T]$*.*

We use this existence theorem in the HANDY model in Chapters 5 and 7, since it doesn't require the Lipschitz continuity of $\boldsymbol{F}$, which turns out to be Lipschitz only on S_1 and S_2, but not on S_0 (see Theorem 3.2 in Section 3.2.1).

Next, we denote by $C^m(\Omega)$ the vector space of the functions that have continuous partial derivatives up to order $m \geq 0$, where $m = 0$ corresponds to continuous functions on $\Omega \in \mathbb{R}^n$. Also, $C^k([0, T] : \Omega)$ denotes the vector space of functions $f : [0, T] \to \Omega$ that have $m \geq 0$ continuous time derivatives, where, $C^0([0, T] : \Omega)$ denotes the continuous functions into Ω.

We denote by $C^{1,1}((0, T), \mathbb{R})$ the vector space of all functions that have Lipschitz continuous first derivatives in $(0, T)$, and similarly for $C^{1,1}[0, T]$,

$$C^{1,1}((0, T), \mathbb{R}) = \{f : (0, T) \to \mathbb{R};\ f' \text{ is Lipschitz continuous on } (0, T)\}.$$

This is the space where the solutions of the HANDY model can be found. We note that if the function $f : [0, T] \to \mathbb{R}$ is $C^1(\mathbb{R})$, then f is locally Lipschitz.

Next, we recall that a set $C \subset \mathbb{R}$ is *compact* if C is closed and bounded. An important fact is that, if $f : C \to \mathbb{R}$ is continuous and C is compact, then f is bounded on C and it attains its maximum and minimum values on C.

The following inequality is of basic importance in proofs of existence and uniqueness theorems.

Lemma 2.5. *(Gronwall's Inequality [17]) Let $f : [0, T] \to \mathbb{R}$ be a continuous and non-negative real-valued function. Suppose $C \geq 0$ and $K > 0$ are two constants such that*

$$f(t) \leq C + \int_0^t Kf(s)ds, \qquad t \in [0, T].$$

Then, for all $t \in [0, T]$,

$$f(t) \leq f(0)e^{Kt} + \frac{C}{K}\left(e^{Kt} - 1\right).$$

2.2 Equilibrium Points and Stability

An *equilibrium point* or a *steady state* $\boldsymbol{x}^*$ of the dynamical system

$$\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}), \tag{2.4}$$

where $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$, is the point $\boldsymbol{x}^* \in \mathbb{R}^n$ such that $\boldsymbol{F}(\boldsymbol{x}^*) = 0$, and 0 is the origin in $\mathbb{R}^n$. We refer to each solution of (2.4) as a **trajectory** $\boldsymbol{x}(t)$ in $\mathbb{R}^n$, for $0 \leq t \leq T$. Therefore, the steady states can be trajectories that are single points. However, such nonlinear systems can exhibit steady oscillations, limit cycles, or chaotic behavior. We use the following concepts in the dissertation, following [7]. First, we consider the various notions of stability of equilibrium points $X^* \in \mathbb{R}^n$ for the differential equation (2.4).

Definition 2.6. *[7] An equilibrium* $\boldsymbol{x}^*$ *is said to be* **stable** *if nearby solutions stay nearby for all future time. More precisely, suppose* $\boldsymbol{x}^* \in \mathbb{R}^n$ *is an equilibrium point for* (2.4)*. Then,* $\boldsymbol{x}^*$ *is a* **stable equilibrium** *if for every neighborhood* $\mathcal{O}$ *of* $\boldsymbol{x}^*$ *in* $\mathbb{R}^n$*, there is a neighborhood* $\mathcal{O}_1$ *of* $\boldsymbol{x}^*$ *in* $\mathcal{O}$ *such that every solution* $\boldsymbol{x}(t)$ *with* $\boldsymbol{x}(0) = \boldsymbol{x}_0$ *in* $\mathcal{O}_1$ *exists and remains in* $\mathcal{O}$ *for all* $t > 0$*.*
An equilibrium $\boldsymbol{x}^*$ *is said to be* **asymptotically stable** *or* **stable and attracting** *if* $\mathcal{O}_1$ *can be chosen so that for every* $\boldsymbol{x}_0 \in \mathcal{O}_1$ *the solution or the trajectory* $\boldsymbol{x}(t)$ *of the system that begins at* $\boldsymbol{x}_0$*, that is* $\boldsymbol{x}(0) = \boldsymbol{x}_0$*, satisfies*

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*.$$

An equilibrium $\boldsymbol{x}^*$ *that is not stable is called* **unstable***. This means that in each neighborhood* $\mathcal{O}$ *of* $\boldsymbol{x}^*$*, there is* $\boldsymbol{x}_0 \in \mathcal{O}$ *such that the trajectory that begins in* $\boldsymbol{x}_0$ *does not stay in* $\mathcal{O}$ *for for all* $t > 0$*.*

Definition 2.7. *Let* $\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x})$ *and suppose that* $\boldsymbol{F}(\boldsymbol{x}_0) = 0$*. Let* $D\boldsymbol{F}_{\boldsymbol{x}_0}$ *denote the* **Jacobian** *matrix of* $\boldsymbol{F}$ *evaluated at* $\boldsymbol{x}_0$*. Then, the linear system of differential equations*

$$\boldsymbol{y}' = D\boldsymbol{F}_{\boldsymbol{x}_0}\boldsymbol{y},$$

is called the **linearized system** *near* $\boldsymbol{x}_0$*. When* $\boldsymbol{x}_0 = 0$*, the linearized system is obtained by simply dropping all of the nonlinear terms in* $\boldsymbol{F}$*. In this work we denote* $D\boldsymbol{F}_{\boldsymbol{x}_0}$ *by* $J(\boldsymbol{x}_0)$*.*

Definition 2.8. *(Eigenvalues and eigenvectors) A nonzero vector* $\boldsymbol{v}_0$ *is called an* **eigenvector** *of* A *if* $A\boldsymbol{v}_0 = \lambda\boldsymbol{v}_0$ *for some* $\lambda \in \mathbb{R}$*. The constant* λ *is called an* **eigenvalue** *of* A*.*

The following is a well-known result connecting the solutions to the eigenvalues and eigenvectors.

Theorem 2.9. *Suppose that $\boldsymbol{v}_0$ is an eigenvector for the matrix A with associated eigenvalue λ. Then, the function $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}_0$ is a solution of the system*

$$\boldsymbol{x}' = A\boldsymbol{x}.$$

It follows that when $Re(\lambda) > 0$ the solution tends to infinity along the direction $\boldsymbol{v}_0$, and when $Re(\lambda) < 0$, the solution converges to 0 along the direction $\boldsymbol{v}_0$. We use these notion in the study od the stability of the steady states of the various HANDY models in this dissertation.

The following theorem introduces the important notion of Liapunov Stability, which is often used in applications to dynamical systems.

Theorem 2.10. *(Liapunov Stability) Let $\boldsymbol{x}^*$ be an equilibrium point for $\boldsymbol{x}' = F(\boldsymbol{x})$. Let $L : \mathcal{O} \to \mathbb{R}$ be differentiable function defined on an open set $\mathcal{O}$ containing $\boldsymbol{x}^*$. Suppose further that:*

(a) $L(\boldsymbol{x}^*) = 0$ and $L(\boldsymbol{x}) > 0$ if $\boldsymbol{x} \neq \boldsymbol{x}^*$;

(b) $L' \leq 0$ in $\mathcal{O}$. Then, $\boldsymbol{x}^*$ is stable.

Furthermore, if L also satisfies,

(c) $L' < 0$ in $\mathcal{O} - \boldsymbol{x}^*$, then $\boldsymbol{x}^*$ is asymptotically stable.

A function L satisfying (a) and (b) is called a **Liapunov function** for $\boldsymbol{x}^*$. If (c) also holds, we call L a **strict Liapunov function** [7].

Let $A(t)$ be a family of $n \times n$ matrices that depends continuously on t. The system $\boldsymbol{x}' = A(t)\boldsymbol{x}$ is linear and non-autonomous. Its solution exists and is unique as is asserted by the following theorem.

Theorem 2.11. *Let $A(t)$ be a continuous family of $n \times n$ matrices defined for $t \in [\alpha, \beta]$. Then, the initial value problem $\boldsymbol{x}' = A(t)\boldsymbol{x}'$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ has a unique solution that is defined on the entire interval $[\alpha, \beta]$.*

2.3 Bifurcation

We turn to the concepts related to bifurcations of solutions. *Bifurcation* in a dynamical system refers to the phenomenon when the system exhibits qualitatively new dynamical behavior as a parameter crosses a threshold value. The parameter value where the bifurcation occurs is called a *bifurcation point.*

A bifurcation occurs when there is a "significant" change in the structure of the solutions of the system as the parameter parameter crosses a threshold value. One of the simplest types of bifurcation occurs when the number of equilibrium solutions changes as the parameter varies.

Definition 2.12 ([7])**.** *Consider the system $\boldsymbol{x}' = \boldsymbol{F}_a\boldsymbol{x}$, where $\boldsymbol{F}_a : \mathbb{R}^n \to \mathbb{R}^n$ depends on the parameter a. If $\boldsymbol{x}_0$ is an equilibrium point of the system, then we have $\boldsymbol{F}_a(\boldsymbol{x}_0) = 0$. If $D\boldsymbol{F}_a(\boldsymbol{x}_0) \neq 0$, then small changes in the parameter value a do not change the local structure near $\boldsymbol{x}_0$: that is, the system is* **structurally stable** *and the differential equation*

$$\boldsymbol{x}' = \boldsymbol{F}_a\boldsymbol{x} + \epsilon(\boldsymbol{x}),$$

has an equilibrium point $\boldsymbol{x}_0(\epsilon)$ that varies continuously with ϵ, for ϵ small.

A glance at the (increasing or decreasing) graphs of $\boldsymbol{F}_a + \epsilon(\boldsymbol{x})$ near $\boldsymbol{x}_0$ shows why this is true.

Ecologically, in the HANDY model that follows, such a bifurcation happens when small changes in δ, the natures depletion factor, have severe consequences of possible famine in the society.

CHAPTER THREE
THE HANDY MODEL

This chapter reviews a slightly modified version of the classical HANDY model, and establishes the existence and uniqueness, boundedness and positivity of the solutions. The computer and simulations of a few scenarios the model predicts are provided in Chapter 4. Modified versions of the model, which form most of the content of this dissertation, are presented in the following chapters, together with their analysis and simulations.

Recall that HANDY stands for the *Human And Nature DYnamics.* This model was published in the pioneering work [14], building upon the famous Lotka-Volterra predator-prey model. For a very detailed construction of the model, and the way the various terms were developed, we refer to the original article. The main motivation of the author in creating HANDY was to explore the main processes that lead to the flourishing or collapse of complex societies under various conditions of population stratification into rich and poor, with substantial inequality. Taking into account the ways wealth is created and natural resources are used.

3.1 The model

The model consist of four nonlinear coupled ODEs: two equations model the populations growth rates of the Commoners or workers, $x_c(t)$, and the Elites or the rich, $x_e(t)$; one equation describes the rate of growth or depletion of natural resources, $y(t)$ (which is a lumped variable for renewable, such as wood, and non-renewable, such as oil, resources), and the rate of growth of wealth, $w(t)$, which includes also food surpluses. Here, t is time that is measured in arbitrary units, which for the sake of simplicity we call 'years.' The models doesn't take into

account the spatial distribution or the possible spatial non-homogeneity of the populations and resources, and deals with sufficiently large populations, so describing the dynamics using a system of ODEs makes sense.
The HANDY model equations are as follows:

$$\begin{aligned}\frac{dx_c}{dt} &= (\beta_c - \alpha_c)x_c,\\ \frac{dx_e}{dt} &= (\beta_e - \alpha_E)x_e,\\ \frac{dy}{dt} &= \gamma y(\lambda - y) - \delta x_c y,\\ \frac{dw}{dt} &= \delta x_c y - C_c - C_e.\end{aligned} \tag{3.1}$$

Here, β_c, β_e are the respective birth rates, assumed to be constant; γ is the nature's regeneration factor, λ is the *saturation level* or the *carrying capacity* of the natural resources, and δ the resources depletion rate constant, all three of which may be constants or given functions. Moreover, the death rates α_c, α_e, are assumed to be such that when the value of wealth is larger than a minimum threshold, the death rates are the 'natural' ones, [16], while when the wealth is below the minimum threshold necessary to sustain each person in the society, hunger or or famine result. To describe α_c, α_e, [14] introduced the averaged consumption functions for the workers C_c and the rich C_e,

$$C_c = \min(1, \frac{w}{w_{th}})sx_C, \qquad C_e = \min(1, \frac{w}{w_{th}})\kappa_1 sx_. \tag{3.2}$$

Here, s is the subsistence income (per capita); κ_1 is the elites' consumption factor, measuring the societal inequality in consumption: when $\kappa_1 = 1$ the society is egalitarian, and when $\kappa_1 = 100$ the average elite member has consumption rate 100 times bigger than an average commoner. As can be seen in [14] and also below, this factor is essential in the stability, flourishing or collapse of the society. Moreover, w_{th} is the minimum subsistence total wealth, or *wealth threshold,* below which the society cannot fully function. It is given by

$$w_{th} = \rho(x_c + \kappa_0 x_e), \tag{3.3}$$

where ρ is the "minimum required consumption per capita," and $\kappa_0 \geq 1$ is the ratio of what the rich pay themselves to what the workers are being paid. It follows from (3.2) that when the total wealth is above the threshold, $w \geq w_{th}$, then the consumption rates are $C_c = sx_c$ and $C_e = \kappa_1 s x_e$, respectively, and when $w < w_{th}$, the consumption rates are below the threshold, $C_c = sx_c w/w_{th}$ and $C_e = \kappa_1 s x_e w/w_{th}$. We return to the death rates. The 'normal' death rate, when there is enough food, is α_m for both populations, and α_M is the death rate when the accumulated wealth (food and related resources) is insufficient and the population is undernourished or when famine takes place. Then, [14] assumes that the death rates depend on the consumption as follows:

$$\begin{aligned}\alpha_c &= \alpha_m + \max(0, 1 - \min(1, \frac{w}{w_{th}}))(\alpha_M - \alpha_m),\\ \alpha_e &= \alpha_m + \max(0, 1 - \kappa_1 \min(1, \frac{w}{w_{th}}))(\alpha_M - \alpha_m).\end{aligned} \tag{3.4}$$

It follows that when $w \geq w_{th}$, we have $\alpha_c = \alpha_e = \alpha_m$. On the other extreme, when $w << w_{th}$, then $\alpha_c = \alpha_e \approx \alpha_M$ and the death rate is higher, since the population is closer to famine. For this reason it is assumed that $\alpha_M > \alpha_m > 0$.

We note that in the original HANDY model $\kappa_0 = \kappa_1 = \kappa$, while here, we allow for the inequality in consumption to be different from the inequality in wealth generation. Indirectly, in this ‘simple’ model κ_0 and κ_1 measure the ‘quality of life’ of the two populations.

To complete the model we need to prescribe the initial conditions. Then, the mathematical problem is the following.

Problem 3.1. *(HANDY model) Find four functions* $(x_c(t), x_e(t), y(t), w(t))$*, defined on* $[0, T]$*, such that*

$$\begin{aligned} x_c' &= \beta_c x_c - \alpha_c x_c, \\ x_e' &= \beta_e x_e - \alpha_e x_e, \\ y' &= \gamma y(\lambda - y) - \delta x_c y, \\ w' &= \delta x_c y - C_c - C_e, \end{aligned} \tag{3.5}$$

where C_c, C_e, w_{th} *and* α_c *and* α_e *are given in* (3.2)*,* (3.3)*, and* (3.4)*, respectively, together with the initial conditions*

$$x_c(0) = x_{c0}, \quad x_e(0) = x_{e0}, \quad y(0) = y_0, \quad w(0) = w_0. \tag{3.6}$$

Here, $T > 0$, and x_{c0}, x_{e0}, y_0, w_0 are non-negative numbers, the initial populations, natural resources and wealth.

The units used to measure the variables in the HANDY model are: Population (Commoners or Elites), in units of ‘individual.’ Nature and Wealth, in units of ‘eco-dollars.’ Time, t, in ‘years’. The structure of the model requires that Nature and Wealth be measured with a unified unit "eco-Dollar," see [14] for explanations. Other parameters and functions in the model have units that are compatible with the above mentioned ones.

The system's Carrying Capacity, χ, and the Maximum Carrying Capacity, χ_M, which can be found below, are both expressed in units of people. They measure the number of individuals in each population that can exist comfortably in the system's equilibrium.

3.2 Model analysis

Before we present any meaningful discussion of the types of behavior the model's solutions may exhibit, we must establish the existence and possible uniqueness of the solutions. To make these solutions relevant to the underlying process that it models, dealing with populations, natural resources and wealth, we establish the positivity of the solutions. The following subsection establishes the existence, uniqueness, and boundedness of the solutions of the HANDY model. That the solutions are positive when the initial conditions are positive is proved in the next subsection.
To avoid the singularity at the origin, we replace (3.3) with

$$w_{th} = \rho(x_c + \kappa_0 x_e + \epsilon), \tag{3.7}$$

where $\epsilon > 0$ is as small as needed. However, in some places ϵ is not needed, so we choose $\epsilon = 0$ there.

3.2.1 Existence and Uniqueness of Solutions

Let $\boldsymbol{z} = (x_c, x_e, y, w)$ and $\boldsymbol{F} : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$\boldsymbol{F}(\boldsymbol{z}) = \begin{pmatrix} \beta_c x_c - \alpha_c x_c \\ \beta_e x_e - \alpha_e x_e \\ \gamma y(\lambda - y) - \delta x_c y \\ \delta x_c y - C_c - C_e, \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}. \tag{3.8}$$

Here, C_c and C_e are given in (3.2), w_{th} is given in (3.3), and α_c and α_e are given in (3.4); they all are Lipschitz functions in $\boldsymbol{z}$, in the domains described shortly.

First, we establish that $\boldsymbol{F}(\boldsymbol{z}) = (F_1(\boldsymbol{z}), F_2(\boldsymbol{z}), F_3(\boldsymbol{z}), F_4(\boldsymbol{z})$ is Lipschitz continuous, then use the local existence theorem, Theorem 2.3. We assume, following [14], that $0 < \alpha_m \leq \beta_e \leq \beta_c \leq \alpha_M < 1$, it follows that

$$\frac{\alpha_M - \alpha_m}{\beta_e - \alpha_m} \geq \frac{\alpha_M - \alpha_m}{\beta_c - \alpha_m} \geq 1. \tag{3.9}$$

We let

$$\mathbb{R}^4_+ = \{\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4; 0 \leq z_1, z_2, z_3, z_4\}.$$

We introduce the regions

$$S_1 = \left\{\boldsymbol{z} \in \mathbb{R}^4_+ : \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)} \leq 1/\kappa_1\right\}, S_2 = \left\{\boldsymbol{z} \in \mathbb{R}^4_+ : \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)} \geq 1\right\}, \tag{3.10}$$

we let $S = S_1 \cup S_2$, and,

$$S_0 = \left\{\boldsymbol{z} \in \mathbb{R}^4_+ : \frac{1}{\kappa_1} < \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)} < 1\right\}. \tag{3.11}$$

We assume everywhere below that $\kappa_0, \kappa_1 \geq 1$. The introduction of ϵ was to avoid division by 0.

It is found that the function $\boldsymbol{F}$ is Lipschitz on S. More precisely,

Theorem 3.2. *Assume that $\kappa_0, \kappa_1 \geq 1$. Then, the function $\boldsymbol{F} : \mathbb{R}^4_+ \to \mathbb{R}^4$,* (5.6)*, is locally Lipschitz continuous in S_1 and in S_2, i.e., there is a positive constant K_R, which depends on R, such that if $\widehat{\boldsymbol{z}} = (\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \widehat{z}_4)$ and $\widetilde{\boldsymbol{z}} = (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4)$ are such* *that* $\|\widehat{\boldsymbol{z}}\| \leq R$ *and* $\|\widetilde{\boldsymbol{z}}\| \leq R$*, and either* $\widetilde{\boldsymbol{z}}, \widehat{\boldsymbol{z}} \in S_1$ *or* $\widetilde{\boldsymbol{z}}, \widehat{\boldsymbol{z}} \in S_2$ *then*

$$\|\boldsymbol{F}(\widehat{\boldsymbol{z}}) - \boldsymbol{F}(\widetilde{\boldsymbol{z}})\| \leq K_R \|\widehat{\boldsymbol{z}} - \widetilde{\boldsymbol{z}}\|. \tag{3.12}$$

However, as can be seen from the proof, not only $\boldsymbol{F}$ is separately Lipschitz on S_1 and on S_2, but F_1 and F_4 are Lipschitz on $S_1 \cup S_0$ and on S_2 (separately), F_2 is Lipschitz on $S_2 \cup S_0$ and on S_1 (separately), and F_3 is Lipschitz everywhere. This structure may be of some interest mathematically, as the question arises what happens at the hyper-surfaces where the transitions happen. Computationally, we found that the algorithms and computer runs were well behaved across the whole region so the transition hyper-surfaces may be of mathematical interest only.

Proof. We study each component of $\boldsymbol{F}$ in turn. We note that all the components of $\widehat{\boldsymbol{z}}$ and $\widetilde{\boldsymbol{z}}$ are nonegative. We write F_1 explicitly,

$$F_1(\boldsymbol{z}) = (\beta_c - \alpha_m) z_1 - \max\left(0, 1 - \min\left(1, \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)}\right)\right)(\alpha_M - \alpha_m) z_1.$$

Next, to simplify the expressions below, we use the notation

$$\widehat{\eta} = \frac{\widehat{z}_4}{\rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)}, \qquad \widetilde{\eta} = \frac{\widetilde{z}_4}{\rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)},$$

and note that $0 \leq \widehat{\eta}, \widetilde{\eta}$. Hence,

$$F_1(\widehat{\boldsymbol{z}}) - F_1(\widetilde{\boldsymbol{z}}) = (\beta_c - \alpha_m)(\widehat{z}_1 - \widetilde{z}_1) -$$

$$-(\alpha_M - \alpha_m) \max(0, 1 - \min(1, \widehat{\eta}))\, \widehat{z}_1 + (\alpha_M - \alpha_m) \max((0, 1 - \min(1, \widetilde{\eta})\, \widetilde{z}_1.$$

We let J_c be

$$J_c = (\max(0, 1 - \min(1, \widehat{\eta}))\widehat{z}_1 - (\max(0, 1 - \min(1, \widetilde{\eta}))\widetilde{z}_1.$$

Then,

$$F_1(\widehat{\boldsymbol{z}}) - F_1(\widetilde{\boldsymbol{z}}) = (\beta_c - \alpha_m)(\widehat{z}_1 - \widetilde{z}_1) - (\alpha_M - \alpha_m)J_c.$$

We now estimate J_c, considering the following two cases.

(i) When $\widehat{z}_4 \geq \rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)$, i.e. $\widehat{\eta} \geq 1$ and $\widetilde{z}_4 \geq \rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)$, i.e., $\widetilde{\eta} \geq 1$, so that $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_2$, then,

$$J_c = 0.$$

(ii) When $\widehat{z}_4 < \rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)$ and $\widetilde{z}_4 < \rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)$, i.e., $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_1 \cup S_0$, then

$$J_c = (1 - \widehat{\eta})\widehat{z}_1 - (1 - \widetilde{\eta})\widetilde{z}_1,$$

Assuming first that $\widehat{\eta} \leq \widetilde{\eta}$, then $|\widehat{z}_1 - (\widetilde{\eta}/\widehat{\eta})\widetilde{z}_1| \leq |\widehat{z}_1 - \widetilde{z}_1|$, and so

$$|J_c| \leq (1 + \widehat{\eta})|\widehat{z}_1 - \widetilde{z}_1|.$$

When $\widehat{\eta} \geq \widetilde{\eta}$, then $|\widetilde{z}_1 - (\widehat{\varepsilon}/\widetilde{\eta})\widehat{z}_1| \leq |\widetilde{z}_1 - \widehat{z}_1|$, and so

$$|J_c| \leq (1 + \widehat{\eta})|\widehat{z}_1 - \widetilde{z}_1|.$$

Therefore, in case (ii),

$$|J_c| \leq 2|\widehat{z}_1 - \widetilde{z}_1|.$$

We conclude that

$$|F_1(\widehat{\boldsymbol{z}}) - F_1(\widetilde{\boldsymbol{z}})| \leq (|\beta_c - \alpha_m| + 2(\alpha_M - \alpha_m)|\widehat{z}_1 - \widetilde{z}_1|.$$

This completes the proof that F_1 is Lipschitz on $S_1 \cup S_0$ and S_2.

The proof that F_2 is Lipschitz is very similar, however, F_2 is Lipschitz on $S_2 \cup S_0$ and S_1. We may write F_2 as

$$F_2(\boldsymbol{z}) = (\beta_e - \alpha_m)z_2 - \max\left(0, 1 - \kappa_1 \min\left(1, \frac{z_4}{\rho(z_1 + \kappa_0 z_2)}\right)\right)(\alpha_M - \alpha_m)z_2.$$

Hence,

$$F_2(\widehat{\boldsymbol{z}}) - F_2(\widetilde{\boldsymbol{z}}) = (\beta_e - \alpha_m)\left(\widehat{z}_2 - \widetilde{z}_2 + \epsilon\right)$$

$$-(\alpha_M - \alpha_m)\max\left(0, 1 - \kappa_1 \min\left(1, \widehat{\eta}\right)\right)\widehat{z}_2 +$$

$$+(\alpha_M - \alpha_m)\max((0, 1 - \kappa_1 \min(1, \widetilde{\eta})\widetilde{z}_2.$$

Where again, we use the notation

$$\widehat{\eta} = \frac{\widehat{z}_4}{\rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)}, \qquad \widetilde{\eta} = \frac{\widetilde{z}_4}{\rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)}.$$

We let J_e be

$$J_e = (\max(0, 1 - \kappa_1 \min\left(1, \widehat{\eta}\right))\widehat{z}_2 - (\max(0, 1 - \kappa_1 \min\left(1, \widetilde{\eta}\right))\widetilde{z}_2.$$

Then,

$$F_2(\widehat{\boldsymbol{z}}) - F_2(\widetilde{\boldsymbol{z}}) = (\beta_e - \alpha_m)(\widehat{z}_2 - \widetilde{z}_2) - (\alpha_M - \alpha_m)J_e.$$

To estimate J_e, we recall that $\kappa_1 \geq 1$, and consider the following two cases:

(i) When $\widehat{z}_4 \geq \rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)/\kappa_1$, and $\widetilde{z}_4 \geq \rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)/\kappa_1$, i.e. $\widehat{\eta} \geq 1/\kappa_1$ and $\widetilde{\eta} \geq 1/\kappa_1$, so that $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_0 \cup S_2$, we have

$$J_e = 0.$$

Indeed, $\max(0, 1 - \kappa_1 \widehat{\eta}) = 0$ and $\max(0, 1 - \kappa_1 \widetilde{\eta}) = 0$.

(ii) When $\widehat{z}_4 < \rho(\widehat{z}_1 + \kappa_0 \widehat{z}_2 + \epsilon)/\kappa_1$ and $\widetilde{z}_4 < \rho(\widetilde{z}_1 + \kappa_0 \widetilde{z}_2 + \epsilon)/\kappa_1$, i.e., $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_1$; then

$$J_e = (1 - \kappa_1 \widehat{\eta})\widehat{z}_2 - (1 - \kappa_1 \widetilde{\eta})\widetilde{z}_2,$$

Hence, assuming first that $\widehat{\eta} \leq \widetilde{\eta}$, then $|\widehat{z}_2 - (\widetilde{\eta}/\widehat{\eta})\widetilde{z}_2| \leq |\widehat{z}_2 - \widetilde{z}_2|$, and so

$$|J_e| \leq (1 + \kappa_1 \widehat{\eta})|\widehat{z}_2 - \widetilde{z}_2|.$$

When $\widehat{\eta} \geq \widetilde{\eta}$, then $|\widetilde{z}_2 - (\widehat{\varepsilon}/\widetilde{\eta})\widehat{z}_2| \leq |\widetilde{z}_2 - \widehat{z}_2|$, and so

$$|J_e| \leq (1 + \kappa_1 \widehat{\eta})|\widehat{z}_2 - \widetilde{z}_2|.$$

Therefore, in this case, since $\kappa_1 \widehat{\eta} < 1$ and $\kappa_1 \widetilde{\eta} < 1$,

$$|J_e| \leq 2|\widehat{z}_2 - \widetilde{z}_2|.$$

We conclude that

$$|F_2(\widehat{\boldsymbol{z}}) - F_2(\widetilde{\boldsymbol{z}})| \leq (|\beta_e - \alpha_m| + 2(\alpha_M - \alpha_m)|\widehat{z}_2 - \widetilde{z}_2|.$$

This shows that F_2 is Lipschitz on $S_0 \cup S_2$ and on S_1.

We turn to estimate F_3. We have,

$$F_3(\boldsymbol{z}) = \gamma z_3(\lambda - z_3) - \delta z_1 z_3 = \gamma\lambda z_3 - \gamma z_3^2 - \delta z_1 z_3.$$

Hence,

$$F_3(\widehat{\boldsymbol{z}}) - F_3(\widetilde{\boldsymbol{z}}) = \gamma\lambda(\widehat{z}_3 - \widetilde{z}_3) - \gamma(\widehat{z}_3^2 - \widetilde{z}_3{}^2) - \delta(\widehat{z}_1\widehat{z}_3 - \widetilde{z}_1\widetilde{z}_3)$$

$$= \gamma\lambda(\widehat{z}_3 - \widetilde{z}_3) - \gamma(\widehat{z}_3 + \widetilde{z}_3)(\widehat{z}_3 - \widetilde{z}_3) - \delta\widehat{z}_1(\widehat{z}_3 - \widetilde{z}_3) - \delta\widetilde{z}_1(\widehat{z}_3 - \widetilde{z}_3).$$

Thus,

$$|F_3(\widehat{\boldsymbol{z}}) - F_3(\widetilde{\boldsymbol{z}})| \leq |\gamma\lambda - \gamma(\widehat{z}_3 + \widetilde{z}_3) - \delta\widehat{z}_1||\widehat{z}_3 - \widetilde{z}_3| + |\delta\widetilde{z}_1||\widehat{z}_3 - \widetilde{z}_3|.$$

$$\leq (\gamma\lambda + 2\gamma R + \delta R)|\widehat{z}_3 - \widetilde{z}_3| + \delta R|\widehat{z}_3 - \widetilde{z}_3| \leq K_{3R}\|\widehat{\boldsymbol{z}} - \widetilde{\boldsymbol{z}}\|.$$

Here, we choose $K_{3R} = \gamma\lambda + 2\gamma R + \delta R$. This concludes the proof that F_3 is Lipschitz.

Finally, to estimate the last component, we note that

$$F_4(\boldsymbol{z}) = \delta z_1 z_3 - \min\left(1, \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)}\right) s z_1 - \min\left(1, \frac{z_4}{\rho(z_1 + \kappa_0 z_2 + \epsilon)}\right) \kappa_1 s z_2$$

Then,

$$F_4(\widehat{\boldsymbol{z}}) - F_4(\widetilde{\boldsymbol{z}}) = \delta(\widehat{z}_1\widehat{z}_3 - \widetilde{z}_1\widetilde{z}_3) - s\widehat{z}_1 \min(1,\ \widehat{\eta}) - \kappa_1 s\widehat{z}_2 \min(1, \widehat{\eta})$$

$$+ s\widetilde{z}_1 \min(1, \widetilde{\eta}) + \kappa_1 s\widetilde{z}_2 \min(1, \widetilde{\eta})) \, .$$

We deal with the terms separately.

$$J_1 = \delta(\widehat{z}_1\widehat{z}_3 - \widetilde{z}_1\widetilde{z}_3) = \delta\widehat{z}_1(\widehat{z}_3 - \widetilde{z}_3) + \delta\widetilde{z}_3(\widehat{z}_1 - \widetilde{z}_1).$$

Therefore,

$$|J_1| \leq \delta R \|\widehat{\boldsymbol{z}} - \widetilde{\boldsymbol{z}}\|.$$

We define now

$$J_2 = -s(\min(1, \widehat{\eta})\, \widehat{z}_1 - \min(1, \widetilde{\eta})\, \widetilde{z}_1),$$

and

$$J_3 = -s\kappa_1(\min(1, \widehat{\eta})\, \widehat{z}_2 - \min(1, \widetilde{\eta})\, \widetilde{z}_2).$$

We consider two cases, again. (i) $1 < \widehat{\eta}, \widetilde{\eta}$, so that $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_2$; (ii) $0 \leq \widehat{\eta}, \widetilde{\eta} < 1$, and $\widehat{\boldsymbol{z}}, \widetilde{\boldsymbol{z}} \in S_1$;

In case (i), we have

$$|J_2| = s|\widehat{z}_1 - \widetilde{z}_1|, \qquad |J_3| = s\kappa_1|\widehat{z}_2 - \widetilde{z}_2|.$$

In case (ii), the following holds

$$|J_2| = s|\widehat{\eta}\widehat{z}_1 - \widetilde{\eta}\widetilde{z}_1|.$$

Noting that when $\widehat{\eta} \leq \widetilde{\eta}$,

$$|\widehat{\eta}\widehat{z}_1 - \widetilde{\eta}\widetilde{z}_1| = \widehat{\eta}|\widehat{z}_1 - (\widetilde{\eta}/\widehat{\eta})\widetilde{z}_1| \leq \widehat{\eta}|\widehat{z}_1 - \widetilde{z}_1|.$$

Similarly, when $\widehat{\eta} \geq \widetilde{\eta}$,

$$|\widehat{\eta}\widehat{z}_1 - \widetilde{\eta}\widetilde{z}_1| = \widetilde{\eta}|(\widehat{\eta}/\widetilde{\eta})\widehat{z}_1 - \widetilde{z}_1| \leq \widetilde{\eta}|\widetilde{z}_1 - \widehat{z}_1|.$$

Therefore,

$$|J_2| \leq s \max(\widehat{\eta}, \widetilde{\eta})|\widehat{z}_1 - \widetilde{z}_1|.$$

Similar arguments yield

$$|J_3| \leq s\kappa_1 \max(\widehat{\eta}, \widetilde{\eta})|\widehat{z}_2 - \widetilde{z}_2|.$$

It follows that

$$|F_4(\widehat{\boldsymbol{z}}) - F_4(\widetilde{\boldsymbol{z}})| \leq K_R^1 |\widehat{\boldsymbol{z}} - \widetilde{\boldsymbol{z}}|.$$

Collecting all the estimates above shows that $\boldsymbol{F}$ is Lipschitz on S_1 and S_2, which completes the proof of the theorem. ■

Now, we are in the position to use the main existence theorem for ODEs, Theorem 2.3, because $\boldsymbol{F}$ is Lipschitz continuous on S_1 and S_2.

Theorem 3.3. *(Local Existence and Uniqueness) Consider Problem* (3.1) *with the initial conditions* $\boldsymbol{z}_0 = (x_{c0}, x_{e0}, y_0, w_0)$*. Assume that* $\boldsymbol{z}_0 \in S_1$*; then, there exists* $T > 0$ *such that the problem has a unique solution* $\boldsymbol{z}(t)$*, for* $0 \leq t < T$*, such that* $\boldsymbol{z}(0) = \boldsymbol{z}_0$ *and* $\boldsymbol{z}(t) \in S_1$*. Similarly, assume that* $\boldsymbol{z}_0 \in S_2$ *then, there exists* $T > 0$ *and the problem has a unique solution* $\boldsymbol{z}(t)$*, for* $0 \leq t < T$*, such that* $\boldsymbol{z}(0) = \boldsymbol{z}_0$ *and* $\boldsymbol{z}(t) \in S_2$*.*

We conclude that Problem 3.1 has a unique local solution if the initial data are in either S_1 or S_2, and then the local solution remains there. We return below to the question of whether the solutions can cross S_0. The computer simulations indicate that the global solutions can be found in all the regions, and they do cross the boundaries of the S regions. Actually, as we show in Chapter 5.1, the HANDY Model with Social Mobility, we can obtain global solutions without the global Lipschitz condition, however, the solutions' uniqueness is unresolved there.

3.2.2 Positivity of the solution

Theorem 3.3 provides the uniqueness of the local solution for the model 3.1. The non-negativity of the solution should be proven so that the model makes sense, since we deal with populations, natural resources and wealth (nonnegative).
We denote by T^* the maximal time on which the solution exists and then the solution is continuous on $[0, T^*)$. We prove the following theorem:

Theorem 3.4. *(Positivity of the solutions) Assume that the initial conditions satisfy* $x_{c0} > 0, x_{e0} \geq 0, y(0) \geq 0,$ *and* $w(0) \geq 0$. *Then, the solution* $\boldsymbol{z} = (x_c(t), x_e(t), y(t), w(t))$ *of Problem 3.1 is non-negative as long as it exists, i.e., on* $[0, T^*)$.

We need the following inequalities, which we assume, below,

$$\alpha_m \leq \alpha_c, \alpha_e \leq \alpha_M. \tag{3.13}$$

Proof. We note that $(0, 0, 0, 0)$ and $(0, 0, \lambda, 0)$ are critical points or steady states of the system, and the first is unstable while the second is stable, as we show below. So, seeking other solutions, we assume that $x_{c0} > 0, x_{e0} \geq 0, y(0) > 0$, and $w(0) \geq 0$. We study each equation in turn, starting with the commoners' equation. Thus,

$$\frac{dx_c}{dt} = (\beta_c - \alpha_c)x_c,$$

and using separation of variables, we obtain

$$\frac{dx_c}{x_c} = (\beta_c - \alpha_c)dt \geq -\alpha_c dt \geq -\alpha_M dt.$$

Thus,

$$\int_{x_{co}}^{x_c} \frac{dx_c}{x_c} \geq -\alpha_M \int_0^t dt.$$

Solving the integral inequality yields,

$$\ln x_c - \ln x_{co} \geq -\alpha_M t,$$

and then

$$x_c(t) \geq x_{co} \exp(-\alpha_M t) > 0,$$

for all $0 \leq t < T^*$, since $x_{co} > 0$.

We use similar steps For the Elites' equation and find,

$$x_e(t) \geq x_{eo} \exp(-\alpha_M t) \geq 0,$$

for all $0 \leq t < T^*$. Here, the right-hand side vanishes when $x_{eo} = 0$.

We turn to the Nature's resources equation,

$$\frac{dy}{dt} = \gamma y(\lambda - y) - \delta x_c y,$$

and since $x_c > 0$ and $\gamma, \delta > 0$, it follows that

$$\frac{dy}{dt} \geq -(\gamma y + \delta x_c) y.$$

Next, since

$$\frac{dy}{dt} \leq \gamma \lambda y,$$

it follows that

$$y(t) \leq y_0 e^{\gamma \lambda t}.$$

Moreover, it is straightforward to see, since $dx_c(t)/dt \le \beta_c x_c$, that

$$x_c(t) \le x_{co} \exp(\beta_c t).$$

Inserting these estimates in the above estimate yields

$$\frac{dy}{dt} \ge -(\gamma y_0 e^{\gamma\lambda t} + \delta x_{co} e^{\beta_c t})y.$$

It follows that

$$y(t) \ge y_0 \exp\left(-(\gamma y_0 e^{\gamma\lambda t} + \delta x_{co} e^{\beta_c t})\right) > 0,$$

for $0 \le t < T^*$, that is, as long as the solution exists.

Finally, we prove the positivity of the wealth solution $w(t)$. Then, on the interval $[0, T^*)$ we have that $y(t) > 0$, and hence,

$$\frac{dw}{dt} = \delta x_c y - \min(1, w/w_{th}) s x_c - \min(1, w/w_{th}) \kappa_1 s x_e.$$

If $w \ge \rho(x_c + \kappa_0 x_e)$ then $w > 0$. So, assume that $w \le \rho(x_c + \kappa_0 x_e)$, hence

$$\frac{dw}{dt} = \delta x_c y - w s x_c / w_{th} - w \kappa_1 s x_e / w_{th}$$

$$\ge -w(s x_c + \kappa_1 s x_e)/w_{th}$$

$$= -w \left(\frac{s x_c + \kappa_1 s x_e}{\rho(x_c + \kappa_0 x_e)} \right) = -w \frac{s}{\rho} \left(\frac{x_c + \kappa_1 x_e}{x_c + \kappa_0 x_e} \right).$$

We have two cases: (i) when $\kappa_0 < \kappa_1$, and (ii) when $\kappa_0 > \kappa_1$. In case (i),

$$\frac{x_c + \kappa_1 x_e}{x_c + \kappa_0 x_e} = \frac{1 + \kappa_1 (x_e/x_c)}{1 + \kappa_0 (x_e/x_c)} \le \frac{\kappa_1}{\kappa_0}.$$

In case (ii),

$$\frac{x_c + \kappa_1 x_e}{x_c + \kappa_0 x_e} = \frac{1 + \kappa_1 (x_e/x_c)}{1 + \kappa_0 (x_e/x_c)} \leq \frac{\kappa_0}{\kappa_1}.$$

Let,

$$K = \max\left(\frac{\kappa_1}{\kappa_0}, \frac{\kappa_0}{\kappa_1}\right).$$

Then, in either case, it is found that

$$\frac{dw}{dt} \geq -w\frac{sK}{\rho}.$$

Therefore,

$$w(t) \geq w_0 \exp\left(\frac{-sK}{\rho}t\right) \geq 0,$$

and the equality holds only when $w_0 = 0$.

This completes the proof of the theorem. ■

The next result shows that the solution is bounded on every finite time interval, and therefore, the solution of the problem exists on every finite time interval.

Theorem 3.5. *The local solution* $\boldsymbol{z} = (x_c, x_e, y, w)$ *exist and is bounded for* $t \in [0, T]$, *for each* $0 < T < \infty$. *Moreover, it satisfies,*

$$0 < x_{c0} \exp(-\alpha_M t) \leq x_c(t) \leq x_{c0} \exp(\beta_c t), \tag{3.14}$$

$$0 \leq x_{e0} \exp(-\alpha_M t) \leq x_e(t) \leq x_{c0} \exp(\beta_e t), \tag{3.15}$$

$$0 < y_0 \exp\left(-(\gamma y_0 e^{\gamma\lambda t} + \delta x_{co} e^{\beta_c t})\right) \leq y(t) \leq y_0 e^{\gamma\lambda t}, \tag{3.16}$$

$$0 \leq w_0 \exp\left(\frac{-s\gamma}{\rho} t\right) \leq w(t) \leq \widehat{K} \exp((\beta_c + \gamma\lambda)t). \tag{3.17}$$

Here,

$$\widehat{K} = \frac{\delta x_{co} y_0 w_0}{\beta_c + \gamma\lambda}.$$

Therefore, the functions are bounded on every finite interval $[0, T]$*, hence* $T^* = \infty$*.*

Proof. First, we deal with the Commoners or workers equation. Thus,

$$\frac{dx_c}{dt} = (\beta_c - \alpha_c)x_c \leq \beta_c x_c,$$

and using separation of variables, we obtain

$$\int_{x_{c0}}^{x_c} \frac{dx_c}{x_c} \leq \beta_c \int_0^t dt = \beta_c t.$$

So,

$$x_c(t) \leq x_{c0} \exp(\beta_c t) \leq x_{c0} \exp(\beta_c T).$$

The other side of the equality was obtained above. The inequality for x_e follows in the same way.

The inequality (3.16) follows from the estimate above. Finally, it is noted that the estimates (3.14) and (3.16) yield

$$\frac{dw}{dt} \leq \delta x_c y \leq \delta(x_{co} \exp(\beta_c t) y_0 e^{\gamma\lambda t} = \delta x_{co} y_0 \exp((\beta_c + \gamma\lambda)t).$$

This yields,

$$w \leq \widehat{K} \exp((\beta_c + \gamma\lambda)t).$$

■

We conclude this section with the following main theoretical result concerning the classical HANDY model, which summarizes the results above.

Theorem 3.6. *The solution $\boldsymbol{z} = (x_c, x_e, y, w)$ of Problem 3.1 exist and is bounded for $t \in [0,T]$, for each $0 < T < \infty$. It is nonnegative when the initial conditions are nonnegative.*

3.3 Stability of the steady states

In this section, we study the stability of the two steady states with zero populations: the origin $\boldsymbol{z}_0 = (0,0,0,0)$ and $\boldsymbol{z}_\lambda = (0,0,\lambda,0)$. To that end, we compute the Jacobian at $\boldsymbol{z}_0$ and $\boldsymbol{z}_\lambda$. Here, we use the expression (3.7) for w_{th}, to avoid the singularity at the origin. Then, we discuss the other possible steady states.

It follows from (5.6) that the Jacobian matrix of the system is given by

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_c} & \frac{\partial F_1}{\partial x_e} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial x_c} & \frac{\partial F_2}{\partial x_e} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial x_c} & \frac{\partial F_3}{\partial x_e} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial w} \\ \frac{\partial F_4}{\partial x_c} & \frac{\partial F_4}{\partial x_e} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial w} \end{pmatrix} = (J_{ij}),$$

where $i, j = 1, 2, 3, 4$.

First, we are interested in the origin. Since we use the form of w_{th} given in (3.7), and $\epsilon > 0$, it follows that it is enough to consider the case $\eta = \frac{w}{w_{th}} < 1$, so that $w < w_{th}$. Therefore,

$$\alpha_c = \alpha_m + \max(0, 1 - \min(1, \eta))(\alpha_M - \alpha_m)$$

$$= \alpha_m + \max(0, 1 - \eta)(\alpha_M - \alpha_m)$$

$$= \alpha_m + (1-\eta)\,(\alpha_M - \alpha_m) = \alpha_M - (\alpha_M - \alpha_m)\frac{w}{w_{th}}.$$

And

$$\alpha_e = \alpha_m + \max(0, 1 - \kappa_1 \min(1,\eta))(\alpha_M - \alpha_m)$$

$$= \alpha_m + (1-\kappa_1\eta)\,(\alpha_M - \alpha_m) = \alpha_M - \kappa_1(\alpha_M - \alpha_m)\frac{w}{w_{th}}.$$

Moreover, in this case, we have

$$C_c = s\frac{w}{w_{th}}x_c, \qquad C_e = \kappa_1 s\frac{w}{w_{th}}x_e.$$

We compute the J_{ij}, starting with $i = 1$,

$$J_{11} = \frac{\partial F_1}{\partial x_c} = \beta_c - \alpha_c - x_c\frac{\partial \alpha_c}{\partial x_c}.$$

It follows that

$$J_{11} = (\beta_c - \alpha_M) + (\alpha_M - \alpha_m)\frac{w}{w_{th}} - x_c\frac{\partial \alpha_c}{\partial x_c}.$$

Next,

$$\frac{\partial \alpha_c}{\partial x_c} = (\alpha_M - \alpha_m)\frac{w}{w_{th}^2}\frac{\partial w_{th}}{\partial x_c} = \rho(\alpha_M - \alpha_m)\frac{w}{w_{th}^2}.$$

Thus,

$$J_{11} = \frac{\partial F_1}{\partial x_c} = \beta_c - \alpha_M + \eta(\alpha_M - \alpha_m)\left(1 - \rho\frac{x_c}{w_{th}}\right).$$

Next,

$$J_{12} = -x_c\frac{\partial \alpha_c}{\partial x_e} = (\alpha_M - \alpha_m)\frac{w x_c}{w_{th}^2}\frac{\partial w_{th}}{\partial x_e} = \eta\rho\kappa_0(\alpha_M - \alpha_m)\frac{x_c}{w_{th}}.$$

It is straightforward to see,

$$J_{13} = 0, \qquad J_{14} = (\alpha_M - \alpha_m)\frac{x_c}{w_{th}}.$$

We turn to the second row, $i = 2$.

$$J_{21} = \frac{\partial F_2}{\partial x_c} = -x_e \frac{\partial \alpha_e}{\partial x_c} = -\eta \kappa_1 \rho (\alpha_M - \alpha_m) \frac{x_e}{w_{th}}.$$

Next,

$$\frac{\partial \alpha_e}{\partial x_e} = \rho \kappa_1 \kappa_0 (\alpha_M - \alpha_m) \frac{w}{w_{th}^2}.$$

Thus,

$$J_{22} = (\beta_e - \alpha_M) + \eta \kappa_1 (\alpha_M - \alpha_m) \left(1 - \rho \kappa_0 \frac{x_e}{w_{th}}\right).$$

Next,

$$J_{23} = 0, \qquad J_{24} = \kappa_1 (\alpha_M - \alpha_m) \frac{x_e}{w_{th}}.$$

The third row, $i = 3$, follows.

$$J_{31} = \frac{\partial F_3}{\partial x_c} = -\delta y, \qquad J_{32} = \frac{\partial F_3}{\partial x_e} = 0.$$

$$J_{33} = \frac{\partial F_3}{\partial y} = \gamma \lambda - 2\gamma y - \delta x_c.$$

$$J_{34} = \frac{\partial F_3}{\partial w} = 0.$$

Finally, we compute row four, $i = 4$. Since

$$F_4 = \delta x_c y - s x_c \eta - \kappa_1 s x_e \eta,$$

we have the following,

$$J_{41} = \frac{\partial F_4}{\partial x_c} = \delta y - \eta s + \eta s (x_c + \kappa_1 x_e) \frac{1}{w_{th}}.$$

$$J_{42} = \frac{\partial F_4}{\partial x_e} = -\eta s \kappa_1 + \eta s \rho \kappa_0 (x_c + \kappa_1 x_e) \frac{1}{w_{th}}.$$

$$J_{43} = \frac{\partial F_4}{\partial y} = \delta x_c.$$

$$J_{44} = \frac{\partial F_4}{\partial w} = -s(x_c + \kappa_1 x_e) \frac{1}{w_{th}}.$$

This completes the construction of the components of the Jacobian matrix J.

Evaluating J at the origin, recalling that $w_{th} \geq \epsilon > 0$ so that $\eta = 0$ and also $\alpha_c = \alpha_e = \alpha_M$ and $C_c = C_e = 0$, we obtain,

$$J(0,0,0,0) = \begin{pmatrix} \beta_c - \alpha_M & 0 & 0 & 0 \\ 0 & \beta_e - \alpha_M & 0 & 0 \\ 0 & 0 & \gamma\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.18}$$

Therefore, the eigenvalues of J at the origin are

$$\widehat{\lambda}_1 = \beta_c - \alpha_M, \quad \widehat{\lambda}_2 = \beta_e - \alpha_M, \quad \widehat{\lambda}_3 = \gamma\lambda, \quad \widehat{\lambda}_4 = 0.$$

Since, by assumption

$$0 < \alpha_m \leq \beta_e \leq \beta_c \leq \alpha_M < 1, \tag{3.19}$$

it follows that

$$\widehat{\lambda}_1 \leq 0, \quad \widehat{\lambda}_2 \leq 0, \quad \widehat{\lambda}_3 > 0, \quad \widehat{\lambda}_4 = 0.$$

We conclude that the origin is unstable since $\gamma\lambda > 0$.

Next, we consider $\boldsymbol{z}_\lambda = (0, 0, \lambda, 0)$. Using the same procedure, we find that

$$J(0,0,\lambda,0) = \begin{pmatrix} \beta_c - \alpha_M & 0 & 0 & 0 \\ 0 & \beta_e - \alpha_M & 0 & 0 \\ -\delta\lambda & 0 & -\gamma\lambda & 0 \\ \delta\lambda & 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

Therefore, the eigenvalues of J are

$$\widehat{\lambda}_1 = \beta_c - \alpha_M \leq 0, \quad \widehat{\lambda}_2 = \beta_e - \alpha_M \leq 0, \quad \widehat{\lambda}_3 = -\gamma\lambda < 0, \quad \widehat{\lambda}_4 = 0.$$

We conclude that $\boldsymbol{z}_\lambda = (0, 0, \lambda, 0)$ is stable and may have three attracting directions. We summarize it as follows.

Proposition 3.7. *The origin, $\boldsymbol{z}_0 = (0, 0, 0, 0)$ in the HANDY model, Problem 3.1, is unstable, and the steady state $\boldsymbol{z}_\lambda = (0, 0, \lambda, 0)$ is stable.*

Theoretically, since the origin is unstable, there are likely to be other equilibria, periodic solutions, limit cycles or strange attractors. So we turn to study the stability of the other possible critical points of the system. To have nonzero equilibria, we must put restrictions on the various system coefficients, which we do as needed. We denote by $\bar{\boldsymbol{z}} = (\bar{x}_c, \bar{x}_e, \bar{y}, \bar{w})$ the equilibrium values, and assume that $\bar{x}_c > 0, \bar{y} > 0$ and $\bar{x}_e, \bar{w} \geq 0$. Indeed, there is no point in an equilibrium with only the rich and no workers to support them, or without natural resources.

We have that $F_1 = (\beta_c - \alpha_c)x_c$, hence to have $F_1 = 0$, when $x_c > 0$, we need

$$\beta_c = \alpha_c = \alpha_m + \max(0, 1 - \min(1, \frac{\bar{w}}{\bar{w}_{th}}))(\alpha_M - \alpha_m)$$

where $\bar{w}_{th} = \rho(\bar{x}_c + \kappa_0 \bar{x}_e + \epsilon)$.

When $F_2 = (\beta_e - \alpha_e)\bar{x}_e = 0$, hence either $\bar{x}_e = 0$ or

$$\beta_e = \alpha_e = \alpha_m + \max(0, 1 - \kappa_1 \min(1, \frac{\bar{w}}{\bar{w}_{th}}))(\alpha_M - \alpha_m).$$

When $F_3 = (\gamma(\lambda - \bar{y}) - \delta\bar{x}_c)\bar{y} = 0$, since $\bar{y} > 0$, we find

$$\bar{y} = \lambda - \frac{\delta}{\gamma}\bar{x}_c.$$

Thus, $0 < \bar{x}_c < \gamma\lambda/\delta$.

Next, when $F_4 = \delta\bar{x}_c\bar{y} - C_c - C_e = 0$, we find that

$$\delta\bar{x}_c\bar{y} = C_c + C_e = \min(1, \frac{\bar{w}}{\bar{w}_{th}})s\bar{x}_c + \min(1, \frac{\bar{w}}{\bar{w}_{th}})\kappa_1 s\bar{x}_e$$

To proceed, we consider the two cases: (i) $\eta = \bar{w}/\bar{w}_{th} < 1$; and $\eta = \bar{w}/\bar{w}_{th} \geq 1$.

We consider case (i) first. The assumption $\eta < 1$ yields

$$\beta_c = \alpha_m + (1 - \eta)(\alpha_M - \alpha_m) = \alpha_M - (\alpha_M - \alpha_m)\eta,$$

$$\beta_e = \alpha_m + (1 - \kappa_1\eta)(\alpha_M - \alpha_m) = \alpha_M - \eta\kappa_1(\alpha_M - \alpha_m),$$

$$C_c = \eta s\bar{x}_c, \qquad C_e = \eta\kappa_1 s\bar{x}_e.$$

Then, since we are interested in solutions where $\bar{x}_c > 0$, we assume below that $\epsilon = 0$, and we need to solve the algebraic system,

$$(\alpha_M - \alpha_m)\eta = \alpha_M - \beta_c, \tag{3.21}$$

$$\kappa_1(\alpha_M - \alpha_m)\eta = \alpha_M - \beta_e, \tag{3.22}$$

$$\bar{y} = \lambda - \frac{\delta}{\gamma}\bar{x}_c, \tag{3.23}$$

$$\bar{w} = \eta\rho(\bar{x}_c + \kappa_0\bar{x}_e), \tag{3.24}$$

$$\delta\bar{x}_c\bar{y} = s(\bar{x}_c + \kappa_1\bar{x}_e)\eta. \tag{3.25}$$

It is seen that various values of the parameters lead to different critical points. First, we replace (3.19), with the assumption

$$0 < \alpha_m < \beta_e \leq \beta_c < \alpha_M < 1, \tag{3.26}$$

Then, (3.21) and (3.22) imply that in this case

$$\eta = \frac{\bar{w}}{\bar{w}_{th}} = \frac{\alpha_M - \beta_c}{\alpha_M - \alpha_m} = \frac{\alpha_M - \beta_e}{\kappa_1(\alpha_M - \alpha_m)} < 1. \tag{3.27}$$

It follows that in this case we must have $\alpha_M - \beta_c = (\alpha_M - \beta_e)/\kappa_1$, which imposes an interesting restriction on these parameter values. In particular, if we assume as in [14] that $\beta_c = \beta_e$, then necessarily $\kappa_1 = 1$, which, when $\kappa_0 = 1$, is the equitable society.

We denote by $\psi = \bar{x}_e/\bar{x}_c$ the ratio of the rich to workers in the steady state, and then straightforward algebraic manipulations yield,

$$\begin{aligned} \bar{x}_c &= \frac{\gamma}{\delta}\left(\lambda - \eta\frac{s}{\delta}(1+\kappa_1\psi)\right), \\ \bar{x}_e &= \psi\bar{x}_c, \\ \bar{y} &= \eta\frac{s}{\delta}(1+\kappa_1\psi), \\ \bar{w} &= \eta\rho(1+\kappa_0\psi)\bar{x}_c. \end{aligned} \tag{3.28}$$

We conclude that η is fixed at the value (3.27), and there is a one-parameter family of steady states that is parametrized with ψ, and moreover, since we require that $\bar{x}_c > 0$, it follows from the first equality in (3.36) that

$$0 \le \psi < \frac{1}{\kappa_1}\left(\frac{\delta\lambda}{\eta s} - 1\right).$$

We turn to the case (ii) when $\eta \ge 1$. Then,

$$C_c = s\bar{x}_c, \qquad C_e = \kappa_1 s\bar{x}_e, \qquad \alpha_c = \alpha_e = \alpha_m.$$

Since we assume that $\bar{x}_c > 0$ we must have $\beta_c = \beta_e = \alpha_m$. Setting again $\psi = \bar{x}_e/\bar{x}_c$, after some algebraic manipulations, we obtain

$$\begin{aligned} \bar{x}_c &= \frac{\gamma}{\delta}\left(\lambda - \frac{s}{\delta}(1+\kappa_1\psi)\right), \\ \bar{x}_e &= \psi\bar{x}_c, \\ \bar{y} &= \frac{s}{\delta}(1+\kappa_1\psi), \\ \bar{w} &= \eta\rho(1+\kappa_0\psi)\bar{x}_c. \end{aligned} \tag{3.29}$$

The steady states in this case are characterized by a two-parameter family of solutions, with $\eta \geq 1$. Moreover, it follows from the first equality in (3.29) that to guarantee $\bar{x}_c > 0$, we must have $\lambda > s(1 + \kappa_1\psi)/\delta$, thus,

$$0 \leq \psi < \frac{1}{\kappa_1}\left(\frac{\lambda\delta}{s} - 1\right). \tag{3.30}$$

Here, the lower end corresponds to $\bar{x}_e = 0$ and the upper end to $\bar{x}_c = 0$.

Remark 3.8. *We note that $\bar{x}_c, \bar{x}_e$ and $\bar{y}$ depend only on ψ, while $\bar{w}$ also depends on η. The fact that $\bar{x}_c, \bar{x}_e$ and $\bar{y}$ do not depend on η is a consequence of the structure of the problem, more precisely, for $\eta \geq 1$ the functions $C_c, C_e, \alpha_c, \alpha_e$ do not depend on η, the ratio of the wealth to the wealth threshold.*

Using the parameters in [14], it is found in case (ii) that $\kappa_1 = 1$, $\eta = 1$ and $\delta\lambda/s = 6.67 \cdot 10^{-6} \cdot 100/5 \cdot 10^{-4} = 1.33$. Then, the steady states exist for $0 \leq \psi < 1.33$, and each $1 \leq \eta$.

To study the stability of the steady states we use the Jacobian matrix derived above. So, we let $\bar{\boldsymbol{z}} = (\bar{x}_c, \bar{x}_e, \bar{y}, \bar{w})$, and evaluate it at the nonzero steady state $\bar{\boldsymbol{z}}$. We first consider case (i) $\eta = \bar{w}/\bar{w}_{th} < 1$, therefore,

$$\eta = \frac{\alpha_M - \beta_c}{\alpha_M - \alpha_m} = \frac{\alpha_M - \beta_e}{\kappa_1(\alpha_M - \alpha_m)}.$$

We have,

$$\frac{\bar{x}_c}{\bar{w}_{th}} = \frac{1}{\rho(1 + \kappa_0\psi)},$$

$$\frac{\bar{x}_e\bar{w}}{\bar{w}_{th}^2} = \frac{\psi\bar{x}_c\bar{w}}{\bar{w}_{th}^2} = \eta\psi\frac{\bar{x}_c}{\bar{w}_{th}} = \frac{\eta\psi}{\rho(1 + \kappa_0\psi)}.$$

It follows from the calculations above that the J_{ij}, for $i, j = 1, 2, 3, 4$, are given by

$$J_{11} = (\beta_c - \alpha_M) + (\alpha_M - \alpha_m)\left(1 - \rho\frac{\bar{x}_c}{\bar{w}_{th}}\right)\frac{\bar{w}}{\bar{w}_{th}} =$$

$$= (\beta_c - \alpha_M) + \frac{\eta\kappa_0\psi(\alpha_M - \alpha_m)}{1 + \kappa_0\psi} = -\frac{(\alpha_M - \beta_c)}{1 + \kappa_0\psi}.$$

$$J_{12} = \rho\kappa_0(\alpha_M - \alpha_m)\frac{\bar{w}\bar{x}_c}{\bar{w}_{th}^2} = \frac{\eta\kappa_0(\alpha_M - \alpha_m)}{1 + \kappa_0\psi} = \frac{(\alpha_M - \beta_c)\kappa_0}{1 + \kappa_0\psi}.$$

$$J_{13} = 0.$$

$$J_{14} = (\alpha_M - \alpha_m)\frac{\bar{x}_c}{\bar{w}_{th}} = \frac{\alpha_M - \alpha_m}{\rho(1 + \kappa_0\psi)}.$$

Here and below, we use the appropriate expression for η. Next,

$$J_{21} = -\kappa_1\rho(\alpha_M - \alpha_m)\frac{\bar{x}_e\bar{w}}{\bar{w}_{th}^2} = -\frac{\kappa_1\eta\psi(\alpha_M - \alpha_m)}{1 + \kappa_0\psi} = -\frac{(\alpha_M - \beta_e)\psi}{1 + \kappa_0\psi}.$$

$$J_{22} = (\beta_e - \alpha_M) + \kappa_1(\alpha_M - \alpha_m)\left(1 - \rho\kappa_0\frac{\bar{x}_e}{\bar{w}_{th}}\right)\frac{\bar{w}}{\bar{w}_{th}}.$$

$$= -(\alpha_M - \beta_e) + \frac{\eta\kappa_1(\alpha_M - \alpha_m)}{1 + \kappa_0\psi} = -\frac{(\alpha_M - \beta_e)\kappa_1\psi}{1 + \kappa_0\psi}.$$

Furthermore,

$$J_{23} = 0,$$

and

$$J_{24} = \kappa_1(\alpha_M - \alpha_m)\frac{\bar{x}_e}{\bar{w}_{th}} = \frac{\kappa_1\psi(\alpha_M - \alpha_m)}{\rho(1 + \kappa_0\psi)}.$$

The third row,

$$J_{31} = -\delta\bar{y} = -\eta s(1 + \kappa_1\psi).$$

$$J_{32} = 0.$$

$$J_{33} = \gamma\lambda - 2\gamma\bar{y} - \delta\bar{x}_c = \gamma\lambda - 2\gamma\eta\frac{s}{\delta}(1+\kappa_1\psi) - \gamma\left(\lambda - \eta\frac{s}{\delta}(1+\kappa_1\psi)\right)$$

$$= -\gamma\eta\frac{s}{\delta}(1+\kappa_1\psi)$$

$$J_{34} = 0.$$

Finally, for $i = 4$,

$$J_{41} = s\eta\left(\kappa_1\psi + \frac{1+\kappa_1\psi}{1+\kappa_0\psi}\right),$$

$$J_{42} = s\eta\left(\kappa_0\frac{1+\kappa_1\psi}{1+\kappa_0\psi} - \kappa_1\right),$$

$$J_{43} = \gamma\left(\lambda - \eta\frac{s}{\delta}(1+\kappa_1\psi)\right),$$

$$J_{44} = -\frac{s(1+\kappa_1\psi)}{\rho(1+\kappa_{0\psi})}.$$

It follows that the Jacobian J has a simpler form,

$$J(\bar{\boldsymbol{z}}) = \tag{3.31}$$

$$\begin{pmatrix} -\frac{(\alpha_M-\beta_c)}{1+\kappa_0\psi} & \frac{(\alpha_M-\beta_c)\kappa_0}{1+\kappa_0\psi} & 0 & \frac{\alpha_M-\alpha_m}{\rho(1+\kappa_0\psi)} \\ -\frac{(\alpha_M-\beta_e)\psi}{1+\kappa_0\psi} & -\frac{(\alpha_M-\beta_e)\kappa_1\psi}{1+\kappa_0\psi} & 0 & \frac{\kappa_1\psi(\alpha_M-\alpha_m)}{\rho(1+\kappa_0\psi)} \\ -\eta s(1+\kappa_1\psi) & 0 & -\gamma\eta\frac{s}{\delta}(1+\kappa_1\psi) & 0 \\ s\eta\left(\kappa_1\psi + \frac{1+\kappa_1\psi}{1+\kappa_0\psi}\right) & s\eta\left(\kappa_0\frac{1+\kappa_1\psi}{1+\kappa_0\psi} - \kappa_1\right) & \gamma\left(\lambda - \eta\frac{s}{\delta}(1+\kappa_1\psi)\right) & -\frac{s(1+\kappa_1\psi)}{\rho(1+\kappa_0\psi)} \end{pmatrix}.$$

For the sake of completeness, we also describe a case where $\eta < 1$ and $\psi = 0$, i.e., no Elites. Indeed, if for example, we assume that $\alpha_M + \alpha_m = 2\beta_c$, and $(2-\kappa_1)\alpha_M + \kappa_1\alpha_m = 2\beta_e$, then

$$\eta = \frac{\bar{w}}{\bar{w}_{th}} = \frac{\alpha_M - \beta_c}{\alpha_M - \alpha_m} = \frac{\alpha_M - \beta_e}{\kappa_1(\alpha_M - \alpha_m)} = \frac{1}{2}. \tag{3.32}$$

Therefore,

$$\begin{pmatrix} -(\alpha_M-\beta_c) & \kappa_0(\alpha_M-\beta_c) & 0 & \frac{1}{\rho}(\alpha_M-\alpha_m) \\ 0 & 0 & 0 & 0 \\ -\dfrac{s}{2} & 0 & -\dfrac{s\gamma}{2\delta} & 0 \\ \dfrac{s}{2} & 0 & \gamma\left(\lambda-\dfrac{s}{2\delta}\right) & -\dfrac{s}{\rho} \end{pmatrix}. \tag{3.33}$$

The Jacobian matrix with the baseline parameter values, and $\kappa_1 = \kappa_0 = 10$ with $(\eta, \psi) = (0.5, 0)$, which yields the Jacobian matrix

$$\begin{pmatrix} -0.04 & 0.3 & 0. & 12. \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.375 & 0 \\ 0 & 0 & 0.625 & -0.1 \end{pmatrix}, \tag{3.34}$$

that has the eigenvalues

$$-0.04, \quad 0, \quad -0.1, \quad -0.375.$$

It is seen that this steady state is stable with three direction in which it is attracting. On the other hand, a computation with $\eta \leq 0.066$ leads to all negative real eigenvalues. Similarly, when $\kappa_1 = 100$ the steady state is stable if $\eta \leq 0.0066$ i.e roughly $w < 2w_{th}/(3\kappa_1)$. For example the pair $(0.06, 0)$ with the same baseline parameter values, leads to the matrix:

$$\begin{pmatrix} -0.04 & 0.036 & 0 & 12. \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.045 & 0 \\ 0 & 0 & 0.955 & -0.1 \end{pmatrix} \tag{3.35}$$

with the eigenvalues

$$-0.04, \quad 0 \quad -0.1 \quad -0.045,$$

and so the steady state is stable.

We turn to the stability of the steady states in case (ii): $\eta \geq 1$. So we let $\bar{\boldsymbol{z}} = (\bar{x}_c, \bar{x}_e, \bar{y}, \bar{w})$, and evaluate J at the nonzero steady state $\bar{\boldsymbol{z}}$. We recall that for $\eta \geq 1$ the steady states are parametrized by a two-parameter family (η, ψ), where ψ is restricted by the bound above. We have,

$$\begin{aligned} \bar{x}_c &= \frac{\gamma}{\delta}\left(\lambda - \frac{s}{\delta}(1+\kappa_1\psi)\right), \\ \bar{x}_e &= \psi \bar{x}_c, \\ \bar{y} &= \frac{s}{\delta}(1+\kappa_1\psi), \\ \bar{w} &= \eta\rho(1+\kappa_0\psi)\bar{x}_c. \end{aligned} \tag{3.36}$$

We return to the original assumption

$$0 < \alpha_m \leq \beta_e \leq \beta_c \leq \alpha_M < 1. \tag{3.37}$$

We note that in this case $\beta_c = \alpha_c = \alpha_m$, and if we want $0 < \psi$, so that there are Elites in the solution, then, we also must have $\beta_e = \alpha_e = \alpha_m$, then $F_1 = F_2 = 0$, and

$$C_c = s\bar{x}_c, \qquad C_e = \kappa_1 s\bar{x}_e.$$

It follows from the calculations above that the J_{ij}, for $i, j = 1, 2, 3, 4$, are given by

$$J_{11} = J_{12} = J_{13} = J_{14} = 0.$$

Similarly,

$$J_{21} = J_{22} = J_{23} = J_{24} = 0.$$

Next,

$$J_{31} = -\delta\bar{y} = -\eta s(1 + \kappa_1\psi).$$

$$J_{32} = 0.$$

$$J_{33} = \gamma\lambda - 2\gamma\bar{y} - \delta\bar{x}_c = \gamma\lambda - 2\gamma\frac{s}{\delta}(1 + \kappa_1\psi) - \gamma\left(\lambda - \frac{s}{\delta}(1 + \kappa_1\psi)\right)$$

$$= -\gamma\frac{s}{\delta}(1 + \kappa_1\psi)$$

$$J_{34} = 0.$$

Finally, since $F_4 = \delta y x_c - s x_c - \kappa_1 s x_e$, we find

$$J_{41} = \delta\bar{y} - s = \kappa_1 s\psi, \qquad J_{42} = -\kappa_1 s,$$

$$J_{43} = \gamma\left(\lambda - \frac{s}{\delta}(1 + \kappa_1\psi)\right), \qquad J_{44} = 0.$$

Substituting theses expressions in J yields, $(\eta \geq 1)$,

$$J(\bar{\boldsymbol{z}}) =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\eta s(1+\kappa_1\psi) & 0 & -\gamma\frac{s}{\delta}(1+\kappa_1\psi) & 0 \\ \kappa_1 s\psi & -\kappa_1 s & \gamma\left(\lambda - \frac{s}{\delta}(1+\kappa_1\psi)\right) & 0 \end{pmatrix}. \tag{3.38}$$

This matrix has a special form and it follows that the eigenvalues are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \qquad \lambda_4 = -\gamma\frac{s}{\delta}(1+\kappa_1\psi).$$

Since $\lambda_4 < 0$, it follows that the steady states are stable.

More specifically, the simulation with the baseline parameter values, $\kappa_1 = \kappa_0 = 10$, and $(\eta, \psi) = (1.0, 0.02)$, (to follow the bound on ψ) and setting $\delta = 6.67 \cdot 10^{-6}$, resulted in the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.001 & 0 & -0.9 & 0 \\ 0.0001 & -0.005 & 0.1 & 0 \end{pmatrix}, \tag{3.39}$$

with eigenvalues $(0, -0.9, 0, 0)$ implying the stability of the non-zero equilibrium state. However, there is only one direction in which the solutions converge to the state. Moreover, with the value $\delta = 6.67 \cdot 10^{-6}$, we obtain

$$0 \leq \psi < \frac{1}{\kappa_1}\left(\frac{\lambda\delta}{s} - 1\right) = 0.1(2 \cdot 10^5\delta - 1) = 0.033,$$

which means that the rich cannot be more than 3.3% of the workers.

For the case of the equality, i.e., $\psi = 0$, so there are no Elites, only workers, the Jacobian matrix in case (ii), $\eta = 1$, is the following:

$$J(\bar{\boldsymbol{z}})_{(\psi=0)} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\eta s & 0 & -\gamma \frac{s}{\delta} & 0 \\ 0 & -\kappa_1 s & \gamma\left(\lambda - \frac{s}{\delta}\right) & 0 \end{pmatrix}. \tag{3.40}$$

The eigenvalues are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \qquad \lambda_4 = -\gamma \frac{s}{\delta}.$$

Again, the state is stable with one negative eigenvalue.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0005 & 0 & -0.75 & 0 \\ 0 & -0.005 & 0.25 & 0 \end{pmatrix}. \tag{3.41}$$

the eigenvalues are $(0, \quad 0, \ -0.75, \quad 0)$

We summarize the stability findings in this section, as related to the equilibrium points, in the following theorem. It was found in the numerical simulations that steady oscillations may occur, too. The existence of steady oscillations and other steady solutions that are not points of equilibrium remains an unresolved theoretical issue that should be addressed.

The stability results for the HANDY model, Problem 3.1, are the following.

Theorem 3.9 (Stability of the steady states)**.** *Assume that*

$$0 < \alpha_m \le \beta_e \le \beta_c \le \alpha_M < 1,$$

and let $\bar{\boldsymbol{z}} = (\bar{x}_c, \bar{x}_e, \bar{y}, \bar{w})$ *denote a steady state of the system.*

Then, the following hold:

The zero steady state $(0,0,0,0)$ *is unstable, while the state* $(0,0,\lambda,0)$ *is stable.*

Considering nonzero steady states, there are two cases.

Case (i) $\eta = \bar{w}/\bar{w}_{th} < 1$*. Then, necessarily,*

$$\eta = \frac{\alpha_M - \beta_c}{\alpha_M - \alpha_m} = \frac{\alpha_M - \beta_e}{\kappa_1(\alpha_M - \alpha_m)}.$$

The steady states form a one-parameter family of solutions,

$$\begin{aligned} \bar{x}_c &= \frac{\gamma}{\delta}\left(\lambda - \eta\frac{s}{\delta}(1+\kappa_1\psi)\right), \\ \bar{x}_e &= \psi\bar{x}_c, \\ \bar{y} &= \eta\frac{s}{\delta}(1+\kappa_1\psi), \\ \bar{w} &= \eta\rho(1+\kappa_0\psi)\bar{x}_c, \end{aligned} \tag{3.42}$$

parametrized by ψ*, where*

$$0 \le \psi < \frac{1}{\kappa_1}\left(\frac{\delta\lambda}{\eta s} - 1\right).$$

The system Jacobin matrix is given in (3.32)*, and the eigenvalues need to be computed in each simulations run.*

Case (ii) $\eta = \bar{w}/\bar{w}_{th} \ge 1$*. Then, the steady states form a two-parameter family of solutions* (η, ψ)*, with*

$$0 \le \psi < \frac{1}{\kappa_1}\left(\frac{\delta\lambda}{s} - 1\right).$$

The solutions are:

$$\begin{aligned}\bar{x}_c &= \frac{\gamma}{\delta}\left(\lambda - \frac{s}{\delta}(1+\kappa_1\psi)\right),\\ \bar{x}_e &= \psi\bar{x}_c,\\ \bar{y} &= \frac{s}{\delta}(1+\kappa_1\psi),\\ \bar{w} &= \eta\rho(1+\kappa_0\psi)\bar{x}_c.\end{aligned} \tag{3.43}$$

The system Jacobin matrix is given in (3.38)*, and the eigenvalues are*

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \qquad \lambda_4 = -\gamma\frac{s}{\delta}(1+\kappa_1\psi).$$

It follows that the steady states are stable.

We note again that in case (ii) only $\bar{w}$ depends on η, moreover, all the steady states are stable. In case (i) there are unstable steady states that seem to generate periodic solutions.

The questions of limit cycles and chaotic behavior in case (i) is unresolved. However, there seems to be some computational support to the existence of such solutions for appropriate sets of parameters.

3.4 Carrying capacity and the depletion factor

This short section follows the ideas in [14]. It follows from Theorem 3.9 that the resources or nature's solution $\bar{y}$ reaches its maximum regeneration when $\bar{y} = \lambda/2$, which makes the choice of the optimal δ, denoted by δ_*, for the first case in the Theorem to be $\delta_* = 2s\eta(1+\kappa_1\psi)/\lambda$, thus the commoners carrying capacity is,

$$\chi_M = \frac{\gamma(\lambda/2)^2}{s\eta(1+\kappa_1\psi)} = \frac{\gamma\lambda^2}{4s\eta(1+\kappa_1\psi)},$$

and the optimal depletion factor for the second case: $\delta_* = 2s(1+\kappa_1\psi)/\lambda$ with the carrying capacity,

$$\chi_M = \frac{\gamma(\lambda/2)^2}{s(1+\kappa_1\psi)} = \frac{\gamma\lambda^2}{4s(1+\kappa_1\psi)}.$$

It is noted that $1 \leq \eta$ doesn't contribute to the carrying capacity.

CHAPTER FOUR

ALGORITHM AND SIMULATIONS OF THE BASIC HANDY MODEL

In this chapter, we describe the numerical algorithm and computer simulations of the classical HANDY model, Problem 3.1, where the inequality constant κ was split into two. All the simulations were done using the explicit Euler method; the program was written in Python3 ([20]) on a MacBook Air laptop; and we used the Jupyter Notebook as the editor [8]. A typical run of $1,000 - 2,500$ 'years' took less than a minute, where 'years' is a convenient but arbitrary time unit.

First, we describe the numerical discretization of the model equations, using a simple explicit Euler scheme thus, the algorithm is a time marching process. Then, we provide the basic information for the baseline simulations, which are essentially those of the original HANDY paper [14], with the minor change resulting from replacing the inequality factor κ with κ_0 and κ_1. Following [14], we present two scenarios with $\kappa_0 = \kappa_1 = 1$, which is the egalitarian case, and the case of large inequality, $\kappa_0 = \kappa_1 = 100$. We also show numerically how the changes in κ_0 and κ_1, separately, affect the model predictions, and both are important, small changes in κ_1 lead to larger changes in the solutions than the small changes induced from κ_0. Furthermore, it is found that the model solutions can cross the boundaries of the the domains S_1 and S_2 where the function $\boldsymbol{F}$ is Lipschitz, into S_0, where it is not. Thus, the three domains, which were constructed to prove the uniqueness of the solutions, don't seem to be necessary.

Finally, we study the approach to a steady state, based on the stability analysis in Section 3.3.

4.1 Numerical algorithm

To discretize the system in Problem 3.1, we divide the basic time interval $[0, T]$ into N subintervals of length Δt, and let the partition points be $t_n = n\Delta t$, for $n = 0, \dots N = T\Delta t$. Then, if $f(t)$ is a function, we let the nodal values be $f^n = f(t_n)$. Next, we approximate the derivative $f' \approx (f^{n+1} - f^n)/\Delta t$ and then

$$f^{n+1} = f^n + \left(\frac{f^{n+1} - f^n}{\Delta t}\right)\Delta t.$$

We apply this scheme to the equations in Problem 3.1, and obtain the following explicit discretized problem.

Problem 4.1. *(Discretized HANDY model) Find four sequences* (x_c^n, x_e^n, y^n, w^n), *for* $n = 0, \dots N$, *where* $N\Delta t = T$, *such that*

$$x_c^{n+1} = x_c^n + (\beta_c x_c^n - \alpha_c^n x_c^n)\Delta t, \tag{4.1}$$

$$x_e^{n+1} = x_e^n + (\beta_e x_e^n - \alpha_e^n x_e^n)\Delta t, \tag{4.2}$$

$$y^{n+1} = y^n + (\gamma y^n(\lambda - y^n) - \delta x_c^n y^n)\Delta t, \tag{4.3}$$

$$w^{n+1} = w^n + (\delta x_c^n y^n - C_c^n - C_e^n)\Delta t, \tag{4.4}$$

where $C_c^n, C_e^n, w_{th}^n, \alpha_c^n$ *and* α_e^n *are given in*

$$C_c^n = \min(1, \frac{w^n}{w_{th}^n}) s x_c^n, \tag{4.5}$$

$$C_e^n = \min(1, \frac{w^n}{w_{th}^n}) \kappa_1 s x_e^n, \tag{4.6}$$

$$w_{th}^n = \rho(x_c^n + \kappa_0 x_e^n), \tag{4.7}$$

$$\alpha_c^n = \alpha_m + \max(0, 1 - \min(1, \frac{w^n}{w_{th}^n}))(\alpha_M - \alpha_m), \tag{4.8}$$

$$\alpha_e^n = \alpha_m + \max(0, 1 - \kappa_1 \min(1, \frac{w^n}{w_{th}^n}))(\alpha_M - \alpha_m). \tag{4.9}$$

Together with the initial conditions

$$x_c^0 = x_{c0}, \quad x_e^0 = x_{e0}, \quad y^0 = y_0, \quad w^0 = w_0, \tag{4.10}$$

where x_{c0}, x_{e0}, y_0, w_0 *are the non-negative initial conditions.*

4.1.1 Algorithm

An algorithm for the discretized Problem 4.1 was constructed and implemented as a Python code using the Euler method above. The steps of the algorithm are as follows:

Step 1 Initiation

set the problem coefficients and input data:

$$\alpha_m,\ \alpha_M,\ \beta_c,\ \beta_e,\ \gamma,\ \delta,\ \kappa_0,\ \kappa_1,\ \lambda,\ \rho,\ s;$$

$$x_{c0},\ x_{e0},\ y_0,\ w_0.$$

set T (the final year) and N (the number of time steps)

set $\Delta t = T/N$.

Step 2

set $n = 0$

set $x_c^0 = x_{c0}, \quad x_e^0 = x_{e0}, \quad y^0 = y_0, \quad w^0 = w_0,$

Step 3 Time marching

compute

C_c^n from (4.5)

C_e^n from (4.6)

w_{th}^n from (4.7)

α_c^n from (4.8)

α_c^n from (4.9)

set $n = n + 1$

compute

x_c^n from (4.1)

x_e^n from (4.2)

y^n from (4.3)

w^n from (4.4)

Step 4 if $n < N$ **go to Step 3**

end

At the end of each run, the program provided graphical depiction of the results, i.e., the linearly interpolated solutions.

4.2 Baseline simulations

The computer simulations of the HANDY model, Problem 3.1, were performed first, to verify our code by comparing the results to those in [14]; secondly to obtain insight into the relative importance of the two coefficients κ_0 and κ_1; and finally, to study the approach to the steady state when $\eta < 1$, since the case $\eta \geq 1$ can be seen in the baseline simulations. In all the simulations graphic displays, the green curve represents the natural resources and the cyan curve the wealth, and the curves in blue and magenta represent the Commoners and Elites, respectively. Table 4.1 lists the 11 model parameters, the four initial conditions and the two numerical

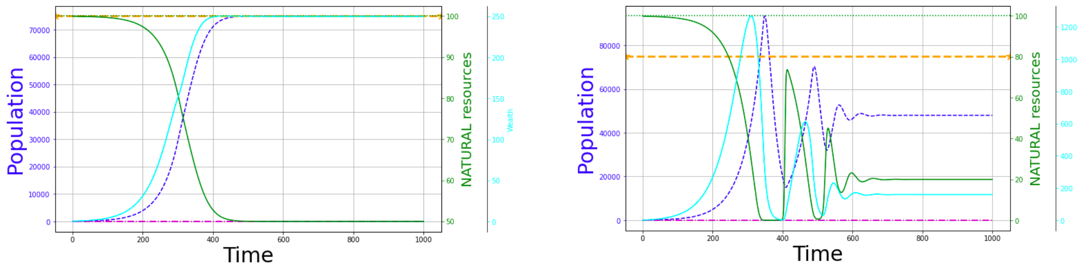
Table 4.1: The values of the parameters and data used in the simulations in this section. The model contains 11 parameters, four initial conditions and two numerical constants.

Symbols	Values	Meaning
Δt	.01	time step
T	1000	time period
$x_c(0)$	100	Commoners initial condition
$x_e(0)$	25	Elites initial condition
$y(0)$	100	Nature's initial condition
$w(0)$	0	wealth initial condition
λ	100	Natures carrying capacity
γ	.01	Natures regeneration factor
δ	$6.67 \cdot 10^{-6}$, $8.33 \cdot 10^{-6}$, $3.33 \cdot 10^{-5}$	Natures depletion factor
β_c	.03	Commoners birth rate
β_e	.03	Elites birth rate
α_m	0.01	'natural' death rate
α_M	0.07	famine death rate
s	$5 \cdot 10^{-4}$	salary per capita
ρ	$5 \cdot 10^{-3}$	min required consumption
κ_0	1,5,10, 25, 50	wealth unbalance threshold
κ_1	1,2,3,4, 5	consumption inequality factor

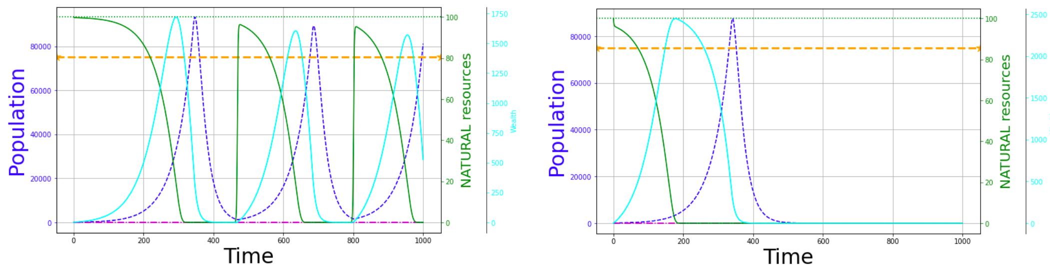
constants. We note here that below, for the sake of convenience we use the word 'year' for the time unit. However, it is as arbitrary as the other units.

4.2.1 Four cases without Elites

We start by comparing our results to those in Section 5.1 in [14], where only workers are present, $x_e = 0$, and the society is not stratified. The results are depicted in Fig. 4.1, where we set $\kappa = \kappa_0 = \kappa_1 = 1$, with $\delta_* = 6.67 \cdot 10^{-6}$ in 4.1(a), $\delta = 2.5\delta_*$ in 4.1(b), $\delta = 4\delta_*$ in 4.1(c), and $\delta = 5.5\delta_*$ in 4.1(d).



(a) Soft landing, monotone behavior (b) Oscillatory approach



(c) Oscillatory solutions (d) Collapse

Figure 4.1: Four scenarios of long-time behavior and approach to a steady state in the HANDY model with no Elites ($x_e = 0$), following Section 5.1 in [14]. The steady states in (a) and (b) are stable, a periodic solution is seen in (c). The state in (d) is unstable, since $y = 0$, however, the instability kicks in much later. The fast recovery of Nature in (b) and especially in (c), where it is periodic, is noticeable.

A 'soft landing' or monotone approach to the steady state is depicted in (a), where $\bar{x}_c = \chi_M \approx 75,000, \bar{x}_e = 0, \bar{y} = 50 = \lambda/2, \bar{w} = 250$. It is seen that over a period of about 300 years, the system changes monotonically from the initial conditions to the steady state, which is stable, and is seen to be attracting in the simulation. In (b) the approach to the steady state $\bar{x}_c = 50,000, \bar{x}_e = 0,\ \bar{y} = \lambda/5 = 20, \bar{w} = 180$ involves damped oscillations over a period of about 300 years, starting about year 300. It involves fast ascents and rapid declines, before settling into the stable steady state, which seems computationally to be attracting.

The periodic or oscillatory solutions in (c), with frequency of about 350 years, may be representing periods of 'boom and bust.' This solution is of considerable interest both theoretically, as it may indicate the existence of a limit cycle, and practically, since these rather large oscillations in the solution may be related to considerable suffering of the population.

The steady states in (a) and (b) are stable (and computationally attracting), with respective eigenvalues

$$-0.510, \quad -0.048, \quad -0.081,\ 0, \quad \text{and} \quad -0.281, \quad -0.030 \pm 0.103i, \quad 0.$$

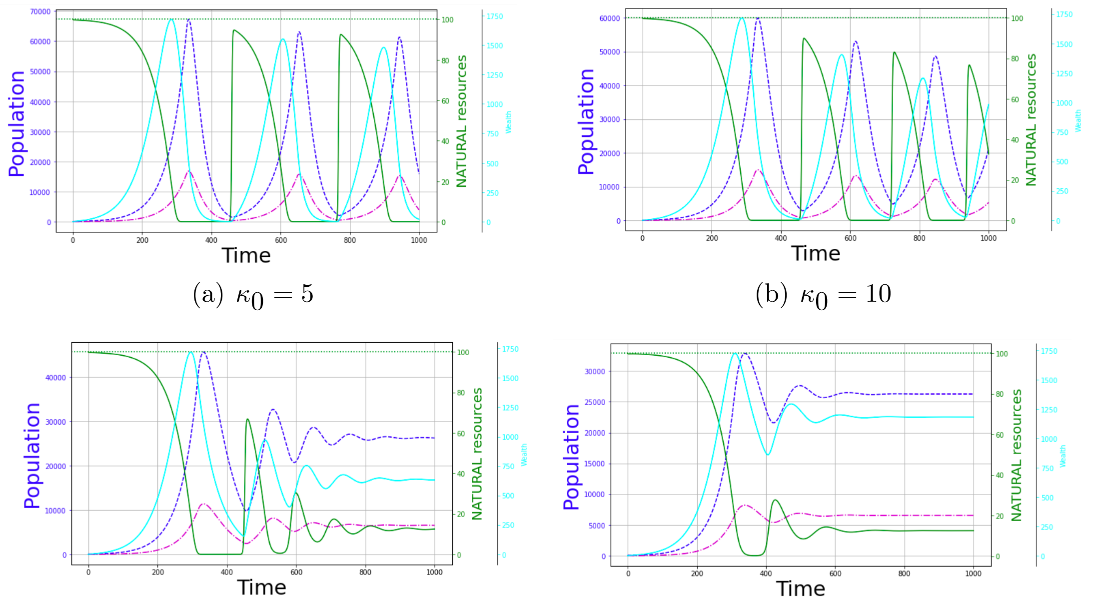
They are seen to converge to the steady state rather quickly. In (c) we see steady oscillations. Finally, in (d), we can see the complete collapse of the system to zero populations, of $(0,\ 0,\ 0,\ 0)$, which is unstable in the long run. Indeed, the eigenvalues are

$$0.037 \pm 0.128i, \quad -0.223, \quad 0,$$

and the state is unstable. The comparison with the results in [14] provides confidence in our numerical method and the results.

4.2.2 Different κ_0 and κ_1

The aim of this set of simulations is to study the relative importance of the *wealth threshold imbalance constant* κ_0 and the *inequality pay factor* κ_1. The simulations study numerically the effects of the different values of the unbalanced wealth threshold with values $\kappa_0 = 1, 5, 10, 25$, and 50. Then, the effects of increasing the inequality factor through the values of $\kappa_1 = 1, 2, 3, 4$ and 5. The simulations show that the changes made of κ_0 and κ_1 have considerable effects on the whole system, including possible populations collapse. We note that the simulations indicate that changes in the inequality pay factor κ_1 affect the system's behavior more than the changes in the wealth threshold imbalance factor κ_0.



(a) $\kappa_0 = 5$

(b) $\kappa_0 = 10$

(c) $\kappa_0 = 25$

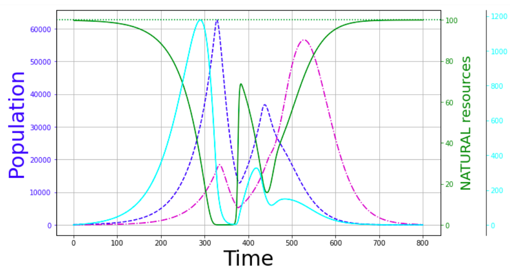
(d) $\kappa_0 = 50$

Figure 4.2: The effects of increasing the wealth threshold imbalance in w_{th} via changes in κ_0. Here, $\kappa_1 = 1$ and $\delta = 3.332 \cdot 10^{-5}$ are both fixed. The oscillations in (a) and (b) decay slowly to the steady states with many 'boom and bust' periods, while the decay in (c) and (d) is faster.

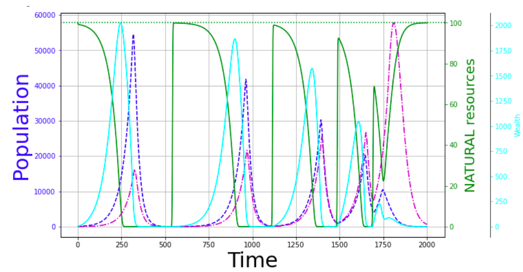
It is seen in Fig. 4.2 that for fixed $\kappa_1 = 1$, the behavior of the solutions depends substantially on κ_0, which changes through the values $\kappa_0 = 5(a), 10(b), 25(c)$, and $50(d)$. The solutions in the first two cases $\kappa_0 = 5(a), 10(b)$ are qualitatively similar, since both exhibit large but damped oscillations. In the case $\kappa_0 = 10(b)$ the damping is faster. In the case $\kappa_0 = 25(c)$, the damping of the cycles is over 400 years, from time 200 to 600, and is much more pronounced, as the system is getting closer to the equilibrium point $(30,000; 8,000; 17; 220)$ over the following 200 'years'. The damping is very pronounced in the case when $\kappa_0 = 50(d)$, and over a period of less about 200 years the natural resources y decay, while the two populations and the wealth increase, following an initial short period of readjustment. Here, the change in κ_0 speeds up the approach to the steady state. That is seen clearly when one compares the graphs in 4.2(c) with those in 4.2(d). The effect on the resources is evident in all the simulations. The natural resources converge in the original case to 50 eco-dollars, which is half of the carrying capacity $\lambda = 100$, as shown in 4.1(a). The behavior is qualitatively different in the cases with $\kappa_0 = 25, 50$, since the natural resources are elevated with every κ_0 increase from the full collapse, i.e. preventing the resources from approaching 0.

This observation also applies to the other system's components. For instance; with $\kappa_0 = 10$ the two populations increase during 500 years to a bit less than $60,000$ and $20,000$, respectively, for the Commoners and Elites, from the original model 4.1(a) of reaching maximum values of less than $60,000$ of Commoners and $15,000$ of Elites. In the case of increasing w_{th} with $\kappa_0 = 50$, the nature depletes to a steady state of less than 20% of its initial carrying capacity $\lambda = 100$. While the population don't exceed $27,000$ for the Commoners and $7,000$ of Elites, 4.2(d). Finally, increasing the wealth w_{th} with $\kappa_0 = 5$ barley changes the simulation from the periodic to oscillatory but damped behavior. The natural resources collapses rapidly, causing the population to grow in a much faster rate of less than $70,000$ for the Commoners and less than $20,000$ for the Elites during the same $1,000$ period of time 4.2(a). Nevertheless, the wealth w shows an inconsistent behavior reaching its maximum value at time $T = 1,000$ coinciding with the change in $\kappa_0 = 5, 10, 25$ and 50 with values less than $w = 1,750, 1,000, 750$ and $1,250$ eco-dollars, respectively. Those small changes coincide with initial values that represent a small population for all the scenarios of the model. We recall that $\delta = 4 \times 8.33e^{-6} = 3.332 \cdot 10^{-5}$ is used in every graph.

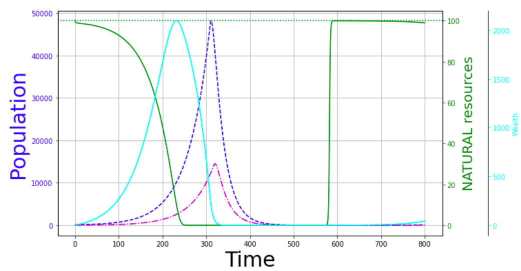
We next study the effects of changing the inequality factor κ_1. The simulations are depicted in Fig. 4.3, and it is seen that as κ_1 increases in small increments, through the values $2, 3, 4, 5$, the qualitative behavior changes. We note that the time scales



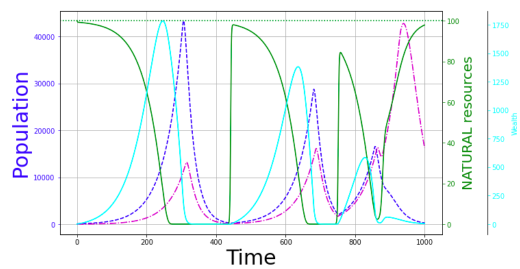
(a) with $\kappa_1 = 2$ with $\delta_* = 2.36\delta_0$



(b) with $\kappa_1 = 3$ with $\delta_* = 6.5\delta_0$



(c) with $\kappa_1 = 4$ with $\delta_* = 7.5\delta_0$



(d) with $\kappa_1 = 5$ with $\delta_* = 4.5\delta_0$

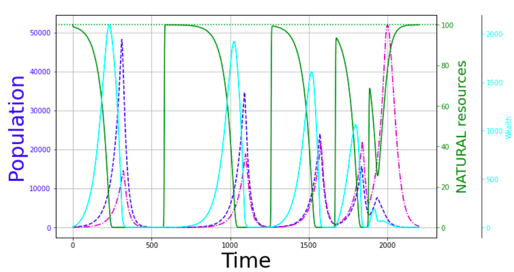
Figure 4.3: Effects of changing the inequality pay factor κ_1 while keeping $\kappa_0 = 1$. As κ_1 increases in small increments, the qualitative behavior changes. The populations collapse in (a) and (c), converging to the steady state $(0, 0, 100, 0)$ that is steady and computationally attracting. The system seems to exhibit chaotic behavior in (b) and (d). Note that the time scales are different to show more clearly the solutions' behavior, and also, the fast regeneration of the resources in (b), (c) and (d).

are different to show more clearly the solutions' behavior. The δ_0 is calculated in every sub-figure according to 3.4; and the values are

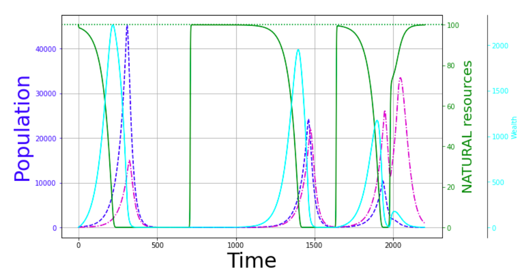
$1 \cdot 10^{-5},\ 1.16 \cdot 10^{-5},\ 1.33 \cdot 10^{-5},\ 1.5 \cdot 10^{-5}$ for $\kappa_1 = 2,\ 3,\ 4,\ 5$ respectively. However; to get more interesting simulations; all the mentioned δ_0 values are multiplied by a factor. Which makes all the $\delta's$ used in the graph of 4.3 is much bigger than the ones used in other graphs.

In (a), where $\kappa_1 = 2$, and (c) where $\kappa_1 = 4$, there are rather large confined oscillations over a period of 400 years, and afterwards the system reaches the steady state in which the natural resources are fully recovered, while the populations collapse, the stable steady state of $(0, 0, 100, 0)$. The numerical solutions indicate that it is also attracting. Here, the nature's depletion factor is $\delta = 2.36 \cdot 1 \cdot 10^{-5}$, which is larger than the calculated value $\delta_0 = 1 \cdot 10^{-5}$. The oscillations in (b), where $\delta_* = 6.5\delta_0$, seem to be chaotic.

A very interesting feature in (b) and (c) is the steep recovery of nature to its steady value of $\lambda = 100$, about year 600 and it happens about year 400 in (d), while the populations are very close to zero. The next set of simulations studies the systems dependence on the nature's depletion factor, where $\delta_0 = 2.36 \cdot 1 \cdot 10^{-5}$. It is seen that a small change in κ_1 through the values $2, 3, 4$ and 5 and the indicated changes in δ, the nature recovers back to its carrying capacity λ, and the populations collapse, the Commoners collapse first and then the Elites follow. The Commoners population reaches its maximum value around year 350 and collapses to recover again to a smaller value than the previous peak until a full collapse.



(a) $\kappa_1 = 4$ with $\delta_* = 7.5\delta_0$



(b) $\kappa_0 = \kappa_1 = 4$ with $\delta_* = 10.4\delta_0$

Figure 4.4: A longer time period of the result in 4.3(c), with $\kappa_0 = 1$ on the left-hand side, and $\kappa_0 = 4$ and a bigger δ on the right-hand side.

On the other hand, the maximum population of Elites is reached close to the end time period of every simulation in 4.3(a), 4.3(b), and 4.3(d) with $\kappa_1 = 2, 3$, and 5, respectively. The case when $\kappa_1 = 4$ is for 800 years with $7.5\delta_o$ nature takes 250 years to recover while the Commoners and Elites populations take almost 400 years to catch up. Further analysis of what is happening in Fig. 4.3(c) required a run of a longer time period. A bit longer than twice the time period with the same parameters of $\kappa_0 = 1$, $\kappa_1 = 4$, and $\delta = 7.5\delta_0$ resulted in a full collapse of the Commoners population, shown in the graphs 4.3(a) 4.3(b), and 4.3(d). Especially the end of each time period in (b) and (d), a pattern is showing the dependence of wealth and the Elites on the Commoners population that when the latter collapse the wealth follows and then the non-workers proceed to a full collapse. The graph in 4.4(a) demonstrates the collapse in 4.3(c) when a longer time run is used. Yet, when increasing $\kappa_0 = 4 = \kappa_1$ and the δ_* to $\delta_* = 10.4\delta_0$ it is evident that the same pattern of collapse is showing in figure 4.4(b).

CHAPTER FIVE

THE HANDY MODEL WITH SOCIAL MOBILITY

This chapter studies the HANDY-SM model, which is a modified version of the HANDY model allowing for social mobility between the Elites and the Commoners. Each year a fraction of the Elites become Commoners or workers, and a fraction of the Commoners 'make it' and become rich. This allows the model to represent better many socioeconomic systems. In addition, we introduce a minor change in the weights of the resources depletion rate δ_y and the wealth increase factor δ_w, noticing that in Chapter 3, following [12], we used $\delta_y = \delta_w = \delta$.
We prove the existence, boundedness and positivity of the solutions. Then, we study the stability of the steady states. Computer simulations of various scenarios, that are solutions of this model, can be found in the Chapter 6.

5.1 The HANDY-SM model

The modified model also consist of four nonlinear coupled ODEs: two equations model the populations growth rates of the Commoners or workers, $x_c(t)$, and the Elites or the rich, $x_e(t)$; one equation describes the rate of growth or depletion of natural resources, $y(t)$ (which is a lumped variable for renewables, such as wood, and non-renewables, such as oil), and the rate of growth of wealth $w(t)$, which includes food surpluses. The time t, as above, is measured in 'years.'
The complete HANDY-SM model, the HANDY model with Social Mobility is the following:

Problem 5.1. *[HANDY-SM] Find four functions* $(x_c(t), x_e(t), y(t), w(t))$*, defined on* $[0, T]$*,* $0 < T < \infty$*, such that*

$$\begin{aligned}
\frac{dx_c}{dt} &= (\beta_c - \alpha_c)x_c + \gamma_e x_e - \gamma_c x_c, \\
\frac{dx_e}{dt} &= (\beta_e - \alpha_e)x_e - \gamma_e x_e + \gamma_c x_c, \\
\frac{dy}{dt} &= \gamma y(\lambda - y) - \delta_y x_c y, \\
\frac{dw}{dt} &= \delta_w x_c y - C_c - C_e,
\end{aligned} \tag{5.1}$$

where C_c *and* C_e *are given in* (5.3)*,* w_{th} *is given in* (5.4)*, and* α_c *and* α_e *are given in* (5.5)*; together with the initial conditions*

$$x_c(0) = x_{c0}, \quad x_e(0) = x_{e0}, \quad y(0) = y_0, \quad w(0) = w_0, \tag{5.2}$$

where $x_{c0} > 0, y_0 > 0$*, and* x_{e0}, w_0 *are non-negative numbers.*

Here, the symbols have the same interpretation as in Chapter 3, i.e., β_c, β_e are the respective birth rates, assumed to be constant; γ is the nature's regeneration factor, λ its saturation level or the carrying capacity of the natural resources, and δ_y the resources depletion rate constant, while δ_w is the resource wealth growth rate constant, all of which may be constants or given functions of time. The new additions, are the rate constants: $\gamma_c \geq 0$ for workers who become rich, $\gamma_e \geq 0$ for the rich who go bankrupt. Also, we let $\delta_y > 0$ be the resources depletion rate constant and δ_w the rate constant of wealth generation. As above, C_c, C_e are the total consumption rates, given by

$$C_c = \min(1, \frac{w}{w_{th}})s x_c, \qquad C_e = \min(1, \frac{w}{w_{th}})\kappa_1 s x_e. \tag{5.3}$$

The wealth threshold is

$$w_{th} = \rho(x_c + \kappa_0 x_e + \epsilon), \tag{5.4}$$

where the $\epsilon > 0$ is a very small number that prevents the threshold w_{th} from becoming zero, since it appears in the denominators in (5.3) and (5.5). Finally, the death rates α_c, α_e, are given by,

$$\begin{aligned} \alpha_c &= \alpha_m + \max(0, 1 - \min(1, \frac{w}{w_{th}}))(\alpha_M - \alpha_m), \\ \alpha_e &= \alpha_m + \max(0, 1 - \kappa_1 \min(1, \frac{w}{w_{th}}))(\alpha_M - \alpha_m). \end{aligned} \tag{5.5}$$

In the original HANDY model ([14]), as was noted above, the choice was

$$\kappa_0 = \kappa_1 = \kappa, \qquad \gamma_c = \gamma_e = 0, \qquad \delta_y = \delta_w = \delta.$$

These represent the expansion of the model to HANDY-SM.

5.2 Model analysis

We establish the boundedness, positivity and then the existence of the solutions to Problem 5.1. Here, the proof of the existence of solutions is different from the one in Chapter 3, it is based on Theorem 2.4, therefore, we do not need to split the domain $\mathbb{R}^4_+$ into the three regions S_0, S_1 and S_2.

We, then, study the steady states and their stability, following the steps in Chapter 3.

5.2.1 Positivity and a priori estimates

Let $\boldsymbol{z} = (x_c, x_e, y, w)$ and $\boldsymbol{F} : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$\boldsymbol{F}(\boldsymbol{z}) = \begin{pmatrix} \beta_c x_c - \alpha_c x_c + \gamma_e x_e - \gamma_c x_c \\ \beta_e x_e - \alpha_e x_e - \gamma_e x_e + \gamma_c x_c \\ \gamma y(\lambda - y) - \delta_y x_c y \\ \delta_w x_c y - C_c - C_e, \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}. \tag{5.6}$$

Here, C_c and C_e are given in (5.3), w_{th} is given in (5.4), and α_c and α_e are given in (5.5).

We assume, following [14], that $0 < \alpha_m < \beta_e \le \beta_c \le \alpha_M < 1$, and then it follows that

$$\frac{\alpha_M - \alpha_m}{\beta_e - \alpha_m} \ge \frac{\alpha_M - \alpha_m}{\beta_c - \alpha_m} \ge 1. \tag{5.7}$$

Everywhere below, we assume that $\kappa_0, \kappa_1 \ge 1$.

First, we establish the necessary a-priori estimates on the solutions of the system, which we write as

$$\boldsymbol{z}' = \boldsymbol{F}(\boldsymbol{z}), \qquad \boldsymbol{z}(0) = \boldsymbol{z}_0 = (x_{c0}, x_{e0}, y_0, w_0). \tag{5.8}$$

Here and below, the prime indicates the time derivative, and each component of $\boldsymbol{z}_0$ is assumed to be positive. Below, we show that the results hold true for $x_{e0} = 0$ and $w_0 = 0$.

Let $0 < T < \infty$, and then $[0, T]$ is an arbitrary finite interval. We begin by showing that starting with positive initial conditions, all the possible solutions are nonnegative and bounded. This leads to estimates that are valid for $[0, T]$ and allow us to prove that a solution exists on this interval.

Proposition 5.2. *Assume that x_{c0}, x_{e0}, y_0, w_0 are all positive. If $\boldsymbol{z} = (x_c, x_e, y, w)$ is any solution to the system* (5.8)*, then all the components of $\boldsymbol{z}$ are nonnegative. Moreover, for $0 \leq t \leq T$, the following estimates hold,*

$$
\begin{aligned}
&0 \leq x_c(t) \leq (x_{c0} + x_{e0}) e^{\beta_c t}, \\
&0 \leq x_e(t) \leq (x_{c0} + x_{e0}) e^{\beta_c t}, \\
&0 < y(t) \leq y_0 e^{\gamma \lambda t}, \\
&0 \leq w(t) \leq (w_0 + M) + M e^{(\gamma\lambda + \beta_c) t}.
\end{aligned}
\tag{5.9}
$$

Here, M is a positive constant that depends only on the problem parameters.

Proof. It follows from the first two equations, by multiplying the first with x_e and the second with x_c, that

$$
\begin{aligned}
x_c' x_e &= (\beta_c - \alpha_c)\, x_c x_e + \gamma_e x_e^2 - \gamma_c x_c x_e, \\
x_e' x_c &= (\beta_e - \alpha_e) x_e x_c - \gamma_e x_e x_c + \gamma_c x_c^2.
\end{aligned}
$$

Adding the two expressions yields

$$
(x_e x_c)' = ((\beta_c - \alpha_c) + (\beta_e - \alpha_e))\, x_c x_e - (\gamma_c + \gamma_e)\, x_e x_c + \gamma_c x_c^2 + \gamma_e x_e^2.
$$

Since initially $x_e(0) x_c(0) > 0$, by continuity, there may be $t_0 > 0$ such that $x_e(t) x_c(t) > 0$ for $0 \leq t < t_0$. Then, if $x_e(t_0) x_c(t_0) = 0$, we obtain

$$
(x_e x_c)'(t_0) = \gamma_c x_c^2 + \gamma_e x_e^2 \geq 0.
$$

If either $x_e(t_0) > 0$ or $x_c(t_0) > 0$, then $(x_e x_c)'(t_0) > 0$, so, by continuity, the function $(x_e x_c)(t)$ is increasing at t_0, which contradicts the assumption that $x_e(t_0)x_c(t_0) = 0$ for the first time.

Now, suppose both vanish at $t_0 > 0$. Then, for $t \geq t_0$, we consider the system

$$\begin{aligned} x_c'(t) &= (\beta_c - \alpha_c)\, x_c + \gamma_e x_e - \gamma_c x_c, \qquad x_c\,(t_0) = 0, \\ x_e'(t) &= (\beta_e - \alpha_e) x_e - \gamma_e x_e + \gamma_c x_c, \qquad x_e\,(t_0) = 0, \end{aligned}$$

which has the unique solution solution $x_c(t) = x_e(t) = 0$ for $t \geq t_0$, so extinction has occurred. It follows that $x_e x_c \geq 0$ on $[0, T]$, for each $0 < T$ and so both functions are nonnegative, as long as they exist.

Next, we obtain upper bounds on x_c and x_e. We add the first two equations and obtain

$$(x_c + x_e)' = (\beta_c - \alpha_c)\, x_c + (\beta_e - \alpha_e)\, x_e \leq \beta_c (x_c + x_e).$$

Therefore,

$$x_c(t) + x_e(t) \leq (x_{c0} + x_{e0}) e^{\beta_c t}.$$

Since x_c and x_e are nonnegative, each one is bounded and thus, we established the first two estimates in (5.9).

Next, to show that y is actually positive, we argue as follows. The equation is

$$y' = \gamma y(\lambda - y) - \delta_y x_c y.$$

Since $x_c \geq 0$ and $\gamma, \delta_y > 0$, it follows that

$$y' \geq -(\gamma y + \delta x_c) y.$$

Since $y' \leq \gamma\lambda y$, we obtain

$$y(t) \leq y_0 e^{\gamma\lambda t}.$$

Using the first estimate in (5.9) and this estimate, we find

$$y' \geq -(\gamma y_0 e^{\gamma\lambda t} + \delta x_{co} e^{\beta_c t})y.$$

Thus,

$$y(t) \geq y_0 \exp\left(-(\gamma y_0 e^{\gamma\lambda t} + \delta(x_{co} + x_{e0})e^{\beta_c t})\right) > 0.$$

This proves both sides of the estimate for y in (5.9).

Finally, we address the last inequality for w. We have,

$$w' = \delta_w x_c y - \left(\min\left(1, \frac{w}{w_{th}}\right) s x_c + \min\left(1, \frac{w}{w_{th}}\right) \kappa_1 s x_e\right)$$

Since $w_0 > 0$, assume that there is $0 < t_0$ such that $w(t) > 0$ for $0 \leq t < t_0$, and if $w(t_0) = 0$ then $w'(t_0) = \delta_w x_c y \geq 0$. But $y > 0$ so if $x_c(t_0) = 0$, then, as discussed above, both x_c and x_e vanish identically for $t_0 \leq t$, which shows that $w(t) = 0$ for $t_0 \leq t$, as well.

Furthermore, it follows from the equation that $w' \leq \delta_w x_c y$, and using the estimates for y and x_c, we obtain

$$w' \leq \delta_w y_0 (x_{c0} + x_{e0}) e^{(\gamma\lambda + \beta_c)t}.$$

Integration over $[0, t]$ yields

$$w(t) \leq (w_0 + M) + M e^{(\gamma\lambda + \beta_c)t},$$

where

$$M = \frac{\delta_w y_0(x_{c0} + x_{e0})}{(\gamma\lambda + \beta_c)}.$$

We conclude that w is nonnegative and bounded. This completes the proof. ■

These estimates allow us to establish the existence of solutions of the system by using Theorem 2.4. Indeed, the a-priori estimates, established in Proposition 5.2, for possible solutions of system (5.1), allow us to apply Theorem 2.4, hence we obtain the existence of the solutions to the *HANDY-SM Model*. Moreover, a careful checking of the proof of Proposition 5.2 shows that the results hold true even when $x_{0e} = 0$ and $w_0 = 0$, by allowing $x_{0e} \to 0$ and $w_0 \to 0$.
We summarize these results in one on the main theorems in this chapter.

Theorem 5.3. *Assume that x_{c0} and y_0 are positive and x_{e0} and w_0 are nonnegative. Then, Model 5.1 has solutions on every finite time interval $[0, T]$. Moreover, the solutions satisfy the estimates* (5.9)*.*

As was mentioned above, Theorem 2.4 does not require the right-hand side $\boldsymbol{F}$ to be Lipschitz continuous. Moreover, the same proof applies to Problem 3.3, so the division into S_0, S_1 and S_2 is of mathematical origin and is not essential for existence. However, Theorem 5.3 does not guarantee the uniqueness of the solution. That has to be done separately. Proceeding as in Chapter 3, we can show uniqueness on the modified domains S_j, for $j = 0, 1, 2$ where the function $\boldsymbol{F}$ is Lipschitz.

5.3 Steady states

This section studies the steady states of system (5.1). First, we determine the steady states, and then establish their stability.
Clearly, the origin $\boldsymbol{z}_0 = (0, 0, 0, 0)$ in $\mathbb{R}^4_+$ and $\boldsymbol{z}_\lambda = (0, 0, \lambda, 0)$ are steady states. As we show below, the origin is unstable while $\boldsymbol{z}_\lambda$ is stable (see Section 3.3). Then, we

turn to find the other possible steady states $\overline{\boldsymbol{z}} = (\overline{x}_c, \overline{x}_e, \overline{y}, \overline{w})$, and assume that $\overline{y} > 0$, since without resources the model doesn't make sense, and we also assume that $x_{c0} > 0$, since without workers there is no wealth generation.

To find the steady states, we solve the nonlinear algebraic system

$$\begin{aligned} 0 &= (\beta_c - \alpha_c)\overline{x}_c + \gamma_e \overline{x}_e - \gamma_c \overline{x}_c, \\ 0 &= (\beta_e - \alpha_e)\overline{x}_e - \gamma_e \overline{x}_e + \gamma_c \overline{x}_c, \\ 0 &= \gamma \overline{y}(\lambda - \overline{y}) - \delta_y \overline{x}_c \overline{y}, \\ 0 &= \delta_w \overline{x}_c \overline{y} - C_c - C_e, \end{aligned} \tag{5.10}$$

where C_c and C_e are given in (5.3), w_{th} is given in (5.4), and α_c and α_e are given in (5.5). Next, we omit the ϵ in w_{th}, since $x_{c0} > 0$, and use

$$\overline{w}_{th} = \rho(\overline{x}_c + \kappa_0 \overline{x}_e).$$

We introduce two nonnegative parameters that simplify the writing in what follows:

$$\eta = \frac{\overline{w}}{\overline{w}_{th}}, \qquad \psi = \frac{\overline{x}_e}{\overline{x}_c}, \tag{5.11}$$

which measure the ratio of wealth to the wealth threshold, and the ratio of Elites to Commoners. Then,

$$C_c = \min(1, \eta) s x_c, \qquad C_e = \min(1, \eta) \kappa_1 s x_e,$$

and

$$\alpha_c = \alpha_m + \max(0, 1 - \min(1, \eta))(\alpha_M - \alpha_m),$$

$$\alpha_e = \alpha_m + \max(0, 1 - \kappa_1 \min(1, \eta))(\alpha_M - \alpha_m).$$

We recall that, by assumption, $0 < \alpha_m \le \beta_e \le \beta_c \le \alpha_M < 1$.

We note that $\kappa_1 \ge 1$ and consider three cases: (i) $\eta < 1/\kappa_1$, (ii) $1/\kappa_1 \le \eta < 1$, and (iii) $1 \le \eta$.

Case (i): $\eta < 1/\kappa_1$. Then, $C_c = \eta s \overline{x}_c$, $C_e = \eta \kappa_1 s \overline{x}_e$, and

$$\alpha_c = \alpha_M - \eta(\alpha_M - \alpha_m), \qquad \alpha_e = \alpha_M - \kappa_1 \eta(\alpha_M - \alpha_m).$$

Since $\overline{x}_c > 0$ by assumption, the first equation in (5.10) implies,

$$0 = \beta_c - \alpha_M + \eta(\alpha_M - \alpha_m) + \gamma_e \psi - \gamma_c.$$

From the second equation, we get either $\overline{x}_e = 0$, i.e., $\psi = 0$ (no Elites), or

$$0 = \beta_e - \alpha_M + \kappa_1 \eta(\alpha_M - \alpha_m) - \gamma_e \psi + \gamma_c.$$

Adding the two expressions and using simple algebra yields ($\psi > 0$),

$$\eta = \frac{(2\alpha_M - \beta_c - \beta_e)}{(\alpha_M - \alpha_m)(\kappa_1 + 1)}. \tag{5.12}$$

Now, the condition $\eta < 1/\kappa_1$ reads

$$\frac{2\alpha_M - \beta_c - \beta_e}{\alpha_M - \alpha_m} < 1 + \frac{1}{\kappa_1}.$$

It follows that when this inequality doesn't hold, case (i) cannot be realized as a nonnegative steady solution. Furthermore, we find that the ratio ψ is determined by the coefficients,

$$\psi = \frac{1}{\gamma_e(\kappa_1 + 1)} \left((\kappa_1 - 1)\alpha_M + (\kappa_1 + 1)\gamma_c + \beta_e - \kappa_1\beta_c \right). \tag{5.13}$$

We note in passing that $\psi = 0$, i.e., no Elites, can happen only when $\gamma_c = 0$, so no upward mobility, and $\beta_e = \beta_c = \alpha_M$. We describe this case below.

The equation for $\overline{y}$ leads to

$$0 = (\gamma(\lambda - \overline{y}) - \delta_y \overline{x}_c)\overline{y},$$

and since we assume that $\overline{y} > 0$, we find

$$0 = \gamma(\lambda - \overline{y}) - \delta_y \overline{x}_c.$$

Hence,

$$\overline{x}_c = \frac{\gamma}{\delta_y}(\lambda - \overline{y}).$$

The equation for w yields

$$0 = \delta_w \overline{x}_c \overline{y} - \eta s \overline{x}_c - \eta \kappa_1 s \overline{x}_e = (\delta_w \overline{y} - \eta s - \eta \kappa_1 s \psi)\overline{x}_c,$$

and since $\overline{x}_c > 0$,

$$0 = \delta_w \overline{y} - \eta s - \eta \kappa_1 s \psi.$$

Therefore,

$$\overline{y} = \frac{\eta s}{\delta_w}(1 + \kappa_1 \psi).$$

This implies that

$$\overline{x}_c = \frac{\gamma}{\delta_y}(\lambda - \overline{y}) = \frac{\gamma}{\delta_y}\left(\lambda - \frac{\eta s}{\delta_w}(1 + \kappa_1 \psi)\right).$$

Next, $\overline{x}_e = \psi \overline{x}_c$. Finally, $\overline{w} = \eta\rho(\overline{x}_c + \kappa_0 \overline{x}_e)$, thus

$$\overline{w} = \eta\rho(1 + \kappa_0\psi)\overline{x}_c.$$

Moreover, since $\overline{x}_c > 0$, we find that $\overline{w} > 0$ and

$$0 < \frac{\gamma}{\delta_y}\left(\lambda - \frac{\eta s}{\delta_w}(1 + \kappa_1 \psi)\right),$$

which provides the following bound on ψ,

$$0 \leq \psi < \frac{1}{\kappa_1}\left(\frac{\lambda \delta_w}{s\eta} - 1\right).$$

We now consider case (ii), $1/\kappa_1 \leq \eta < 1$. Proceeding as above, we find

$$C_c = \eta s \overline{x}_c, \qquad C_e = \eta \kappa_1 s \overline{x}_e,$$

and

$$\alpha_c = \alpha_M - \eta(\alpha_M - \alpha_m), \qquad \alpha_e = \alpha_m.$$

Similarly, since $\overline{x}_c > 0$, we obtain from the first equation in (5.1),

$$0 = \beta_c - \alpha_M + \eta(\alpha_M - \alpha_m) + \gamma_e \psi - \gamma_c.$$

The second equation yields either $\overline{x}_e = 0$, which is the case with $\psi = 0$, or

$$0 = (\beta_e - \alpha_m - \gamma_e)\psi + \gamma_c.$$

Thus, ψ is also fixed in this case,

$$\psi = \frac{\gamma_c}{\alpha_m - \beta_e + \gamma_e}. \tag{5.14}$$

Therefore, $0 \leq \psi$ only when $0 < \alpha_m - \beta_e + \gamma_e$. Otherwise, case (ii) doesn't yield any steady states. Then, η is determined as,

$$\eta = \frac{\alpha_M - \beta_c}{\alpha_M - \alpha_m} - \frac{(\alpha_m - \beta_e)\gamma_c}{(\alpha_M - \alpha_m)(\alpha_m - \beta_e + \gamma_e)}. \tag{5.15}$$

In this case, the right-hand side must satisfy $1/\kappa_1 \leq \eta < 1$, otherwise case (ii) cannot have steady solutions.

Proceeding as in case (i), the equations for y and w lead to the same expressions but with the new η and ψ.

We summarize the findings in cases (i) and (ii) as follows.

Proposition 5.4. *Assume that either (i) $0 \leq \eta < 1/\kappa_1$; or (ii) $1/\kappa_1 < \eta < 1$ and $0 < \alpha_m - \beta_e + \gamma_e$. Then, the unique steady state solution $\overline{\boldsymbol{z}} = (\overline{x}_c, \overline{x}_e, \overline{y}, \overline{w})$, is given by*

$$\begin{aligned}
\overline{x}_c &= \frac{\gamma}{\delta_y}\left(\lambda - \frac{\eta s}{\delta_w}(1 + \kappa_1\psi)\right), \\
\overline{x}_e &= \psi\overline{x}_c, \\
\overline{y} &= \frac{\eta s}{\delta_w}(1 + \kappa_1\psi), \\
\overline{w} &= \eta\rho(1 + \kappa_0\psi)\overline{x}_c.
\end{aligned} \tag{5.16}$$

In case (i) η is given in (5.12) *and ψ in* (5.13)*, while in case (ii) η is given in* (5.15) *and ψ in* (5.14)*.*

We now consider case (iii), $1 \leq \eta$. We note that in this case the equations for $\overline{x}_c$ and $\overline{x}_e$ are not coupled to those for $\overline{y}$ and $\overline{w}$. Proceeding as above, we obtain

$$C_c = s\overline{x}_c, \qquad C_e = \kappa_1 s\overline{x}_e, \qquad \alpha_c = \alpha_e = \alpha_m.$$

Since $\overline{x}_c > 0$, the first equation in (5.1) leads to

$$0 = \beta_c - \alpha_m + \gamma_e\psi - \gamma_c.$$

From the second equation, we get either $\overline{x}_e = 0$, which is the case with $\psi = 0$, or

$$0 = (\beta_e - \alpha_m - \gamma_e)\psi + \gamma_c.$$

Hence,

$$\psi = \frac{\gamma_c}{\alpha_m - \beta_e + \gamma_e} = \frac{1}{\gamma_e}(\alpha_m - \beta_c + \gamma_c). \tag{5.17}$$

Therefore, $0 \leq \psi$ only when $0 < \alpha_m - \beta_e + \gamma_e$ and $0 < \alpha_m - \beta_c + \gamma_c$. Otherwise, case (iii) doesn't yield any steady states. Then, the equation for $\overline{y}$ yields

$$0 = \gamma(\lambda - \overline{y}) - \delta_y\overline{x}_c.$$

Thus,

$$\overline{x}_c = \frac{\gamma}{\delta_y}(\lambda - \overline{y}).$$

Finally, the equation for w yields

$$0 = \delta_w \overline{y} - s - \kappa_1 s\psi.$$

Therefore,

$$\overline{y} = \frac{s}{\delta_w}(1 + \kappa_1\psi).$$

This implies that

$$\overline{x}_c = \frac{\gamma}{\delta_y}(\lambda - \overline{y}) = \frac{\gamma}{\delta_y}\left(\lambda - \frac{s}{\delta_w}(1 + \kappa_1\psi)\right).$$

Next, $\overline{x}_e = \psi\overline{x}_c$. Finally, $\overline{w} = \eta\rho(\overline{x}_c + \kappa_0\overline{x}_e)$, thus

$$\overline{w} = \eta\rho(1 + \kappa_0\psi)\overline{x}_c.$$

Now, by assumption $\overline{x}_c > 0$, hence $\overline{w} > 0$ and

$$0 < \frac{\gamma}{\delta_y}\left(\lambda - \frac{s}{\delta_w}(1 + \kappa_1\psi)\right),$$

which provides the following bound on ψ,

$$0 \leq \psi < \frac{1}{\kappa_1}\left(\frac{\lambda\delta_w}{s} - 1\right).$$

We summarize case (iii) as follows.

Proposition 5.5. *Let* $1 \leq \eta$*, and assume that* $0 < \alpha_m - \beta_e + \gamma_e$*,* $0 < \alpha_m - \beta_c + \gamma_c$*, and*

$$\frac{\gamma_c}{\alpha_m - \beta_e + \gamma_e} = \frac{1}{\gamma_e}(\alpha_m - \beta_c + \gamma_c).$$

Then, the steady state solutions $\overline{\boldsymbol{z}} = (\overline{x}_c, \overline{x}_e, \overline{y}, \overline{w})$*, form a one-parameter family of solutions parametrized by* η*, where* $1 \le \eta$*, and are given by*

$$\begin{aligned}
\overline{x}_c &= \frac{\gamma}{\delta_y}\left(\lambda - \frac{s}{\delta_w}(1+\kappa_1\psi)\right), \\
\overline{x}_e &= \psi\overline{x}_c, \\
\overline{y} &= \frac{s}{\delta_w}(1+\kappa_1\psi), \\
\overline{w} &= \eta\rho(1+\kappa_0\psi)\overline{x}_c.
\end{aligned} \tag{5.18}$$

Here, ψ *is given in* (5.17)*.*

Remark 5.6. *We note that when* $1 \le \eta$ *only the wealth* $\overline{w}$ *of the solution depends on* η*, while* ψ *is fixed by the system parameters. Thus,* $(\overline{x}_c, \overline{x}_e, \overline{y})$ *are uniquely determined, so they have the same values, independently of the wealth, as long as* $1 \le \eta$*, i.e., there is sufficient wealth to have the populations be above the famine threshold.*

5.4 Stability of the steady states

We study the stability of the steady states. To that end, we compute the Jacobian matrix J of $\boldsymbol{F}$, proceeding by the components.

$$J_{11} = \frac{\partial F_1}{\partial x_c} = (\beta_c - \alpha_c - \gamma_c) - x_c\frac{\partial \alpha_c}{\partial x_c};$$

$$J_{12} = \frac{\partial F_1}{\partial x_e} = \gamma_e - x_c\frac{\partial \alpha_c}{\partial x_e};$$

$$J_{13} = \frac{\partial F_1}{\partial y} = 0; \qquad J_{14} = \frac{\partial F_1}{\partial w} = -x_c\frac{\partial \alpha_c}{\partial w}.$$

Next,

$$J_{21} = \frac{\partial F_2}{\partial x_c} = \gamma_c - x_e \frac{\partial \alpha_e}{\partial x_c};$$

$$J_{22} = \frac{\partial F_2}{\partial x_e} = (\beta_e - \alpha_e - \gamma_e) - x_e \frac{\partial \alpha_e}{\partial x_e};$$

$$J_{23} = \frac{\partial F_2}{\partial y} = 0; \qquad J_{24} = \frac{\partial F_2}{\partial w} = -x_e \frac{\partial \alpha_e}{\partial w}.$$

Furthermore,

$$J_{31} = \frac{\partial F_3}{\partial x_c} = -\delta_y y; \qquad J_{32} = \frac{\partial F_3}{\partial x_e} = 0;$$

$$J_{33} = \frac{\partial F_3}{\partial y} = \gamma\lambda - 2\gamma y - \delta_y x_c; \qquad J_{34} = \frac{\partial F_3}{\partial w} = 0.$$

Finally,

$$J_{41} = \frac{\partial F_4}{\partial x_c} = \delta_w y - \frac{\partial C_c}{\partial x_c} - \frac{\partial C_e}{\partial x_c}; \qquad J_{42} = \frac{\partial F_4}{\partial x_e} = -\frac{\partial C_c}{\partial x_e} - \frac{\partial C_e}{\partial x_e};$$

$$J_{43} = \frac{\partial F_4}{\partial y} = \delta_w x_c; \qquad J_{44} = \frac{\partial F_4}{\partial w} = -\frac{\partial C_c}{\partial w} - \frac{\partial C_e}{\partial w}.$$

To compute the partial derivatives of $\alpha_c, \alpha_e, C_c, C_e$ we again need to consider the three cases (i) $\eta < 1/\kappa_1$; (ii) $1/\kappa_1 \leq \eta < 1$; and (iii) $1 \leq \eta$.

We begin with case (i), when $\eta < 1/\kappa_1$. Then,

$$C_c = \eta s x_c, \quad C_e = \eta \kappa_1 s x_e, \quad \alpha_c = \alpha_M - \eta(\alpha_M - \alpha_m), \quad \alpha_e = \alpha_M - \kappa_1 \eta(\alpha_M - \alpha_m).$$

Moreover, in this case,

$$\eta = \frac{(2\alpha_M - \beta_c - \beta_e)}{(\alpha_M - \alpha_m)(\kappa_1 + 1)}.$$

Recalling that $\eta = w/\rho(x_c + \kappa_0 x_e)$ ($\epsilon = 0$), we find:

$$\frac{\partial \alpha_c}{\partial x_c} = -(\alpha_M - \alpha_m)\frac{\partial \eta}{\partial x_c} = \frac{\eta(\alpha_M - \alpha_m)}{(1 + \kappa_0\psi)x_c}; \qquad \frac{\partial \alpha_c}{\partial x_e} = \frac{\eta\kappa_0(\alpha_M - \alpha_m)}{(1 + \kappa_0\psi)x_c};$$

$$\frac{\partial \alpha_e}{\partial x_c} = -\kappa_1(\alpha_M - \alpha_m)\frac{\partial \eta}{\partial x_c} = \frac{\kappa_1 \eta(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)x_c}; \qquad \frac{\partial \alpha_e}{\partial x_e} = \frac{\eta\kappa_0\kappa_1(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)x_c};$$

$$\frac{\partial \alpha_c}{\partial w} = -\frac{(\alpha_M - \alpha_m)}{\rho(1+\kappa_0\psi)x_c}; \qquad \frac{\partial \alpha_e}{\partial w} = -\frac{\kappa_1(\alpha_M - \alpha_m)}{\rho(1+\kappa_0\psi)x_c};$$

$$\frac{\partial C_c}{\partial x_c} = \eta s + s x_c \frac{\partial \eta}{\partial x_c} = \frac{\eta s \kappa_0 \psi}{(1+\kappa_0\psi)}; \quad \frac{\partial C_c}{\partial x_e} = s x_c \frac{\partial \eta}{\partial x_e} = -\frac{\eta s \kappa_0}{(1+\kappa_0\psi)};$$

$$\frac{\partial C_c}{\partial w} = \frac{s}{\rho(1+\kappa_0\psi)}; \qquad \frac{\partial C_e}{\partial w} = \frac{s\kappa_1\psi}{\rho(1+\kappa_0\psi)};$$

$$\frac{\partial C_e}{\partial x_c} = \kappa_1 s x_e \frac{\partial \eta}{\partial x_c} = -\frac{\eta s \kappa_1 \psi}{(1+\kappa_0\psi)}; \quad \frac{\partial C_e}{\partial x_e} = \frac{\eta s \kappa_1}{(1+\kappa_0\psi)}.$$

Collecting these results yields,

$$J_{11} = (\beta_c - (\alpha_M - \eta(\alpha_M - \alpha_m)) - \gamma_c) - \frac{\eta(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)}$$

$$= (\beta_c - \alpha_M - \gamma_c) + \frac{\eta\kappa_0\psi(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)};$$

$$J_{12} = \gamma_e - \frac{\eta\kappa_0(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)}; \qquad J_{13} = 0; \qquad J_{14} = \frac{(\alpha_M - \alpha_m)}{\rho(1+\kappa_0\psi)}.$$

Next,

$$J_{21} = \gamma_c - \frac{\eta\kappa_1(\alpha_M - \alpha_m)\psi}{(1+\kappa_0\psi)};$$

$$J_{22} = (\beta_e - \alpha_M - \gamma_e) + \frac{\eta\kappa_1(\alpha_M - \alpha_m)}{(1+\kappa_0\psi)}; \qquad J_{23} = 0; \qquad J_{24} = \frac{\kappa_1\psi(\alpha_M - \alpha_m)}{\rho(1+\kappa_0\psi)}.$$

Furthermore,

$$J_{31} = -\delta_y y = -\frac{\eta s \delta_y}{\delta_w}(1+\kappa_1\psi); \qquad J_{32} = 0;$$

$$J_{33} = \gamma\lambda - 2\gamma y - \delta_y x_c = -\frac{\gamma\eta s}{\delta_w}(1+\kappa_1\psi); \qquad J_{34} = 0.$$

Finally,

$$J_{41} = \eta s\left((1+\kappa_1\psi) + \frac{\psi(\kappa_1 - \kappa_0)}{(1+\kappa_0\psi)}\right); \qquad J_{42} = \frac{\eta s(\kappa_0 - \kappa_1)}{(1+\kappa_0\psi)};$$

$$J_{43} = \frac{\gamma\delta_w}{\delta_y}\left(\lambda - \frac{\eta s}{\delta_w}(1+\kappa_1\psi)\right) = \frac{\gamma}{\delta_y}(\delta_w\lambda - \eta s(1+\kappa_1\psi)); \qquad J_{44} = -\frac{s(1+\kappa_1\psi)}{\rho(1+\kappa_0\psi)}.$$

To simplify the notations in J, we let

$$\Delta\alpha = \alpha_M - \alpha_m, \qquad \phi_0 = (1+\kappa_0\psi)^{-1}, \qquad \phi_1 = (1+\kappa_1\psi).$$

Then, the Jacobian matrix in case (i) is given by

$$J_{(i)}(\bar{\boldsymbol{z}}) = \tag{5.19}$$

$$\begin{pmatrix} (\beta_c - \alpha_M - \gamma_c) + \eta\kappa_0\psi\Delta\alpha\phi_0 & \gamma_e - \eta\kappa_0\Delta\alpha\phi_0 & 0 & \Delta\alpha\phi_0\rho^{-1} \\ \gamma_c - \eta\kappa_1\psi\Delta\alpha\phi_0 & (\beta_e - \alpha_M - \gamma_e) + \eta\kappa_1\Delta\alpha\phi_0 & 0 & \kappa_1\psi\Delta\alpha\phi_0\rho^{-1} \\ -\eta s\phi_1(\delta_y/\delta_w) & 0 & -\gamma\eta s\phi_1(1/\delta_w) & 0 \\ \eta s\left(\phi_1 + \psi(\kappa_1-\kappa_0)\phi_0\right) & \eta s(\kappa_0-\kappa_1)\phi_0 & \frac{\gamma}{\delta_y}\left(\lambda\delta_w - \eta s\phi_1\right) & -s\phi_1\phi_0\rho^{-1} \end{pmatrix}.$$

It follows that at the origin, $(0,0,0,0)$, where $\eta = \overline{w} = 0$ and $\alpha_c = \alpha_e = \alpha_M$, hence $\Delta\alpha = 0$, we have

$$J(0,0,0,0) = \tag{5.20}$$

$$\begin{pmatrix} \beta_c - \alpha_M - \gamma_c & \gamma_e & 0 & 0 \\ \gamma_c & \beta_e - \alpha_M - \gamma_e & 0 & 0 \\ 0 & 0 & \gamma\lambda & 0 \\ 0 & 0 & 0 & -\frac{s(1+\kappa_1)}{\rho(1+\kappa_0)} \end{pmatrix}.$$

We conclude that the eigenvalues are

$$\widehat{\lambda}_1 = \beta_c - \alpha_c - \gamma_c, \quad \widehat{\lambda}_2 = \beta_e - \alpha_e - \gamma_e, \quad \widehat{\lambda}_3 = \gamma\lambda, \quad \widehat{\lambda}_4 = -\frac{s(1+\kappa_1)}{\rho(1+\kappa_0)}.$$

Since $\widehat{\lambda}_3 = \gamma\lambda > 0$, the origin is unstable.

We next study the stability of $(0,0,\lambda,0)$. Similarly to the above, we find

$$J(0,0,\lambda,0) = \tag{5.21}$$

$$\begin{pmatrix} \beta_c-\alpha_c-\gamma_c & \gamma_e & 0 & 0 \\ \gamma_c & \beta_e-\alpha_e-\gamma_e & 0 & 0 \\ -\delta_y\lambda & 0 & -\gamma\lambda & 0 \\ \delta_w\lambda & 0 & 0 & -\frac{s(1+\kappa_1)}{\rho(1+\kappa_0)} \end{pmatrix}.$$

The eigenvalues are

$$\widehat{\lambda}_1 = \beta_c-\alpha_c-\gamma_c, \quad \widehat{\lambda}_2 = \beta_e-\alpha_e-\gamma_e, \quad \widehat{\lambda}_3 = -\gamma\lambda, \quad \widehat{\lambda}_4 = -\frac{s(1+\kappa_1)}{\rho(1+\kappa_0)}.$$

Since it was assumed above that $\beta_c-\alpha_c-\gamma_c \leq 0$ and $\beta_e-\alpha_e-\gamma_e \leq 0$, then, this state is stable, and if by assumption the inequalities are strict, then it is stable and attracting (asymptotically stable).

We consider case (ii) next, thus,

$$C_c = \eta s x_c, \quad C_e = \eta\kappa_1 s x_e, \quad \alpha_c = \alpha_M - \eta(\alpha_M-\alpha_m), \quad \alpha_e = \alpha_m.$$

It follows that the only changes from case (i) are those related to α_e. Hence, we find that the only difference in the J matrix, from the matrix in case (i) above, is in the following components:

$$J_{21} = \gamma_c; \qquad J_{22} = \beta_e-\alpha_m-\gamma_e; \qquad J_{23} = 0; \qquad J_{24} = 0.$$

$$J_{(ii)}(\bar{\boldsymbol{z}}) =$$

$$
\begin{pmatrix}
(\beta_c - \alpha_M - \gamma_c) + \eta\kappa_0\psi\Delta\alpha\phi_0 & \gamma_e - \eta\kappa_0\Delta\alpha\phi_0 & 0 & \Delta\alpha\phi_0\rho^{-1} \\
\gamma_c & \beta_e - \alpha_m - \gamma_e & 0 & 0 \\
-\eta s\phi_1(\delta_y/\delta_w) & 0 & -\gamma\eta s\phi_1(1/\delta_w) & 0 \\
\eta s\left(\phi_1 + \psi(\kappa_1 - \kappa_0)\phi_0\right) & \eta s(\kappa_0 - \kappa_1)\phi_0 & \frac{\gamma}{\delta_y}\left(\lambda\delta_w - \eta s\phi_1\right) & -s\phi_1\phi_0\rho^{-1}
\end{pmatrix}.
$$

Finally, in case (iii), where $\eta > 1$, we have,

$$C_c = sx_c, \quad C_e = \kappa_1 s x_e, \quad \alpha_c = \alpha_m, \quad \alpha_e = \alpha_m.$$

Then, the Jacobian is given by

$$J_{11} = \frac{\partial F_1}{\partial x_c} = \beta_c - \alpha_m - \gamma_c; \qquad J_{12} = \frac{\partial F_1}{\partial x_e} = \gamma_e;$$

$$J_{13} = \frac{\partial F_1}{\partial y} = 0; \qquad J_{14} = \frac{\partial F_1}{\partial w} = 0.$$

Next,

$$J_{21} = \frac{\partial F_2}{\partial x_c} = \gamma_c; \qquad J_{22} = \frac{\partial F_2}{\partial x_e} = \beta_e - \alpha_m - \gamma_e;$$

$$J_{23} = \frac{\partial F_2}{\partial y} = 0; \qquad J_{24} = \frac{\partial F_2}{\partial w} = 0.$$

Furthermore,

$$J_{31} = \frac{\partial F_3}{\partial x_c} = -\delta_y y = -\frac{\delta_y}{\delta_w} s(1 + \kappa_0\psi); \qquad J_{32} = \frac{\partial F_3}{\partial x_e} = 0;$$

$$J_{33} = \frac{\partial F_3}{\partial y} = \gamma\lambda - 2\gamma y - \delta_y x_c = -\frac{\gamma}{\delta_w} s(1 + \kappa_0\psi); \qquad J_{34} = \frac{\partial F_3}{\partial w} = 0.$$

Finally,

$$J_{41} = \frac{\partial F_4}{\partial x_c} = \delta_w y - \frac{\partial C_c}{\partial x_c} - \frac{\partial C_e}{\partial x_c} = \delta_w y - s = s\kappa_1\psi;$$

$$J_{42} = \frac{\partial F_4}{\partial x_e} = -\frac{\partial C_c}{\partial x_e} - \frac{\partial C_e}{\partial x_e} = -s\kappa_1;$$

$$J_{43} = \frac{\partial F_4}{\partial y} = \delta_w x_c = \frac{\gamma}{\delta_y}(\lambda\delta_w - s(1+\kappa_1\psi));$$

$$J_{44} = \frac{\partial F_4}{\partial w} = -\frac{\partial C_c}{\partial w} - \frac{\partial C_e}{\partial w} = 0.$$

Hence, the Jacobian matrix in case (iii) is given by

$$J_{(iii)}(\bar{\boldsymbol{z}}) = \begin{pmatrix} \beta_c - \alpha_m - \gamma_c & \gamma_e & 0 & 0 \\ \gamma_c & \beta_e - \alpha_m - \gamma_e & 0 & 0 \\ -s\phi_1\frac{\delta_y}{\delta_w} & 0 & -\gamma s\frac{\phi_1}{\delta_w} & 0 \\ s\kappa_1\psi & -s\kappa_1 & \frac{\gamma}{\delta_y}(\lambda\delta_w - s\phi_1) & 0 \end{pmatrix}. \tag{5.22}$$

Thus, we have closed form expressions for the Jacobian matrices in the three cases. In each of the simulations we also included the information about which case is simulated and its stability, when it converges to a steady state.

CHAPTER SIX

ALGORITHM AND SIMULATIONS OF THE HANDY MODEL WITH SOCIAL MOBILITY

This chapter depicts a few numerical simulations of societies modeled in 5.1. It explores some of the causes for societies' possible collapse, related to the concerns about social trends raised by several researchers, e.g, [15]. They claim that addressing only the problem of over-depleting the natural resources may not prevent a collapse. Indeed, [19] claims that *"Conservation does not produce sustainability."* It may be of interest to investigate the contributions of consumption rates inequality to the decline. And the effects of the social gap caused by pay inequality are rated as one potential factor of collapse according to [3]. To study the effects of inequality in shaping the societies' future, we note that our simulations indicate, indeed, that there can be a collapse not only because of the overuse of natural resources, represented in our model by the rate constant δ_y. To that end, this chapter uses the depletion factor $\delta_y = 2 \times 6.15\, 10^{-6}$, which is twice as large as in the original HANDY model, and is kept fixed in the first simulations, but increased in the second simulations to three times of δ_y. This allows to study the effects of increasing δ_y when, in addition, inequality is imposed. Moreover, we also vary the two mobility factors (γ_c, γ_e) and κ_1 to obtain a better picture of their effects on the society's dynamics. In all the simulations graphic displays, the green curve represents the natural resources and the cyan curve the wealth, while the curves in blue and magenta represent the populations of Commoners and Elites, respectively.

6.1 Algorithm

An algorithm for the solutions of the discretized model (5.1), using the Euler method and a python [20] code edited in Jupyter notebook [8], was constructed, as an appropriate modification of the one in Chapter 4. The steps of the algorithm are similar to those of the algorithm in Chapter 4, with the additions of the mobility factors as the code input. The steps follow:

Step 1. Initiation

set the problem coefficients:

$$\alpha_m,\ \alpha_M,\ \beta_c,\ \beta_e\ ,\delta_y,\ \delta_w,\ \kappa_0,\ \kappa_1,\ \lambda,\ \rho,\ s,\ \gamma_c\ ,\gamma_e$$

$$x_{c0},\ x_{e0},\ y_0,\ w_0.$$

set T (the final year) and N (the number of time steps)

set $\Delta t = T/N$.

Step 2.

set $n = 0$

set $x_c^0 = x_{c0}, \quad x_e^0 = x_{e0}, \quad y^0 = y_0, \quad w^0 = w_0,$

Step 3. Time marching

compute

C_c^n from (5.3)

C_e^n from (5.3)

w_{th}^n from (5.4)

α_c^n from (5.5)

α_e^n from (5.5)

set $n = n + 1$

compute

x_c^n from (5.1)

x_e^n from (5.1)

y^n from (5.1)

w^n from (5.1)

Step 4. if $n < N$ **go to Step 3**

end

6.2 Simulations

This section presents some of the simulations of the HANDY-SM model that we run. Since the model has 14 parameters and four initial conditions, the parameters belong to a box in a 14-dimensional space, so trying to find out the system sensitivity to each one and all of them is hopeless. Nevertheless, some of the parameters have values that can be reasonably estimated, so the dimensions of the box are actually less than 14. Since the main interest here is in the effects of inequality, the mobility and depletion of the resources, we concentrate on the effects of the $\gamma's$ and the $\kappa's$, while the depletion factor is kept fixed, but high. It is of interest to perform a partial sensitivity analysis with respect to the δ's as well. The first issue in each simulation was to calculate the value of η, which determined the appropriate characterization of the related steady state and its stability. We recall that the three cases were (i) $0 \leq \eta < 1/\kappa_1$ (ii) $1/\kappa_1 \leq \eta < 1$; and (iii) $1 < \eta$. The eigenvalues of the Jacobian at the steady states in the simulations were computed to determine the stability of the states. First, we chose the factor $\kappa_0 = 10$ and then changed the value of κ_1. Second, we provide the simulations that show the

Table 6.2: The parameters' values and the data used in the simulations of the HANDY-SM model.

Symbols	Values	Meaning
Δt	.05	time step
$x_c(0)$	10	Commoners initial condition
$x_e(0)$	2	Elites initial condition
$y(0)$	100	Nature's initial condition
$w(0)$	0	wealth initial condition
λ	100	Natures carrying capacity
γ	.01	Natures regeneration factor
δ_y	$1.23 \cdot 10^{-5}$	Natures depletion factor
δ_w	$1.2 \times \delta_y$	wealth rate of growth
α_m	0.01	death rate
α_M	0.07	maximum famine death rate
β_c	.03	Commoners birth rate
β_e	.03	Elites birth rate
s	$5 \cdot 10^{-4}$	salary per capita
ρ	10^{-4}	minimum required consumption
κ_0	10	wealth unbalanced threshold rate
κ_1	2,4,6,and 8	inequality factor
γ_c	changes from $2.8 \cdot 10^{-2}$ to $8 \cdot 10^{-2}$	Commoners mobility factor
γ_e	changes	Elites mobility factor

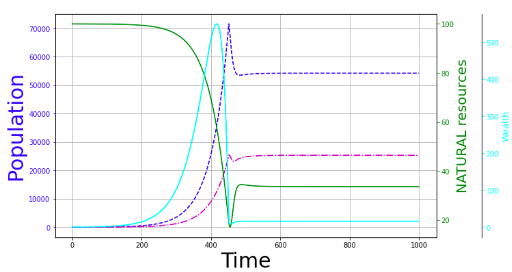
dependence of the societies dynamics, when the mobility factors γ_c and γ_e were changed, while keeping the social inequality factor $\kappa_1 = 2$ fixed. Additionally, δ_w, which represents the "cost rate," was kept different from the value in [14], where it was equal to δ_y, while here we used the value $\delta_w = 1.2 \times \delta_y$. Below, we provide further discussion of the effects of these changes.

Table 6.2 shows the parameters used in the simulations of the HANDY-SM model. Fig. 6.1 focuses on κ_1, which took the values $2, 5, 6$ and 8, and that was less by 2 than the w_{th} that is a case already discussed in [14]. This was the case of inequitable society with $\kappa = 10$. All the other parameters had the values given in table 6.2.

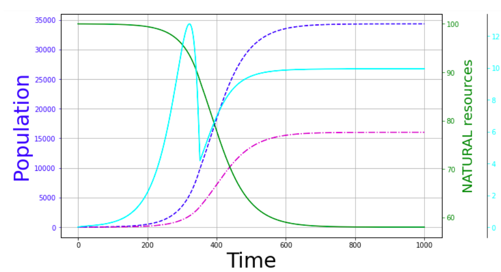
In 6.1(a) the values $(\kappa_1, \eta) = (2, 0.4)$ belong in case (i), the corresponding steady state was almost $(54,000;\ 25,000;\ 33;\ 15)$, and the eigenvalues were:

$$-2.38, \quad -0.108 \pm 0.108i, \quad -0.096$$

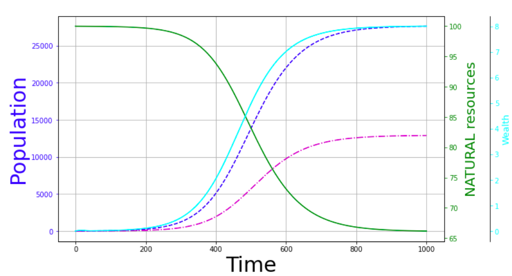
The system was stable and attracting, and the two conjugate complex numbers implied that the approach to the steady state was oscillatory. However, the oscillations were small. We note the very fast growth at about year 450, followed by a decay and quick convergence to the steady state.



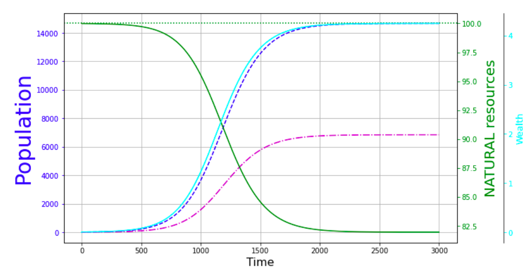
(a) $\kappa_1 = 2$



(b) with $\kappa_1 = 5$



(c) with $\kappa_1 = 6$



(d) $\kappa_1 = 8$

Figure 6.1: Effects of varying the inequality factor $\kappa_1 = 2$ while keeping both $\kappa_0 = 10$, $\delta_y = 2 \times 6.15 \cdot 10^{-6}$ and $(\gamma_c, \gamma_e) = (0.028, 0.08)$ fixed. The populations in all the subfigures seem to be stable, only in 6.1(d) it took longer time period to stabilize.

In scenario 6.1(b), the values $(\kappa_1, \eta) = (5, .22)$ made the simulation belong to case (ii) with the eigenvalues:

$$-3.35, \quad -0.076 \pm 0.093i, \quad -0.14,$$

with the steady state values at almost $(34,000; 16,000; 57; 10)$. It may be interpreted as when the Elites consumption rate is five times bigger than that of the Commoners, it pressures the Commoners to work harder to supply the demand, as compared to the steady state reached in Fig. 6.1(a).
The simulation 6.1(c), with $(\kappa_1, \eta) = (6, 0.19)$, also belongs to case (ii), where the steady state is $(27,000; 13,000; 65; 8)$, with the following eigenvalues:

$$-3.68, \quad -0.068 \pm 0.09i, \quad -0.15.$$

Each time the Elites consumption rate increased, both populations and the wealth decreased, while the nature increased. We note that δ_w, the wealth growth factor was larger than the value in the [14], making the wealth grow faster. A case (ii) of $(\kappa_1, \eta) = (8, 0.148)$ is depicted in 6.1(d), with the steady state of about $(14,600; 6,841; 82; 4)$, where the eigenvalues are

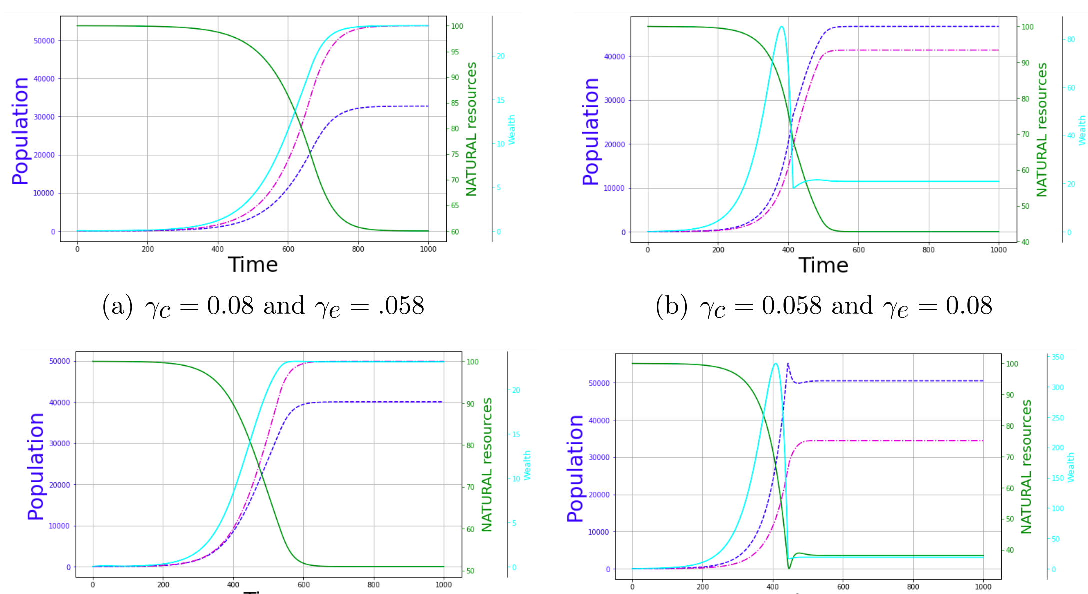
$$-4.34, \quad -0.06 \pm 0.09i, \quad -0.16.$$

The eigenvalues have negative real parts so the state is stable and attracting, however, the convergence to the state is very slow since they are close to zero.
Fig. 6.2 shows the results of varying three values of (γ_c, γ_e), while keeping $(\kappa_0, \kappa_1) = (10, 2)$, $\delta_y = 1.23 \cdot 10^{-5}$ and $\delta_w = 1.2\delta_y$ fixed. All the scenarios in the figure belong in case (i), and all are stable. First, in 6.2(a) $(\gamma_c, \gamma_e) = (0.08, 0.058)$,

which means that the rate of workers who become rich was larger than the rate of Elites who went bankrupt. The eigenvalues were

$$-2.383, \quad -0.147 \pm 0.12644i, \quad -0.05,$$

with steady state point of $(32,000; 53,000; 60; 23)$. Comparing the wealth in 6.1(a) and 6.2(a), which changed drastically from 15 to 23 when both had the same pair $(\kappa_0, \kappa_1) = (10, 2)$, while the pair $(\gamma_c, \gamma_e) = (.028, .08)$ changed to $(\gamma_c, \gamma_e) = (.08, .058)$, and the resources from 33 to 60 were the two main changes with the exchange of the $\gamma' s$.



(a) $\gamma_C = 0.08$ and $\gamma_E = .058$

(b) $\gamma_C = 0.058$ and $\gamma_E = 0.08$

(c) $\gamma_C = .08$, $\gamma_E = 0.08$

(d) $\gamma_C = 0.028$ with $\gamma_E = 0.058$

Figure 6.2: Effects of varying γ_c and γ_e, while keeping $\kappa_0 = 10$ and $\kappa_1 = 2$. The behavior is qualitatively similar, however the numbers are different. In all cases the variables converge to the steady states, monotonically, except for the wealth in (b) and (d) and the workers and nature in (d), over a period of 200 years. Note that there are more rich than workers in (a) and (c), while the order is reversed in (b) and (d).

When $(\gamma_c, \gamma_e) = (0.058, 0.08)$ in 6.2(b), the corresponding eigenvalues are

$$-2.383, \quad -0.132 \pm 0.114i, \quad -0.07,$$

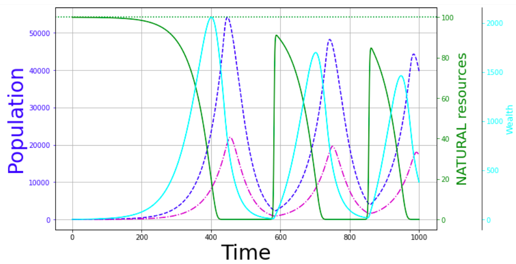
and thus, the stable and attracting steady state values of almost $(47,000; 41,000; 42; 20)$. Similarly to 6.2(a), the wealth reaches the value 23 for the pair $(\gamma_c, \gamma_e) = (0.08, 0.08)$ in 6.2(c) with the eigenvalues

$$-2.383, \quad -0.148 \pm 0.118i, \quad -0.07.$$

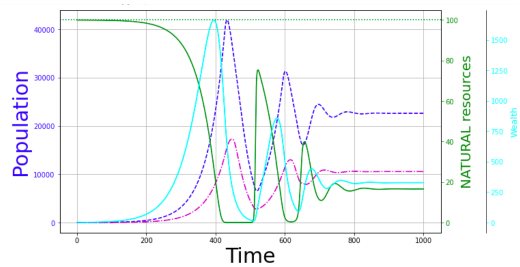
A growth of the Elites at $48,000$, as compared to $41,000$ of Commoners was noticed with equal mobility $\gamma's$. While natural resources ended with 49 eco-dollars as compared to 42 eco-dollars in the previous sub-figure 6.2(b). Finally, in the case depicted in 6.2(d), nature stabilized also on 42, while the wealth at the value of 20. The eigenvalues for this scenario were

$$-2.383, \quad -0.132 \pm 0.114i, \quad -0.079.$$

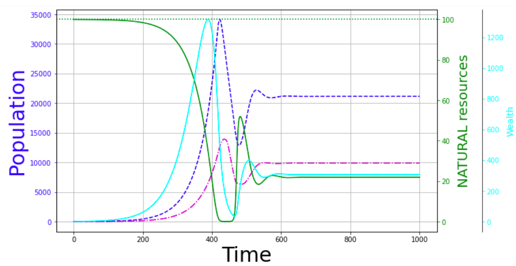
Fig. 6.3 depicts a different behavior with the same choices of $(\gamma_c, \gamma_e) = (0.028, 0.08)$ from the last sub-figure. With changing κ_1 in the pair $(\kappa_0, \kappa_1) = (10, \kappa_1)$, as was done above, with bigger choices for both $\delta_y = 6 \times 6.15 \cdot 10^{-6}$ and $\rho = 5 \cdot 10^{-3}$. Here, similarly, κ_1 varied through the values 2 to 8, with an increment of 2 in each graph, and $\kappa_0 = 10$ was fixed. The graph changes from oscillatory (very slowly approaching equilibrium) in 6.3(a) to faster approaching a steady state in 6.3(d). The graphs in 6.3(b) and 6.3(c) increasing κ_1 changes the graph from a damped oscillation to reaching an almost a steady state after a short period to a very small natural



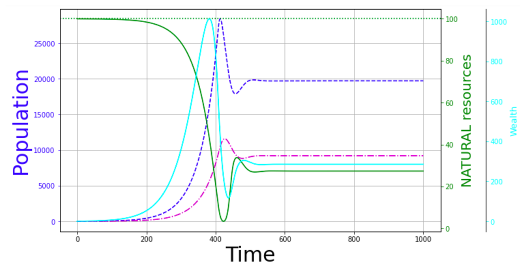
(a) $\kappa_1 = 2$



(b) $\kappa_1 = 4$



(c) $\kappa_1 = 6$



(d) $\kappa_1 = 8$

Figure 6.3: Effects of varying κ_1 while keeping $\kappa_0 = 10$ and $\delta_y = 6 \times 6.15 \cdot 10^{-6}$. The damped oscillations in (a) take a long time to decay, while the approach to the steady states in (b), (c) and (d) is increasingly faster. All the steady states are stable and attracting. Note the different scales in the sub-figures.

resources value. With every increase of κ_1, both nature and wealth increase faster from close to collapse to a steady state.

It is seen that the variations of the mobility factors and the inequality factors affect the system behavior, especially the numbers and the rates of convergence to the equilibria, even though qualitatively, the graphs have similar structures.

CHAPTER SEVEN

THE HANDY MODEL WITH MIDDLE CLASS - I

This chapter extends substantially the HANDY-SM model by adding a Middle class, which consists of managers, professionals, and small business owners, among others, leading to the HANDY-MC-I model. This makes the model more realistic and more complex, at the same time. A rate equation for the new Middle class population is added and appropriate modifications of the various coefficients and the other equations are done. The social mobility between the Elites, Middle class and Commoners is also taken into account. Thus, each year a fraction of the Elites become workers or Middle-class, a fraction of the Middle class become Elites and others become Commoners, and a small fraction of the Commoners 'make it' and become Elites, and others become Middle-class. This extension allows the model to represent better the socioeconomic behavior in the West, where the Middle class is substantial. In addition, as was done above, we use different weights of the resources depletion rate δ_y and the wealth increase factor δ_w, whereas in Chapter 3, we used $\delta_y = \delta_w = \delta$.

We note that in Chapter we describe another version of the model, the HANDY-MC-II model, which includes the Middle class and where the renewable and nonrenewable resources are separated.

The computer simulations of various scenarios involving the HANDY-MC-I model can be found in the following chapter.

7.1 The HANDY-MC-I Model

We use the same symbols as above, and add $x_{mc}(t)$ as the number of individuals in the middle class at time t, α_{mc} the death rate and C_{mc} the consumption rate. The indices m or mc in a quantity refer to its value for the Middle class. The extended model is the following.

The HANDY model with Middle class is as follows.

Problem 7.1. *HANDY-MC-I. Find five functions $x_c(t), x_{mc}(t), x_e(t), y(t), w(t)$, defined on the time interval $[0,T]$, such that*

$$\begin{cases} x_c' = (\beta_c - \alpha_c)x_c + \gamma_{mc}^c x_{mc} + \gamma_e^c x_e - (\gamma_c^e + \gamma_c^{mc})x_c, \\ x_{mc}' = (\beta_{mc} - \alpha_{mc})x_{mc} + \gamma_c^{mc} x_c + \gamma_e^{mc} x_e - (\gamma_{mc}^c + \gamma_{mc}^e)x_{mc}, \\ x_e' = (\beta_e - \alpha_e)x_e + \gamma_c^e x_c + \gamma_{mc}^e x_{mc} - (\gamma_e^c + \gamma_e^{mc})x_e, \\ y' = \gamma y(\lambda - y) - \delta_y(x_c + x_{mc})y, \\ w' = \delta_w(x_c + x_{mc})y - C_c - C_e - C_{mc}. \end{cases} \tag{7.1}$$

Together with the initial conditions

$$x_c(0) = x_{c0},\ x_{mc}(0) = x_{mc0},\ x_e(0) = x_{e0},\ y(0) = y_0,\ w(0) = w_0. \tag{7.2}$$

The coefficient functions C_c, C_{mc}, C_e are given in (7.3)*, $\alpha_c, \alpha_{mc}, \alpha_e$ are given in* (7.5)*, and w_{th} is given in* (7.4)*.*

Here, $x_{c0}, x_{mc0}, x_{e0}, y_0, w_0$ are given nonnegative numbers.

The various coefficient functions are as follows.

$$C_c = \min(1, \frac{w}{w_{th}})sx_c, \quad C_{mc} = \min(1, \frac{w}{w_{th}})\kappa_m s x_{mc}, \quad C_e = \min(1, \frac{w}{w_{th}})\kappa_e s x_e. \tag{7.3}$$

The wealth threshold is given by

$$w_{th} = \rho(x_c + \kappa_0 x_e + \kappa_1 x_{mc} + \varepsilon), \tag{7.4}$$

where, as above $\varepsilon > 0$ is a small number that guarantees that w_{th} doesn't vanish. Also, we let $\eta = w/w_{th}$ and $\Delta\alpha = \alpha_M^* - \alpha_m^*$, then,

$$\begin{cases} \alpha_c = \alpha_m^* + \max(0, 1-\eta)\Delta\alpha \\ \alpha_{mc} = \alpha_m^* + \kappa_m \max(0, 1-\eta)\Delta\alpha \\ \alpha_e = \alpha_m^* + \kappa_e \max(0, 1-\eta)\Delta\alpha. \end{cases} \tag{7.5}$$

Here, we modified slightly the definitions, as compared to Chapter 5, and introduced the notation α_M^*, α_m^* so that it is easier to differentiate them from α_{mc} denoting the Middle-class death rate.

7.2 HANDY-MC-I model analysis

Let $\boldsymbol{z} = (x_c, x_e, x_{mc}, y, w)$ and $\boldsymbol{F} : \mathbb{R}^5 \to \mathbb{R}^5$ be given by

$$\boldsymbol{F}(\boldsymbol{z}) = \begin{pmatrix} (\beta_c - \alpha_c)x_c + \gamma_{mc}^c x_{mc} + \gamma_e^c x_e - (\gamma_c^e + \gamma_c^{mc})x_c, \\ (\beta_{mc} - \alpha_{mc})x_{mc} + \gamma_c^{mc} x_c + \gamma_e^{mc} x_e - (\gamma_{mc}^c + \gamma_{mc}^e)x_{mc}, \\ (\beta_e - \alpha_e)x_e + \gamma_c^e x_c + \gamma_{mc}^e x_{mc} - (\gamma_e^c + \gamma_e^{mc})x_e, \\ \gamma y(\lambda - y) - \delta_y(x_c + x_{mc})y, \\ \delta_w(x_c + x_{mc})y - C_c - C_e - C_{mc}. \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix}. \tag{7.6}$$

Here, C_c, C_{mc} and C_e are given in (7.3), w_{th} is given in (7.4), and α_c, α_{mc} and α_e are given in (7.5).

We write the system in the form

$$\boldsymbol{z}' = \boldsymbol{F}(\boldsymbol{z}), \qquad \boldsymbol{z}(0) = \boldsymbol{z}_0. \tag{7.7}$$

Here, $\boldsymbol{z}_0 = (x_{c0}, x_{mc0}, x_{e0}, y_0, w_0)$.

Again, the prime indicates the time derivative, and each component of $\boldsymbol{z}_0$ is assumed to be positive. Below, we show that the results hold true for $x_{mc0} = x_{e0} = 0$ and $w_0 = 0$, too.

We assume, following [14] and Chapter 5, that

$$0 < \alpha_m^* \leq \beta_e \leq \beta_{mc} \leq \beta_c \leq \alpha_M^* < 1,$$

thus,

$$\frac{\alpha_M^* - \alpha_m^*}{\beta_e - \alpha_m^*} \geq \frac{\alpha_M^* - \alpha_m^*}{\beta_{mc} - \alpha_m^*} \geq \frac{\alpha_M^* - \alpha_m^*}{\beta_c - \alpha_m^*} \geq 1. \tag{7.8}$$

Everywhere below, we assume that $\kappa_m, \kappa_e \geq 1$.

Let $0 < T < \infty$, then $[0, T]$ is an arbitrary finite interval. As in the previous models' analysis, starting with positive initial conditions, we show that all the possible solutions are nonnegative and bounded. Leading to estimates that are valid for $[0, T]$ and allow us to prove that a solution exists on this interval.

Proposition 7.2. *Assume that $x_{c0}, x_{mc0}, x_{e0}, y_0, w_0$ are all positive. If (x_c, x_{mc}, x_e, y, w) is any solution to the system (7.1), then all the components are nonnegative. Moreover, the following estimates hold for $0 \leq t \leq T$,*

$$\begin{aligned} 0 \leq x_c(t) &\leq (x_{c0} + x_{mc0} + x_{e0}) e^{\beta_c t}, \\ 0 \leq x_{mc}(t) &\leq (x_{c0} + x_{mc0} + x_{e0}) e^{\beta_c t}, \\ 0 \leq x_e(t) &\leq (x_{c0} + x_{mc0} + x_{e0}) e^{\beta_c t}, \\ 0 < y(t) &\leq y_0 e^{\gamma \lambda t}, \\ 0 \leq w(t) &\leq (w_0 + M) + M e^{(\gamma\lambda + \beta_c)t}. \end{aligned} \tag{7.9}$$

As in the analysis in Chapter 5, M is a positive constant that depends only on the problem parameters.

Proof. It follows from the first three equations, by multiplying the first with $x_e x_{mc}$, the second with $x_c x_e$ and the third by $x_c x_{mc}$, that,

$$\begin{aligned}
x_c' x_{mc} x_e &= (\beta_c - \alpha_c)\, x_c x_{mc} x_e + \gamma_{mc}^c x_{mc}^2 x_e + \gamma_e^c x_e^2 x_{mc} - (\gamma_c^e + \gamma_c^{mc}) x_c x_{mc} x_e, \\
x_{mc}' x_c x_e &= (\beta_{mc} - \alpha_{mc}) x_{mc} x_c x_e + \gamma_c^{mc} x_c^2 x_e + \gamma_e^{mc} x_e^2 x_c - (\gamma_{mc}^c + \gamma_{mc}^e) x_{mc} x_c x_e, \\
x_e' x_c x_{mc} &= (\beta_e - \alpha_e) x_e x_c x_{mc} + \gamma_c^e x_c^2 x_{mc} + \gamma_{mc}^e x_{mc}^2 x_c - (\gamma_e^c + \gamma_e^{mc}) x_{mc} x_e x_c.
\end{aligned}$$

Adding the three expressions yields

$$(x_e x_c x_{mc})' = ((\beta_c - \alpha_c) + (\beta_{mc} - \alpha_{mc}) + (\beta_e - \alpha_e))\, x_c x_{mc} x_e$$

$$+x_{mc}^2 \left(\gamma_{mc}^c x_e + \gamma_{mc}^e x_c\right) + x_c^2 \left(\gamma_c^e x_{mc} + \gamma_c^{mc} x_e\right) + x_e^2 \left(\gamma_e^c x_{mc} + \gamma_e^{mc} x_c\right)$$

$$-x_c x_{mc} x_e \left(\gamma_e^c + \gamma_{mc}^c + \gamma_c^{mc} + \gamma_e^{mc} + \gamma_c^e + \gamma_{mc}^e\right).$$

Since initially $x_e(0)x_c(0)x_{mc}(0) > 0$, by continuity, there is $t_0 > 0$, which is the first of this kind, such that $x_e(t)x_c(t)x_{mc}(t) > 0$ for $0 \le t < t_0$.
If any one of $x_c(t_0), x_e(t_0), x_{mc}(t_0)$ is positive, say $x_c(t_0) > 0$ and the other two vanish at t_0, then we multiply the equation for x_{mc} with x_c, thus, at t_0, $x_c x_{mc}'(t_0) = \gamma_c^{mc} x_c^2 > 0$, and since $x_c(t_0) > 0$, we obtain that

$$x_{mc}'(t_0) = \gamma_c^{mc} x_c > 0,$$

therefore, $x_{mc}(t_0) > 0$, a contradiction. The argument with the other two variables is similar. If two of the variables vanish at t_0, say $x_c(t_0) > 0$ and $x_{mc}(t_0) > 0$ and

$x_e(t_0) = 0$, then we have, using the equation for $x_c x_{mc} x'_e$ above,

$$x_{mc} x_c x'_e(t_0) = \gamma_c^e x_c^2 x_{mc} + \gamma_{mc}^e x_{mc}^2 x_c > 0.$$

Therefore, $x'_e(t_0) > 0$, which implies that $x_e(t_0) > 0$. We conclude that either $x_e(t_0) = x_c(t_0) = x_{mc}(t_0) = 0$, or there is no such t_0. In the latter case, the three variables are always positive. In the first case, we consider the initial value problem for $t_0 \le t \le T$, with the initial conditions $x_e(t_0) = 0$, $x_c(t_0) = 0, x_{mc}(t_0) = 0$, and find that the unique solution is $x_e(t) = 0, x_c(t) = 0, x_{mc}(t) = 0$ for $t_0 \le t$. Therefore, we established that $x_e(t) \ge 0, x_c(t) \ge 0, x_{mc}(t) \ge 0$ for $0 \le t \le T$, so the three functions are nonnegative, as long as they exist.

Next, we obtain upper bounds on x_c, x_{mc} and x_e. We add the first three equations and obtain

$$(x_c + x_{mc} + x_e)' = (\beta_c - \alpha_c)\, x_c + (\beta_e - \alpha_{mc})\, x_{mc} + (\beta_e - \alpha_e)\, x_e \le \beta_c (x_c + x_{mc} + x_e).$$

Therefore, by Gronwall's Lemma 2.5, we find

$$x_c(t) + x_{mc}(t) + x_e(t) \le (x_{c0} + x_{mc0} + x_{e0}) e^{\beta_c t}.$$

Since x_c, x_{mc} and x_e are nonnegative, each one is bounded and thus, we established the first two estimates in (7.9).

Next, to show that y is actually positive, we argue as follows. The equation is

$$y' = \gamma y(\lambda - y) - \delta_y (x_c + x_{mc}) y.$$

Since $x_c \geq 0$, $\gamma > 0$ and $\delta_y > 0$, it follows that

$$y' \geq -(\gamma y + \delta_y(x_c + x_{mc}))y.$$

Then, since $y' \leq \gamma\lambda y$, we obtain

$$y(t) \leq y_0 e^{\gamma\lambda t}.$$

Using the first in (7.9) and this estimate yields

$$y' \geq -(\gamma y_0 e^{\gamma\lambda t} + \delta_y(x_{c0} + x_{mc0}))y.$$

Thus,

$$y(t) \geq y_0 \exp\left(-(\gamma y_0 e^{\gamma\lambda t} + \delta_y(x_{co} + x_{mc0}))\right) > 0.$$

This proves both sides of the estimate for y in (7.9).

Finally, we address the last inequality for w. We have,

$$w' = \delta_w(x_c + x_{mc})y - \left(\min\left(1, \frac{w}{w_{th}}\right) s x_c + \min\left(1, \frac{w}{w_{th}}\right) \kappa_m s x_{mc}\right.$$

$$\left. + \min\left(1, \frac{w}{w_{th}}\right) \kappa_e s x_e\right)$$

Since $w_0 > 0$, assume that there is $0 < t_0$ such that $w(t) > 0$ for $0 \leq t < t_0$, and if $w\left(t_0\right) = 0$ then $w'\left(t_0\right) = \delta_w(x_c + x_{mc})y \geq 0$. But $y > 0$ so if $x_c(t_0) = x_{mc}(t_0) = 0$, then, as discussed above, all x_c,x_{mc} and x_e vanish identically for $t_0 \leq t$, which shows that $w(t) = 0$ for $t_0 \leq t$, as well. Moreover, it follows from the equation that

$$w' \leq \delta_w(x_c + x_{mc})y,$$

and using the estimates for y and x_c obtained previously, we obtain

$$w' \leq \delta_w y_0 (x_{c0} + x_{mc0}) e^{(\gamma\lambda + \beta_c)t}.$$

Integration over $[0, t]$ yields

$$w(t) \leq (w_0 + M) + M e^{(\gamma\lambda + \beta_c)t},$$

where

$$M = \frac{\delta_w y_0 (x_{c0} + x_{mc0})}{(\gamma\lambda + \beta_c)}.$$

We conclude that w is nonnegative and bounded. This completes the proof of the Proposition. ■

The estimates established in Proposition 7.6 and provided in (7.9) allow us to establish the existence of solutions to the system by using Theorem 2.4.
We summarize the results for the existence of the solutions to *HANDY-MC-I Model.*

Theorem 7.3. *Assume that x_{c0} and y_0 are positive and x_{cm0}, x_{e0} and w_0 are nonnegative. Then, Model 7.1 has nonnegative and bounded solutions on every finite time interval* $[0, T]$. *Moreover, the solutions satisfy the estimates* (7.9).

We note again that the theorem doesn't guarantee the uniqueness of the solutions.

7.3 Steady states

We let $\overline{\boldsymbol{z}} = (\overline{x}_c, \overline{x}_{mc}, \overline{x}_e, \overline{y}, \overline{w})$ denote a steady state of the system (7.1). It has the following two 'simple' steady states,

$$\overline{\boldsymbol{z}}_0 = (0, 0, 0, 0, 0), \qquad \overline{\boldsymbol{z}}_\lambda = (0, 0, 0, \lambda, 0),$$

and there are nonzero states when $x_c > 0$ and $y > 0$.

We just set up the case $0 \leq \eta < 1$. We let $\Delta\alpha = \alpha_M^* - \alpha_m^*$, and use the notation,

$$B_c = \beta_c - \alpha_m^* - \gamma_c^e - \gamma_c^{mc},\ B_{mc} = \beta_{mc} - \alpha_m^* - \gamma_{mc}^c - \gamma_{mc}^e,\ B_e = \beta_e - \alpha_m^* - \gamma_e^c - \gamma_e^{mc}. \quad (7.10)$$

Then, omitting the bars over the variables, the steady states are the solutions of the following system:

$$\begin{aligned}
0 &= (B_c - (1-\eta)\Delta\alpha)x_c + \gamma_{mc}^c x_{mc} + \gamma_e^c x_e, \\
0 &= \gamma_c^{mc} x_c + (B_{mc} - \kappa_m(1-\eta)\Delta\alpha)x_{mc} + \gamma_e^{mc} x_e, \\
0 &= \gamma_c^e x_c + \gamma_{mc}^e x_{mc} + (B_e - \kappa_e(1-\eta)\Delta\alpha)x_e, \\
0 &= \gamma y(\lambda - y) - \delta_y(x_c + x_{mc})y, \\
0 &= \delta_w(x_c + x_{mc})y - \eta s(x_c + \kappa_m x_{mc} + \kappa_e x_e).
\end{aligned} \quad (7.11)$$

While the general case of $0 \leq \eta < 1$ is quite complicated, and is left for future research, we consider below the case when η vanishes, as it is related to the 'simple' steady states. But first, we present shortly the case when $\eta \geq 1$, which turns out to have no point steady states. Then, for $\eta \geq 1$ the steady states, are the solutions of the system,

$$\begin{aligned}
0 &= B_c x_c + \gamma_{mc}^c x_{mc} + \gamma_e^c x_e, \\
0 &= \gamma_c^{mc} x_c + B_{cm} x_{mc} + \gamma_e^{mc} x_e, \\
0 &= \gamma_c^e x_c + \gamma_{mc}^e x_{mc} + B_e x_e, \\
0 &= \gamma y(\lambda - y) - \delta_y(x_c + x_{mc})y, \\
0 &= \delta_w(x_c + x_{mc})y - s(x_c + \kappa_m x_{cm} + \kappa_e x_e).
\end{aligned} \quad (7.12)$$

The first three equations are not coupled with the last two, and to proceed, we let D be the determinant

$$D = det \begin{pmatrix} B_c & \gamma_{mc}^c & \gamma_e^c \\ \gamma_c^{mc} & B_{cm} & \gamma_e^{mc} \\ \gamma_c^e & \gamma_{mc}^e & B_e \end{pmatrix}.$$

It follows that when $D \neq 0$, the unique solution is

$$x_c = x_{mc} = x_e = 0.$$

And then, the full solutions are the two 'simple solutions,'

$$\boldsymbol{z}_0 = (0, 0, 0, 0, 0), \qquad \boldsymbol{z}_0 = (0, 0, 0, \lambda, 0).$$

But this means that when we use the version of w_{th} with ϵ, then $\eta = 0$. We conclude that when $\eta \geq 1$ and the system is non-degenerate $(D \neq 0)$ there are no steady state solutions.

The case $\eta \geq 1$ and the system is degenerate, $D = 0$, seems to be exceptional and it is not clear that its analysis is warranted at this stage.

Therefore, the meaningful steady states happen when $\eta < 1$. However, this doesn't exclude oscillatory solutions, limit cycles and chaotic solutions when $\eta \geq 1$, which show up in simulations. These will be investigated in the future.

7.4 Stability of the origin

Here, we only study the stability of the two 'simple' steady states

$$\overline{\boldsymbol{z}}_0 = (0, 0, 0, 0, 0), \qquad \overline{\boldsymbol{z}}_\lambda = (0, 0, 0, \lambda, 0).$$

These are related to $\eta < 1$. The stability of the other steady states will be considered elsewhere, especially in view of the note in the previous section. To that end, we compute the Jacobian matrix J of $\boldsymbol{F}$, proceeding component by component.

$$J_{11} = \frac{\partial F_1}{\partial x_c} = (\beta_c - \alpha_c - \gamma_c^{mc} - \gamma_c^e) - x_c \frac{\partial \alpha_c}{\partial x_c};$$

$$J_{12} = \frac{\partial F_1}{\partial x_{mc}} = \gamma_{mc}^c - x_c \frac{\partial \alpha_c}{\partial x_{mc}}; \quad J_{13} = \frac{\partial F_1}{\partial x_e} = \gamma_e^c - x_c \frac{\partial \alpha_c}{\partial x_e};$$

$$J_{14} = \frac{\partial F_1}{\partial y} = 0; \quad J_{15} = \frac{\partial F_1}{\partial w} = -x_c \frac{\partial \alpha_c}{\partial w}.$$

Next,

$$J_{21} = \frac{\partial F_2}{\partial x_c} = \gamma_c^{mc} - x_{mc} \frac{\partial \alpha_{mc}}{\partial x_c}$$

$$J_{22} = \frac{\partial F_2}{\partial x_{mc}} = (\beta_{mc} - \alpha_{mc} - \gamma_{mc}^c - \gamma_{mc}^e) - x_{mc} \frac{\partial \alpha_{mc}}{\partial x_{mc}};$$

$$J_{23} = \frac{\partial F_2}{\partial x_e} = \gamma_e^{mc} - x_{mc} \frac{\partial \alpha_{mc}}{\partial x_e};$$

$$J_{24} = \frac{\partial F_2}{\partial y} = 0; \qquad J_{25} = \frac{\partial F_2}{\partial w} = -x_{mc} \frac{\partial \alpha_{mc}}{\partial w}.$$

$$J_{31} = \frac{\partial F_3}{\partial x_c} = \gamma_c^e - x_e \frac{\partial \alpha_e}{\partial x_c}; \quad J_{32} = \frac{\partial F_3}{\partial x_{mc}} = \gamma_{mc}^e - x_e \frac{\partial \alpha_e}{\partial x_{mc}};$$

$$J_{33} = \frac{\partial F_3}{\partial x_e} = (\beta_e - \alpha_e - \gamma_e^{mc} - \gamma_e^c) - x_e \frac{\partial \alpha_e}{\partial x_e};$$

$$J_{34} = 0; \qquad J_{35} = \frac{\partial F_3}{\partial w} = -x_e \frac{\partial \alpha_e}{\partial w}.$$

Furthermore,

$$J_{41} = \frac{\partial F_4}{\partial x_c} = -\delta_y y; \quad J_{42} = \frac{\partial F_4}{\partial x_{mc}} = -\delta_y y; \quad J_{43} = \frac{\partial F_4}{\partial x_e} = 0;$$

$$J_{44} = \frac{\partial F_4}{\partial y} = \gamma\lambda - 2\gamma y - \delta_y x_c - \delta_y x_{mc}; \qquad J_{45} = \frac{\partial F_4}{\partial w} = 0.$$

Finally,

$$J_{51} = \frac{\partial F_5}{\partial x_c} = \delta_w y - \frac{\partial C_c}{\partial x_c} - \frac{\partial C_{mc}}{\partial x_c} - \frac{\partial C_e}{\partial x_c};$$

$$J_{52} = \frac{\partial F_5}{\partial x_{mc}} = \delta_w y - \frac{\partial C_c}{\partial x_{mc}} - \frac{\partial C_{mc}}{\partial x_{mc}} - \frac{\partial C_e}{\partial x_{mc}};$$

$$J_{53} = \frac{\partial F_5}{\partial x_e} = -\frac{\partial C_c}{\partial x_e} - \frac{\partial C_{mc}}{\partial x_e} - \frac{\partial C_e}{\partial x_e};$$

$$J_{54} = \frac{\partial F_5}{\partial y} = \delta_w(x_c + x_{mc}); \qquad J_{55} = \frac{\partial F_5}{\partial w} = -\frac{\partial C_c}{\partial w} - \frac{\partial C_{mc}}{\partial w} - \frac{\partial C_e}{\partial w}.$$

We turn to compute the partial derivatives of $\alpha_c, \alpha_{mc}, \alpha_e, C_c, C_{mc}, C_e$. As above, we let $\eta = w/\rho(x_c + \kappa_1 x_{mc} + \kappa_0 x_e)$ $(\epsilon = 0)$ and $\Delta\alpha = \alpha_M^* - \alpha_m^*$. However, in view of Section 7.2, we deal here only with the case $\eta < 1$. We find,

$$C_c = \eta s x_c, \qquad C_{mc} = \eta \kappa_m s x_{cm}, \qquad C_e = \eta \kappa_e s x_e,$$

$$\alpha_c = \alpha_m^* + (1-\eta)\Delta\alpha, \quad \alpha_{mc} = \alpha_m^* + \kappa_m(1-\eta)\Delta\alpha, \quad \alpha_e = \alpha_m^* + \kappa_e(1-\eta)\Delta\alpha.$$

To simplify the notation we let $\psi_0 = x_e/x_c$ and $\psi_1 = x_{mc}/x_c$. We proceed to compute the partial derivatives of the coefficient functions αs and Cs. First, we compute

$$\frac{\partial \eta}{\partial x_c} = -\frac{\eta}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c}; \qquad \frac{\partial \eta}{\partial x_{mc}} = -\frac{\eta\kappa_1}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c};$$

$$\frac{\partial \eta}{\partial x_e} = -\frac{\eta\kappa_0}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c}; \qquad \frac{\partial \eta}{\partial w} = \frac{1}{\rho(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c}.$$

Then, we find,

$$\frac{\partial \alpha_c}{\partial x_c} = -\Delta\alpha\frac{\partial \eta}{\partial x_c} = \frac{\eta\Delta\alpha}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c};$$

$$\frac{\partial \alpha_c}{\partial x_{mc}} = \frac{\eta\kappa_1\Delta\alpha}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c}; \qquad \frac{\partial \alpha_c}{\partial x_e} = \frac{\eta\kappa_0\Delta\alpha}{(1 + \kappa_0\psi_0 + \kappa_1\psi_1)x_c}.$$

Next,

$$\frac{\partial \alpha_{mc}}{\partial x_c} = -\kappa_m \Delta\alpha \frac{\partial \eta}{\partial x_c} = \frac{\eta \kappa_m \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c};$$

$$\frac{\partial \alpha_{mc}}{\partial x_{mc}} = \frac{\eta \kappa_1 \kappa_m \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c}; \qquad \frac{\partial \alpha_{mc}}{\partial x_e} = \frac{\eta \kappa_0 \kappa_m \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c}.$$

Furthermore,

$$\frac{\partial \alpha_e}{\partial x_c} = -\kappa_e \Delta\alpha \frac{\partial \eta}{\partial x_c} = \frac{\eta \kappa_e \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c};$$

$$\frac{\partial \alpha_e}{\partial x_{mc}} = \frac{\eta \kappa_e \kappa_1 \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c}; \qquad \frac{\partial \alpha_e}{\partial x_e} = \frac{\eta \kappa_0 \kappa_e \Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c}.$$

Next,

$$\frac{\partial \alpha_c}{\partial w} = -\frac{\Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c}; \quad \frac{\partial \alpha_{mc}}{\partial w} = -\frac{\kappa_m \Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c};$$

$$\frac{\partial \alpha_e}{\partial w} = -\frac{\kappa_e \Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)x_c};$$

$$\frac{\partial C_c}{\partial x_c} = \eta s + s x_c \frac{\partial \eta}{\partial x_c} = \frac{\eta s(\kappa_0\psi_0+\kappa_1\psi_1)}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_c}{\partial x_{mc}} = s x_c \frac{\partial \eta}{\partial x_{mc}} = -\frac{\eta s \kappa_1}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_c}{\partial x_e} = s x_c \frac{\partial \eta}{\partial x_e} = -\frac{\eta s \kappa_0}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_c}{\partial w} = \frac{s}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad \frac{\partial C_{mc}}{\partial x_c} = -\frac{\eta s \kappa_m \psi_1}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_{mc}}{\partial x_{mc}} = \frac{\eta s \kappa_m (1+\kappa_0\psi_0)}{(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad \frac{\partial C_{mc}}{\partial x_e} = -\frac{\eta s \kappa_0 \kappa_m \psi_1}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_{mc}}{\partial w} = \frac{s \kappa_m \psi_1}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad \frac{\partial C_e}{\partial x_c} = -\frac{\eta s \kappa_e \psi_0}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_e}{\partial x_{mc}} = -\frac{\eta s \kappa_e \kappa_1 \psi_0}{(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad \frac{\partial C_e}{\partial x_e} = \frac{\eta s \kappa_e (1+\kappa_1\psi_1)}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$\frac{\partial C_e}{\partial w} = \frac{s\kappa_e\psi_0}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}.$$

These results yield, when $0 \leq \eta < 1$, and the B's are given in (7.10), the following components of J:

$$J_{11} = (B_c - (1-\eta)\Delta\alpha) - \frac{\eta\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{12} = \gamma^c_{mc} - \frac{\eta\kappa_1\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad J_{13} = \gamma^c_e - \frac{\eta\kappa_0\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{14} = 0; \quad J_{15} = \frac{\Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}.$$

Next,

$$J_{21} = \gamma^{mc}_c - \frac{\eta\kappa_m\psi_1\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{22} = (B_{mc} - (1-\eta)\kappa_m\Delta\alpha) - \frac{\eta\kappa_1\kappa_m\psi_1\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{23} = \gamma^{mc}_e - \frac{\eta\kappa_0\kappa_m\psi_1\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_0)}; \qquad J_{24} = 0;$$

$$J_{25} = \frac{\kappa_m\psi_1\Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}.$$

Furthermore,

$$J_{31} = \gamma^e_c - \frac{\eta\kappa_e\psi_0\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \quad J_{32} = \gamma^e_{mc} - \frac{\eta\kappa_e\kappa_1\psi_0\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{33} = (B_e - (1-\eta)\kappa_e\Delta\alpha) - \frac{\eta\kappa_0\kappa_e\psi_1\Delta\alpha}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{34} = 0; \qquad J_{35} = \frac{\kappa_e\psi_0\Delta\alpha}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

Then,

$$J_{41} = -\delta_y\overline{y}; \qquad J_{42} = -\delta_y\overline{y}; \qquad J_{43} = 0; \qquad J_{45} = 0.$$

$$J_{44} = \gamma\lambda - 2\gamma\overline{y} - \delta_y(1+\psi_1)\overline{x}_c.$$

Finally,

$$J_{51} = \delta_w\overline{y} + \eta s\frac{(\kappa_e - \kappa_0)\psi_0 + (\kappa_m - \kappa_1)\psi_1}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{52} = \delta_w\overline{y} + \eta s\frac{(\kappa_1 - \kappa_m) + (\kappa_e\kappa_1 - \kappa_m\kappa_0)\psi_0}{(1+\kappa_0\psi_0+\kappa_1\psi_1)};$$

$$J_{53} = \eta s\frac{(\kappa_0 - \kappa_e) + (\kappa_m\kappa_0 - \kappa_e\kappa_1)\psi_1}{(1+\kappa_0\psi_0+\kappa_1\psi_1)}; \qquad J_{54} = \delta_w\overline{x}_c(1+\psi_1);$$

$$J_{55} = -\frac{s(1+\kappa_m\psi_1+\kappa_e\psi_0)}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)}.$$

These expressions allow us to construct the Jacobian matrix J when $0 \leq \eta < 1$. To simplify the notations of J, we let $\phi_0 = (1+\kappa_0\psi_0+\kappa_1\psi_1)^{-1}$ and $\phi_\rho = (\rho(1+\kappa_0\psi_0+\kappa_1\psi_1))^{-1}$. We obtain,

$$J_\eta(\bar{\boldsymbol{z}}) = \tag{7.13}$$

$$\begin{pmatrix} J_{11} & \gamma_{mc}^c - \eta\kappa_1\Delta\alpha\phi_0 & \gamma_e^c - \eta\kappa_0\Delta\alpha\phi_0 & 0 & \Delta\alpha\phi_\rho \\ \gamma_c^{mc} - \eta\kappa_m\psi_1\Delta\alpha\phi_0 & J_{22} & \gamma_e^{mc} - \eta\kappa_0\kappa_m\psi_1\Delta\alpha\phi_0 & 0 & \kappa_m\psi_1\Delta\alpha\phi_\rho \\ \gamma_c^e - \eta\kappa_e\psi_0\Delta\alpha\phi_0 & \gamma_{mc}^e - \eta\kappa_e\kappa_1\psi_0\Delta\alpha\phi_0 & J_{33} & 0 & \kappa_e\psi_0\Delta\alpha\phi_{rho} \\ -\delta_y\overline{y} & -\delta_y\overline{y} & 0 & J_{44} & 0 \\ J_{51} & J_{52} & J_{53} & \delta_w\overline{x}_c(1+\psi_1) & J_{55} \end{pmatrix}.$$

Where,

$$J_{11} = (B_c - (1-\eta)\Delta\alpha) - \eta\Delta\alpha\phi_0,$$

$$J_{22} = B_{mc} - (1-\eta)\kappa_m\Delta\alpha - -\eta\kappa_1\kappa_m\psi_1\Delta\alpha\phi_0,$$

$$J_{33} = (B_e - (1-\eta)\kappa_e\Delta\alpha) - \eta\kappa_0\kappa_e\psi_1\Delta\alpha,$$

and

$$J_{44} = \gamma\lambda - 2\gamma\overline{y} - \delta_y(1+\psi_1)\overline{x}_c.$$

Moreover, J_{51}, J_{52}, J_{53} and J_{55} can be found above.

Finally, to study the stability of the two 'simple' steady states,

$$\boldsymbol{z}_0 = (0,0,0,0,0), \qquad \boldsymbol{z}_\lambda = (0,0,0,\lambda,0).$$

we substitute these values in J, note that we may assume that $\eta = w/w_{th} = 0$, since $w = 0$, and we assumed above that there was $\epsilon > 0$ in w_{th}, and also, we may choose $0 \le \psi_0, \psi_1$, somewhat arbitrarily, as $x_c \to 0$. Thus,

$$J_\eta(\boldsymbol{z}_0) = \tag{7.14}$$

$$\begin{pmatrix} B_c - \Delta\alpha & \gamma^c_{mc} & \gamma^c_e & 0 & \Delta\alpha\phi_\rho \\ \gamma^{mc}_c & B_{mc} - \kappa_m\Delta\alpha & \gamma^{mc}_e & 0 & \kappa_m\psi_1\Delta\alpha\phi_\rho \\ \gamma^e_c & \gamma^e_{mc} & B_e - \kappa_e\Delta\alpha & 0 & \kappa_e\psi_0\Delta\alpha\phi_r ho \\ 0 & 0 & 0 & \gamma\lambda & 0 \\ 0 & 0 & 0 & 0 & -\frac{s(1+\kappa_m\psi_1+\kappa_e\psi_0)}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)} \end{pmatrix}.$$

It is seen that $\lambda_4 = \gamma\lambda > 0$, which implies that the steady state $\boldsymbol{z}_0 = (0,0,0,0,0)$ is *unstable*. Computing $J_\eta(\boldsymbol{z}_\lambda)$, noting that $\overline{y} = \lambda$, yields

$$J_\eta(\boldsymbol{z}_\lambda) = \tag{7.15}$$

$$\begin{pmatrix} B_c - \Delta\alpha & \gamma^c_{mc} & \gamma^c_e & 0 & \Delta\alpha\phi_\rho \\ \gamma^{mc}_c & B_{mc} - \kappa_m\Delta\alpha & \gamma^{mc}_e & 0 & \kappa_m\psi_1\Delta\alpha\phi_\rho \\ \gamma^e_c & \gamma^e_{mc} & B_e - \kappa_e\Delta\alpha & 0 & \kappa_e\psi_0\Delta\alpha\phi_r ho \\ 0 & 0 & 0 & -\gamma\lambda & 0 \\ 0 & 0 & 0 & 0 & -\frac{s(1+\kappa_m\psi_1+\kappa_e\psi_0)}{\rho(1+\kappa_0\psi_0+\kappa_1\psi_1)} \end{pmatrix}.$$

It is seen that here $\lambda_4 = -\gamma\lambda < 0$, which implies that the steady state $\boldsymbol{z}_\lambda$, depending on the other system coefficients, is likely to be stable or even stable and attracting (asymptotically stable).

CHAPTER EIGHT

HANDY-MC-I MODEL SIMULATIONS

This chapter describes our computer simulations of Model 7.1. The model consists of three populations, the natural resources and the total wealth. It has 20 coefficients and five initial conditions, making it rather complex. Here, both the workers and the middle class contribute to the wealth, while the rich just consume the wealth. It is found that there is a wide range of the solutions' behavior, from monotone approach to the steady states, to a few oscillations and then approach to the steady states, to almost periodic, to what seems to be a chaotic behavior. Section 8.1 presents the modified algorithm used in the simulations, which are depicted and commented upon in Section 8.2. In each one of the simulations, we also present the eigenvalues of J, (7.13), which indicate the stability of the relevant steady state.

8.1 Algorithm

The Algorithm for the model simulations is similar to those in Chapters and , with the additional equation for the middle class population, and the appropriate modifications in the coefficient functions of their consumption rate, and rate of death. The algorithm is as follows.

Step 1. Initiation

set the problem coefficients:

$$\alpha_m^*,\ \alpha_M^*,\ \beta_c,\ \beta_e,\ \beta_m,\ \delta_y,\ \delta_w,\ \kappa_0,\ \kappa_1,\ \lambda,\ \rho,\ s$$

$$\kappa_m,\ \kappa_e,\ \gamma_c^e,\ \gamma_e^c,\ \gamma_{mc}^e,\ \gamma_e^{mc},\ \gamma_c^{mc},\ \gamma_{mc}^c,$$

$$x_{c0},\ x_{e0},\ x_{mc0},\ y_0,\ w_0.$$

set T (the final year) and N (the number of time steps)

set $\Delta t = T/N$.

Step 2.

set $n = 0$

set $x_c^0 = x_{c0}, \quad x_e^0 = x_{e0}, \quad x_{mc}^0 = x_{mc0}, \quad y^0 = y_0, \quad w^0 = w_0,$

Step 3. Time marching

compute

C_c^n from (7.3)

C_e^n from (7.3)

C_{mc}^n from (7.3)

w_{th}^n from (7.4)

α_c^n from (7.5)

α_e^n from (7.5)

α_{mc}^n from (7.5)

set $n = n + 1$

compute

x_c^n from (7.1)

x_e^n from (7.1)

x_{mc}^n from (7.1)

y^n from (7.1)

w^n from (7.1)

Step 4. if $n < N$ go to Step 3

end

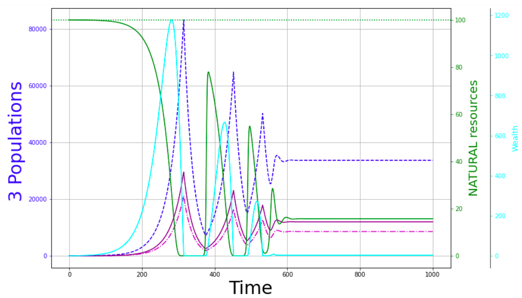
8.2 Simulations

Researchers in [5] and [9] suggest that the middle class in the West comprises about 50% of the population, whereas the Elites posses almost 90% of the society's wealth. It is claimed in [18] that the inequality and inequitable distribution of wealth between population classes has a considerable negative effect on the vulnerable population, essentially, the workers and the poor. Some of the consequences are the widening of the education gap, lowering the GDP growth, and leading to less productive jobs, such as temporary, on call, and part-time jobs. This parallels the claims in [1] that increasing the social mobility increases the range of income for those who are considered in the middle and furthers to those in the lower income stages. Indeed, the simulations highlight the differences of what large or small social mobility factors have on shaping the population classes. These observations are directly related to the possible flourishing, stagnation, or collapse of such societies. In all the simulations graphic displays, the green curve represents the natural resources, the cyan curve the wealth, the in blue, purple and magenta curves represent the Commoners, Middle class, and Elites, respectively.

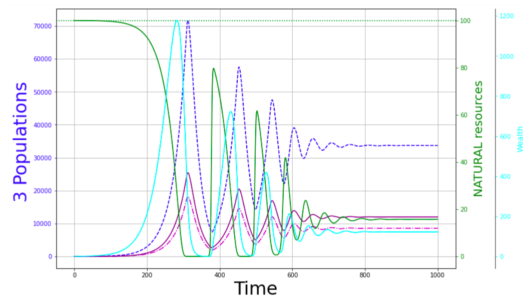
The simulations use two sets of parameter values shown in the table 8.3, starting with smaller mobility factors and unequal birth rates, while keeping the Middle class birth rate as the average of the other populations birth rates. This is depicted in the set of figures 8.1. When the three classes have equal consumption rates of $\kappa_m = \kappa_e = 1$ and the same level of contribution to the society's wealth threshold, the results are seen in Fig. 8.1(a). The oscillations decay and reach a steady state in less than 600 years, where the system's $\eta = 0.714$, with steady state numbers $(33,000; 12,000; 8,555; 15; 3)$, which is stable and attracting, and the eigenvalues are:

$$-5.016, \quad -0.102 \pm 0.216i, \quad -0.01, \quad -0.024.$$

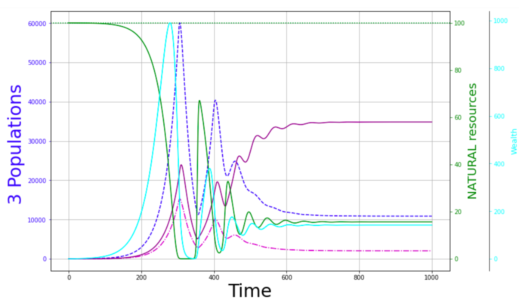
The real parts of the eigenvalues are negative, and the approach to the steady state involves oscillations since there is a pair of complex-conjugate numbers. The convergence to the steady state is clearly visible.



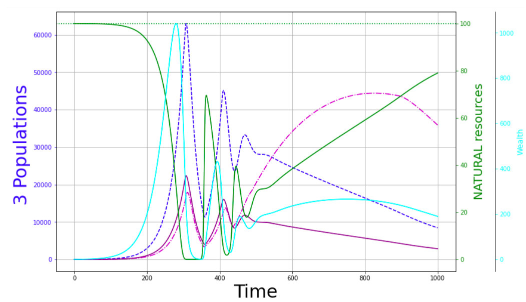
(a) $\kappa_m = \kappa_e = 1$



(b) $\kappa_0 = \kappa_1 = 2$ contributing to the increase of w_{th}.



(c) Middle class consumption is 2



(d) Elites double the consumption

Figure 8.1: HANDY-MC-I simulations with the parameters listed in table 8.3 and $\delta_y = 1.845 \cdot 10^{-5}$. The steady states are stable and attracting. Notice the recovery of the natural resources in (d), once the populations decline.

Increasing the wealth threshold due to the contribution from the Middle class and Elites, with a factor of 2, yields the steady state values $(33,000; 12,000;$ $8,555; 15; 4)$, changing only the wealth value, 8.1(b), with the eigenvalues:

$$-4.271, \quad -0.079 \pm 0.175i, \quad -0.043, \quad -0.03.$$

We note that a slight change in the wealth leads to a noticeable change in the eigenvalues. The convergence to the almost the same steady state is clearly seen, but with more damped oscillations.

Fig. 8.1(c) shows the case when the Middle class increases its consumption rate to 2, while all the classes contribute equally to the wealth threshold. The raise in consumption puts the scenario in case (ii). It reaches the steady state values of $(10,000; 34,000; 20; 15; 1.8)$. We note that the wealth decreases to less than 2. The system's eigenvalues are,

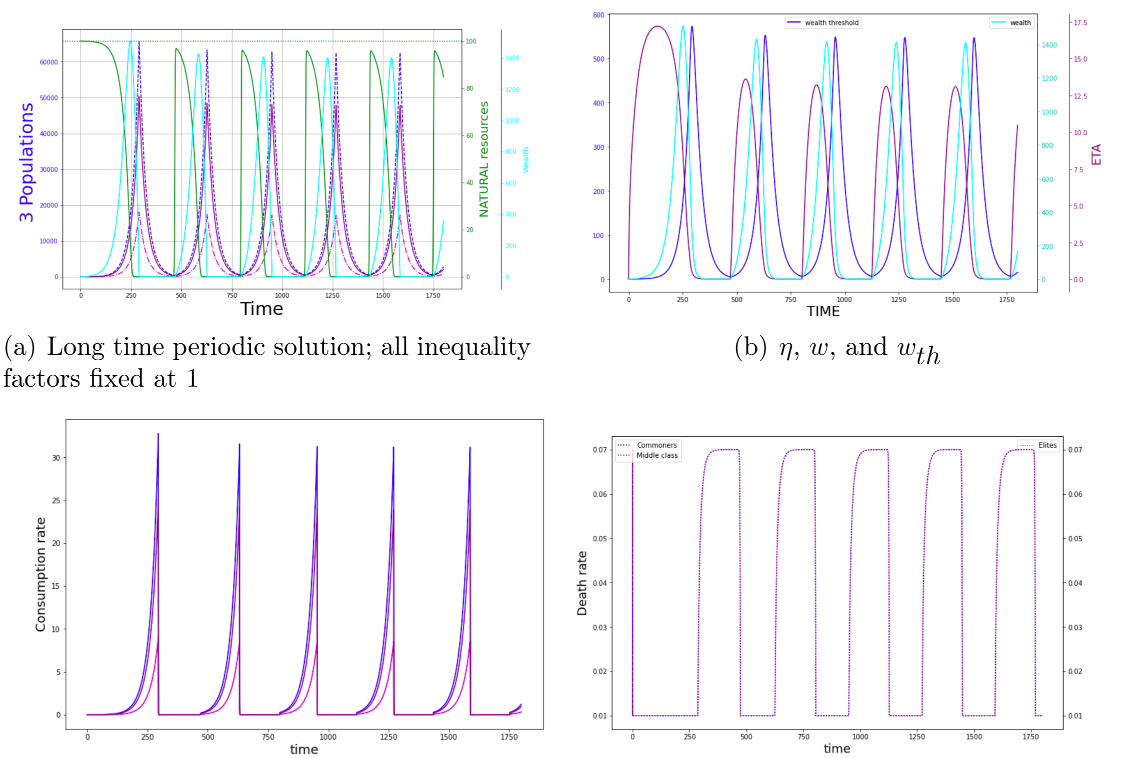
$$-5.428, \quad -0.11 \pm 0.224, \quad -0.024, \quad 0.$$

Therefore, the steady state is stable, moreover, the simulations depict it to be stable and attracting.

The last figure in the set, Fig. 8.1(d), shows the case when the Elites raise their consumption rate to 2. The steady state of the simulation is reached after almost double the time period shown in the figure with the values
$(790; 264; 4,164; 98; 0.255)$ and eigenvalues

$$-5.422, \quad -0.113 \pm 0.215i, \quad -0.008, \quad -0.006.$$

To obtain further insight into the model predictions, we used a larger depletion factor, $\delta_y = 3.168 \cdot 10^{-5}$. The resulting periodic behavior is shown in Fig. 8.2. Table 8.3 has the modified parameters. In Fig. 8.2(a) the periodic behavior over a longer time period can be seen, clearly. Here, $\eta = 0.66$, and so it is case (i). Fig. 8.2(b) shows how η (purple) changes during the time period, and so do the wealth and the wealth threshold. The consumption rates are shown in 8.2(c) or the choice of the



(a) Long time periodic solution; all inequality factors fixed at 1

(b) η, w, and w_{th}

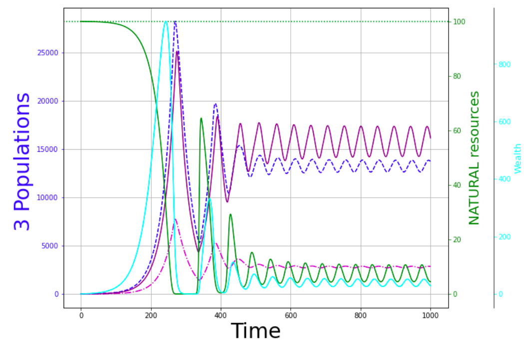
(c) Consumption rates of the populations

(d) α_c, α_{mc} and α_e

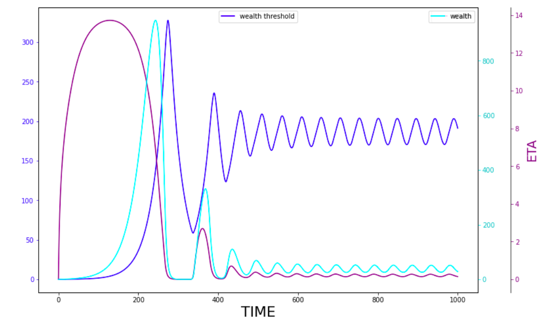
Figure 8.2: HANDY-MC-I model with periodic solutions. The populations, natural resources and wealth are depicted in (a), the wealth w, wealth threshold w_{th} and their ratio η in (b), the consumption rates C_c, C_{mc} and C_e in (c), and the death rates in (d). The parameters are those in table 8.3, the depletion factor is fixed $\delta_y = 3.168 \cdot 10^{-5}$. It seems that the system approaches a limit cycle.

maximum in the three classes death functions shown in Fig. 8.2(d). It seems that the system very quickly reaches a limit cycle.

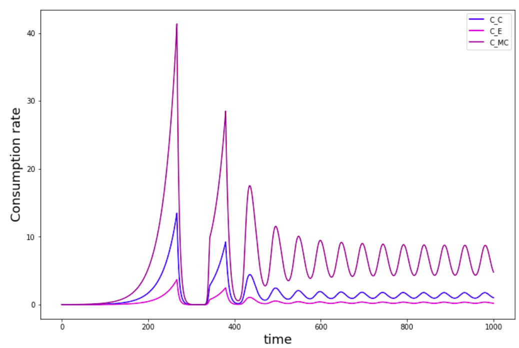
Another periodic scenario is shown in Fig. 8.3. There, all the mobility factors and δ_y were exactly the same as in Fig. 8.1. But, the wealth threshold w_{th} was increased so that both the Middle class and the Elites contributed 100 times more than the workers. Also, what changed is the Middle class increased their consumption rate fourfold. Meanwhile, the Commoners and Elites consumed the same rate of 1, putting the scenario in case (i) for all populations with $\eta = 0.01$. The system was



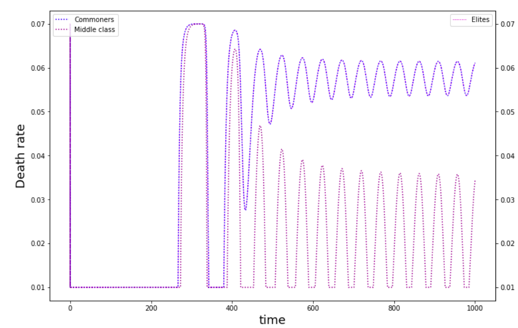
(a) Two populations with the same consumption rate, while the Middle class has a rate of $\kappa_m = 4$



(b) η, w_{th} and w



(c) Middle class consumption at $\kappa_m = 4$

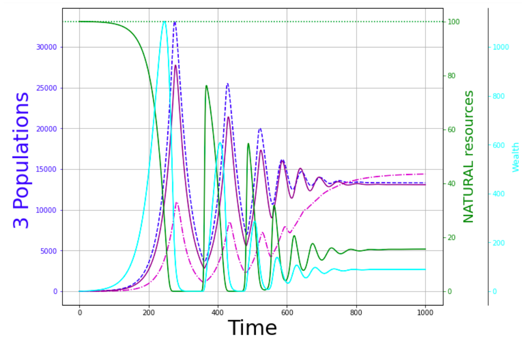


(d) Death rates comparison

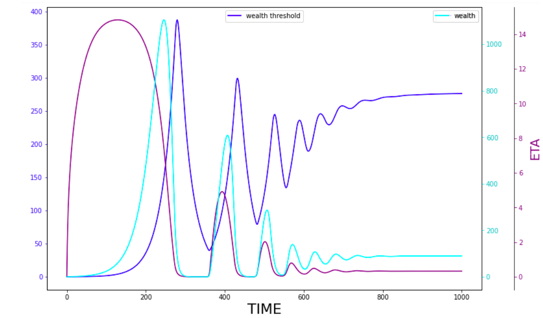
Figure 8.3: HANDY-MC-I simulations in which the Middle class consumption rate is $\kappa_m = 4$, and the Commoners' and Elites rates are equal to 1, i.e. $\kappa_e = 1$. The solutions seem, after a transient, to approach a limit cycle.

periodic and seemed to approach a limit cycle, too. Comparing the figures presented in the combined Fig. 8.3(b) for η, w_{th} and w, the three populations' consumption in Fig. 8.3(c), and deaths in Fig. 8.3(d). While both the death rates for the Commoners and Elites had the same higher values, the Middle class death rate was much lower.

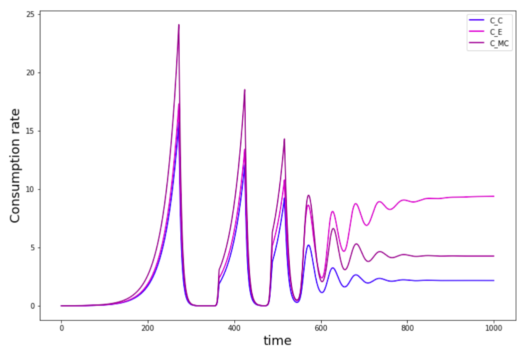
The simulations in Fig. 8.4 were obtained by varying the consumption gap, while δ_y, and $\kappa_0 = \kappa_1 = 100$ were kept fixed. In this set of figures the consumption rate of the Elites was increased to $\kappa_e = 4$ and the Middle class to $\kappa_m = 2$. The oscillation



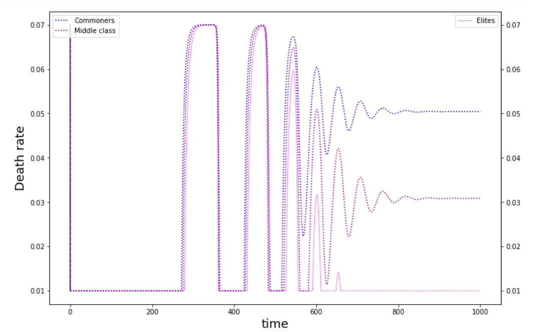
(a) Different consumption rates and the same δ_y fixed



(b) η,w, and w_{th} with different consumption rates.



(c) Consumption rates



(d) α_c, α_{mc} and α_e vs time .

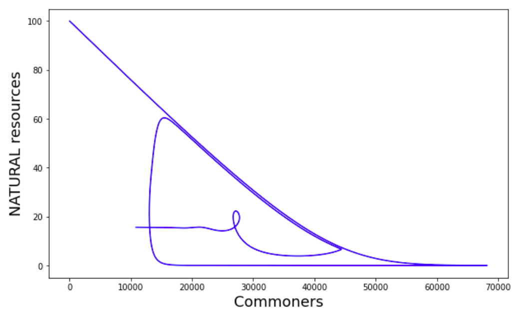
Figure 8.4: Damped oscillatory behavior, with a depletion factor fixed at $\delta_y = 3.168 \cdot 10^{-5}$, with the three populations having different consumption rates, and $\kappa_e = 4$ and $\kappa_m = 2$. The solutions converge to the steady states, which seems to stable and attracting.

reached the steady state in about $1,000$ years, with the values $(13,800; 13,100; 12,500; 14; 82)$. Since $\eta = 0.01$, it was case (i), with the eigenvalues,

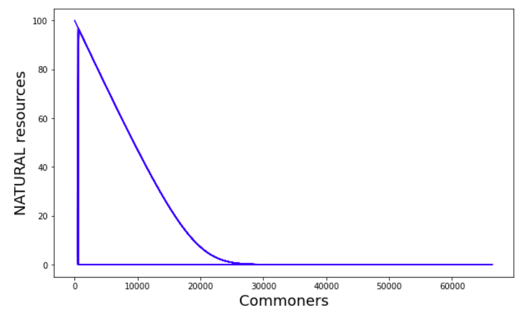
$$-0.379, \quad -0.0225 \pm 0.016i, \quad -0.087 \pm 0.013i.$$

Thus, the steady state is stable and attracting. Moreover, the populations' consumption and death functions have three different values at the steady state.

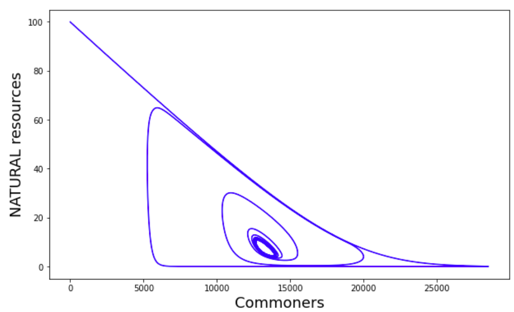
The death functions reaches the values (0.05, 0.03, 0.01), respectively, for (α_c, α_{mc}, α_e) depending on the related population's consumption rate.



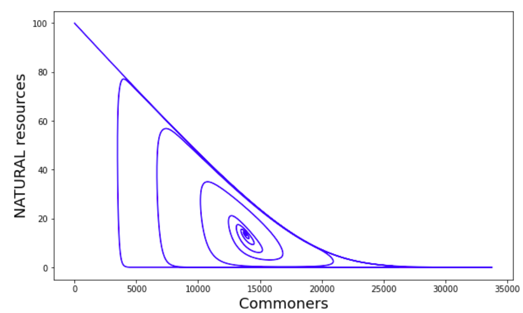
(a) Commoners vs. resources in Fig. 8.1



(b) Commoners vs. resources in Fig. 8.2



(c) Commoners vs. resources in Fig. 8.3



(d) Commoners vs. resources in Fig. 8.4

Figure 8.5: The graphs of Natural resources vs. Commoners in the four previous scenarios. The approach to steady states that are stable and attracting can be seen in (a) and (d). In (b) and (c) the solutions converge to limit cycles. The limit cycle in (c) is more regular, while the one in (b) has an unusual structure of almost a triangle.

Next, Fig. 8.5 depicts plots of the Natural resources vs. the Commoners, which is a different way to display the long-term behavior of the solutions of the four scenarios presented above. It is seen in Fig. 8.5(a) that the solutions in Fig. 8.1 converge to a stable and attracting steady state. On the other hand, it is seen in Fig. 8.5(b) that the system in Fig. 8.2 converges to a limit cycle, however the limit cycle has an unusual, almost triangular shape. Fig. 8.5(c) shows a more regular convergence of the solution in Fig. 8.3 to a limit cycle, which can be seen clearly. Finally, Fig. 8.5(d) depicts the convergence of the system in Fig. 8.4, where the solutions converge to the steady state that is stable and attracting.

Table 8.3: The values of the parameters and data used in the simulations in the HANDY-MC-I model

Symbols	Values	Meaning
Δt	ANY used =.01 and .001	time step
τ_F	1000 and 1800	time period
$x_c(0)$	10	Commoners initial condition
$x_{mc}(0)$	1	Middle class initial condition
$x_e(0)$	1	Elites initial condition
$y(0)$	100	Nature's initial condition
$w(0)$	0	wealth initial condition
λ	100	Natures carrying capacity
γ	0.01	Natures regeneration factor
δ_y	$1.845 \cdot 10^{-5}$ and $3.168 \cdot 10^{-5}$	Natures depletion factor
α_M	0.08 and 0.07	Maximum expected death rate
α_m	0.01	Normal death rate
β_c	0.05 and 0.03	Commoners birth rate
β_e	0.01 and 0.03	Elites birth rate
β_{mc}	$(\beta_c + \beta_e)/2$	Middle class birth rate
s	$5 \cdot 10^{-4}$	salary per capita
ρ	$1 \cdot 10^{-4}$	minimum required consumption
κ_{we}	1,2 and 100	wealth unbalanced threshold rate
κ_{wm}	1,2 and 100	Inequality factor
κ_e	1	Elites rate of consumption
κ_m	1 ,2 , and 4	Middle class rate of consumption
γ_c^e	0.008 and .01	Commoners to Elites mobility factor
γ_e^c	0.0008 and .001	Elites to Commoners mobility factor
γ_c^{mc}	0.005 and .01	Commoners to Middle class mobility
γ_{mc}^c	0.005 and .04	mobility factor
γ_{mc}^e	0.0002 and .002	Middle class to Elites mobility factor
γ_e^{mc}	0.002 and .03	Elites mobility factor
δ_w	$1.2 \cdot \delta_y$	wealth rate of growth

CHAPTER NINE
HANDY-MC MODEL WITH RENEWABLE AND NON-RENEWABLE NATURAL RESOURCES

This short chapter extends the HANDY-MC-I model by splitting the natural resources into renewables, such as wood, and wind and solar energy, and the non-renewables, such as coal, oil and gas. This model is the most complex in this dissertation, but also the most realistic. Following the model, we present a few typical simulations. The analysis and more extensive simulations will be done in future works. The extension consists of splitting the resources variable y into y_r, which describes the renewable resources and y_n, which denotes the non-renewables. The latter can only decrease, being depleted, while the former can regenerate and rebound from almost collapse. This adds another rate equation to the system. In the simulations we also show a zoom of the time period of almost complete collapse, and verify that the system does not flatten out at the origin. This provides a closer view to what happens in a collapse.

9.1 The HANDY-MC-II model

This final model consist of a system of six nonlinear coupled ODEs: three equations model the growth rates of the Commoners or workers, $x_c(t)$, Middle-class, $x_{mc}(t)$, and the Elites or the rich, $x_e(t)$; two equations describe the rates of growth or depletion of renewable natural resources $y_r(t)$, and the depletion of non-renewable resources, y_n; and the rate of growth of wealth $w(t)$, which includes also food surpluses.

The HANDY-MC-II model, is the following:

Problem 9.1. *[HANDY-MC-II] Find six functions* $(x_c(t), x_{mc}(t), x_e(t), y_r(t), y_n(t), w(t))$*, defined on* $[0, T]$*, for* $T > 0$*, such that*

$$\begin{aligned}
\frac{dx_c}{dt} &= (\beta_c - \alpha_c)x_c + \gamma_{mc}^c x_{mc} + \gamma_e^c x_e - (\gamma_c^{mc} + \gamma_c^e)x_c,\\
\frac{dx_{mc}}{dt} &= (\beta_{mc} - \alpha_{mc})x_{mc} + \gamma_c^{mc} x_c + \gamma_e^{mc} x_e - (\gamma_{mc}^e + \gamma_{mc}^c)x_{mc},\\
\frac{dx_e}{dt} &= (\beta_e - \alpha_e)x_e + \gamma_c^e x_c + \gamma_{mc}^e x_{mc} - (\gamma_e^{mc} + \gamma_e^c)x_e,\\
\frac{dy_r}{dt} &= \gamma y_r(\lambda - y_r) - (\delta_{rc} x_c + \delta_{rm} x_{mc})y_r,\\
\frac{dy_n}{dt} &= -(\delta_{nc} x_c + \delta_{nm} x_{mc})y_n,\\
\frac{dw}{dt} &= (\delta_{wc} x_c + \delta_{wm} x_{mc})(y_r + y_n) - C_c - C_{mc} - C_e, \qquad (9.1)
\end{aligned}$$

where C_c, C_{mc} *and* C_e *are given in* (9.3)*,* w_{th} *in* (9.4)*, and* α_c, α_{mc} *and* α_e *in* (9.5)*; together with the initial conditions*

$$x_c(0) = x_{c0},\ x_{mc}(0) = x_{m0},\ x_e(0) = x_{e0},\ y_r(0) = y_{r0},\ y_n(0) = y_{n0},\ w(0) = w_0. \quad (9.2)$$

Here x_{c0}, y_{r0}, y_{n0} are positive and x_{mc0}, x_{e0}, w_0 are non-negative numbers.

The symbols have similar interpretations as in Chapter 7. The respective birth rates are $\beta_c, \beta_{mc}, \beta_e$; γ is the nature's regeneration factor; λ its saturation level or the carrying capacity of the renewable natural resources, all assumed to be positive constants. The $\delta_{rc}, \delta_{rm}, \delta_{nc}, \delta_{nm}$ are the depletion rate constants for the renewable and non-renewable resources used to generate wealth by the workers and the middle-class, respectively. Next, δ_{wc}, δ_{wm} are the rate constants of wealth generation by the workers and middle-class, respectively, using all the resources. The transitions rate constants among the populations are $\gamma_{mc}^c, \gamma_e^c$ for the loss of status of the middle-class and the rich, who become workers; $\gamma_c^{mc}, \gamma_c^e$ for the workers who move to the middle-class or become rich; γ_e^{mc} for the rich who slide to the middle-class; γ_{mc}^e for the middle-class who 'make it' and become rich.

Next, C_c, C_{mc}, C_e are the total consumption rates, given by

$$C_c = \min(1, \frac{w}{w_{th}}) s x_c, \quad C_{mc} = \min(1, \frac{w}{w_{th}}) \kappa_m s x_{mc}, \quad C_e = \min(1, \frac{w}{w_{th}}) \kappa_e s x_e. \quad (9.3)$$

As above, κ_m, κ_e measure the inequality in consumption or spending.

The wealth threshold is

$$w_{th} = \rho(x_c + \kappa_{wm} x_{mc} + \kappa_{we} x_e + \epsilon), \quad (9.4)$$

where κ_{wm}, κ_{we} measure the inequality in the wealth threshold, which measures the threshold below which there is insufficient wealth to support the populations, and the $\epsilon > 0$ is a very small number added so as to prevent the threshold w_{th} from vanishing, since it appears in the denominators. As in the previous models, we denote by $\eta = w/w_{th}$ the ratio of the current wealth to the minimum needed, and note that it is well defined since $w_{th} > 0$. Therefore, we may write

$$C_c = \min(1, \eta) s x_c, \quad C_{mc} = \min(1, \eta) \kappa_m s x_{mc}, \quad C_e = \min(1, \eta) \kappa_e s x_e.$$

Finally, the death rates $\alpha_c, \alpha_{mc}, \alpha_e$, are given by,

$$\begin{aligned} \alpha_c &= \alpha_m^* + \max(0, 1 - \min(1, \eta))(\alpha_M^* - \alpha_m^*), \\ \alpha_{mc} &= \alpha_m^* + \max(0, 1 - \kappa_m \min(1, \eta))(\alpha_M^* - \alpha_m^*), \\ \alpha_e &= \alpha_m^* + \max(0, 1 - \kappa_e \min(1, \eta))(\alpha_M^* - \alpha_m^*). \end{aligned} \quad (9.5)$$

Here, we assume that

$$\kappa_e \geq 1, \kappa_m \geq 1,$$

in all the simulations, since the death rates of the Middle class and the Elites, in case of insufficient resources, or famine, $\eta < 1$, are likely to be lower than those of the Commoners.

The analysis of the model, its steady states and their stability will be studied in the future.

9.2 Algorithm

The algorithm for the model follow the exact steps of the previous chapter specifically in section 8.1 with added equation to clarify the steps.

Step 1. Initiation

set the problem coefficients:

$$\alpha_m,\ \alpha_M,\ \beta_c,\ \beta_e\ ,\beta_m\ ,,\ \delta_{wc}\ ,\delta_{wm},\ \kappa_{we},\ \kappa_{wm},\ \lambda,\ \rho,\ s,$$

$$\gamma_c^e\ ,\gamma_e^c\ \gamma_{mc}^e\ ,\gamma_e^{mc}\ ,\gamma_c^{mc}\ ,\gamma_{mc}^c\ ,\delta_{rc}, \delta_{rm}, \delta_{nc}, \delta_{nm}$$

$$x_{c0},\ x_{e0},\ x_{mc0},\ y_{r0},\ y_{n0},\ w_0.$$

set T (the final year) and N (the number of time steps)

set $\Delta t = T/N$.

Step 2.

set $n = 0$

set $x_c^0 = x_{mc0}, \quad x_e^0 = x_{e0}, \quad x_{mc}^0 = x_{mc0}, \quad y_r^0 = y_{r0}, \quad y_n^0 = y_{n0} \quad w^0 = w_0,$

Step 3. Time marching

compute

C_c^n from (9.3)

C_e^n from (9.3)

C_{mc}^n from (9.3)

w_{th}^n from (9.4)

α_c^n from (9.5)

α_e^n from (9.5)

α_{mc}^n from (9.5)

set $n = n + 1$

compute

x_c^n from (9.1)

x_e^n from (9.1)

x_{mc}^n from (9.1)

y_r^n from (9.1)

y_n^n from (9.1)

w^n from (9.1)

Step 4. if $n < N$ go to Step 3

end

9.3 Simulations

The HANDY-MC-II model is the most complex in this work, with six dependent variables and 25 coefficients. Table 9.4 shows the parameters and coefficients used in the simulations of the model 9.1. In all the simulations displays, the curves and the colors are as follows: green represents the renewable natural resources y_r; black is the nonrenewables y_n; cyan is the wealth w; and blue, purple and magenta represent the Commoners, Middle class, and the Elites, respectively.

We turn to the simulations of the model with three classes and two sources for wealth accumulation, and their depletion. The cases presented here, had birth rates that were rather arbitrarily chosen of $\beta_c = 0.35$, $\beta_e = 0.3$ and $\beta_{mc} = 0.325$. These large birth rates explain the large populations seen in the figures. The time interval in the displays was chosen so that the details and overall behavior can be seen clearly.

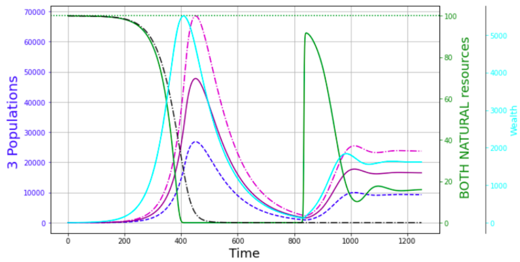
The scenarios shown in Fig. 9.1 were for a society in which the Elites' needs, as expressed in their fraction of the wealth threshold, were 1000 times those of the Commoners, while the needs of the Middle class were 'only' 100 times as large. The consumption rate constants were equal and set as 1. The purpose was to study the dependence of the solutions on the ratio of the wealth growth factor δ_{wc} and the depletion rate constant of the renewable resources, used by the workers, and varied as $\delta_{wc} = 0.8\,\delta_{rc}$ in Fig. 9.1(a); $\delta_{wc} = 1.2\,\delta_{rc}$ in Fig. 9.1(c) and $\delta_{wc} = 1.8\,\delta_{rc}$ in Fig. 9.1(e). Furthermore, δ_{rc} had the value $4.662 \cdot 10^{-5}$, while the Middle class renewables depletion rate constant was $\delta_{rm} = 2.46 \cdot 10^{-5}$, both relatively large depletion rate constants. It is seen that the behavior in the three cases is qualitatively similar. However, increasing the ratio from 0.8 to 1.2 to 1.8, increases the time for the system to recover after the first peak and the complete depletion of the nonrenewables, from 100 to 300 to 600 'years.' In Fig. 9.1(a) the near collapse is at year 830, and the Commoners drop from about $40,000$ at the peak to 1000, the Middle class drops from about peak of $67,000$ to $1,600$, and the Elites drop from a peak at $110,000$ to $2,400$. Moreover, the wealth dropped from about $5,000$ to 125. For each on of the scenarios we also depicted a zoom of the behavior of the system near the time of first collapse, to show in more details what might happen at such time intervals. We note that in all three scenarios the renewables recovered with a very steep growth (green very steep curves), and the recovery of the populations followed some time later. We conclude that by changing the ratio, the time scale when the populations are low, following a collapse, is increased. It seems likely that a real society that experiences such near collapse, may not recover for social and political reasons, as well as effects of other societies and populations, such as neighboring countries.

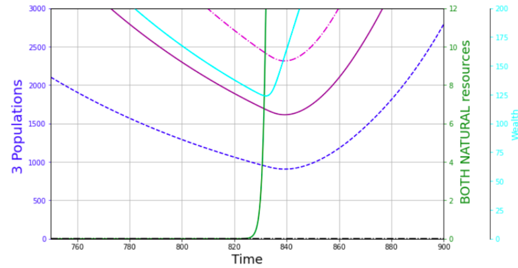
Next, when $\delta_{wc} = 1.2\delta_{rc}$ in Fig. 9.1(d), the system almost collapses to about $(100, 200, 350, 0, 0, 15)$ near the year 1100. While with δ_{wc} to $1.8\delta_{rc}$, the system almost collapses after a longer time period of time, at the year 1520 to very small numbers $(5, 7, 9, 0, 0, 2)$, shown in 9.1(f). In all cases, it is seen that the recovery period is growing from about 100 years in Fig. 9.1(a), to about 500 years in Fig. 9.1(c), to $1,000$ years in Fig. 9.1(e). It is clear that such long times for recovery may not be optional for real societies, as other national processes are likely to take place.

It is noted that once the nonrenewables were fully depleted, the society faced a rather hard collapse, and from that time on, the simulations were very similar to those of the HANDY-MC-I model in Fig. 7.1, since only renewables remained. Also, the approach to the steady state was much faster in Fig. 9.1(a) than in Fig. 9.1(c), while in Fig. 9.1(e) the system had many oscillations with decreasing amplitudes, as it converged to the steady state. The stability analysis done in Chapter 7.1 showed that the origin of the system was an unstable steady state. Although we did not perform such analysis for the HANDY-MC-II model, yet, it is very likely that its origin, i.e., the steady state $(0, 0, 0, 0, 0, 0)$, was unstable. The purpose of the zooms in Fig. 9.1 was to show that the collapse was not complete, so the quantities did not

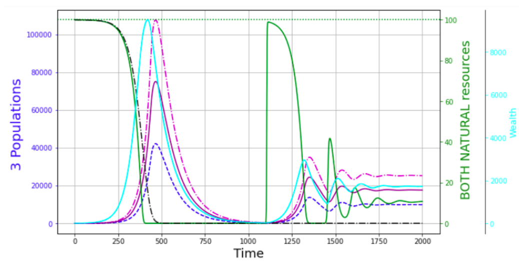
Table 9.4: The values of the parameters and data used in the simulations of the HANDY-MC-II model

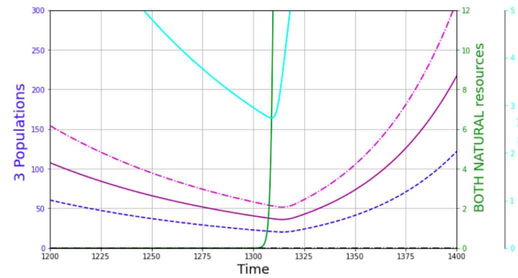
Symbols	values	meaning
Δt	.05	time step (year)
τ_F	1,400	time period (years)
$x_c(0)$	10	Commoners initial condition
$x_{mc}(0)$	2	Middle-class initial condition
$x_e(0)$	2	Elites initial condition
$y_r(0)$	100	Nature's renewable initial condition
$y_n(0)$	100	Nature's non-renewable initial condition
$w(0)$	0	wealth initial condition
λ	100	Natures carrying capacity
γ	.01	Natures regeneration factor
δ_{rc}	$3 \times 6.15 \cdot 10^{-6}$	Renewables depletion factor by workers,
δ_{rm}	$3 \times 6.15 \cdot 10^{-6}$	Renewables depletion factor by Middle class,
δ_{nc}	$6.15 \cdot 10^{-7}$	Non-Renewables depletion factor by workers
δ_{nm}	$6.15 \cdot 10^{-7}$	Non-Renewables depletion factor by Middle class
α_M^*	0.06	maximum expected death rate
α_m^*	0.01	normal death rate
β_c	.03	Commoners birth rate
β_{mc}	.03	Middle-class birth rate
β_e	.03	Elites birth rate
s	$5 \cdot 10^{-4}$	salary per capita
ρ	10^{-4}	minimum required consumption per capita
κ_e	10	wealth threshold contribution rate from the rich
κ_m	3	wealth threshold contribution rate from Middle class
κ_{we}	1 or 4	Inequality factor of the rich
κ_{wm}	1 or 4	Inequality factor of the Middle class
γ_{mc}^c	0.6	Middle-class to Commoners mobility factor
γ_c^{mc}	0.5	Commoners to Middle-class mobility factor
γ_e^c	0.3	Elites to Commoners mobility factor
γ_c^e	0.06	Commoners to Elites mobility factor
γ_{mc}^e	0.1	Middle-class to Elites mobility factor
γ_e^{mc}	0.03	Elites to Middle class mobility factor
δ_{wm}	$1.05\,\delta_{wc}$	wealth growth rate by Middle class
δ_{wc}	$2.2 \times \delta_{rc}$	wealth growth rate by Commoners



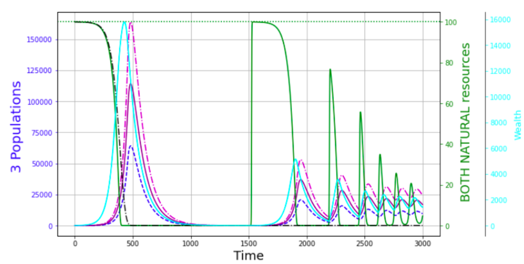


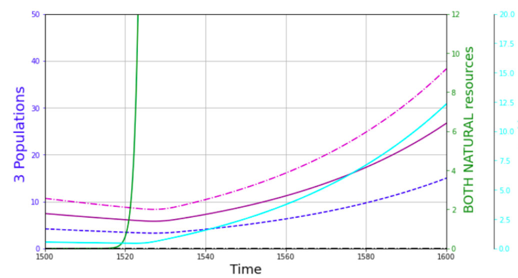
(a) $\delta_{wc} = 0.8\,\delta_{rc}$ (b) zoom on the collapse period in (a)





(c) $\delta_{wc} = 1.2 \cdot \delta_{rc}$ (d) zoom on the collapse period (c)

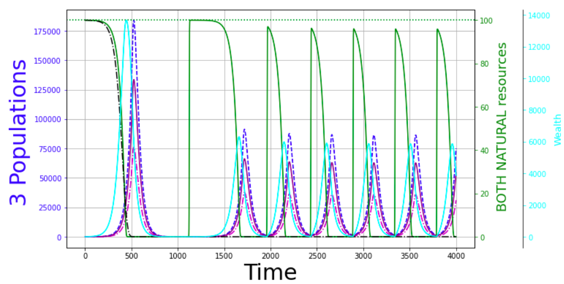




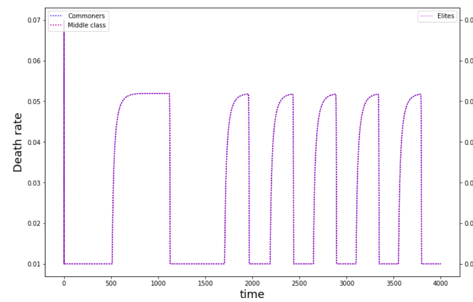
(e) $\delta_{wc} = 1.8\delta_{rc}$ (f) zoom on the collapse period (e)

Figure 9.1: The populations with different fractions of the wealth threshold. The depletion factor is fixed at $\delta_{rc} = 4.662 \cdot 10^{-5}$ and the Middle class depletes at the rate $\delta_{rm} = 2.46 \cdot 10^{-5}$. Note the steep recovery of the renewables, and also that the length of the time period close to collapse is longer with every increase in δ_{wc}. The zoom details on the right-hand side of each figure provide a close look at the system when it is closest to collapse. In all cases the nonrenewables are completely depleted about year 500. Notice the different rates of convergence to the steady states.

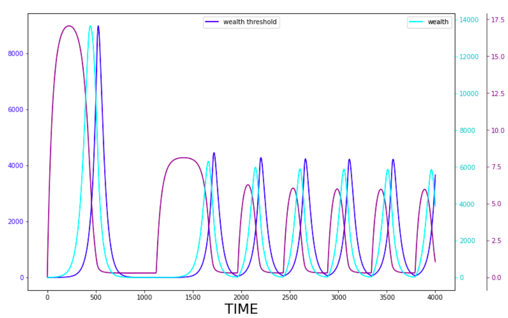
reach the zero values, and the system recovered, with rates dependent on the ratio of the two factors; wealth growth to the renewables depletion factors $\delta_{wc}/\,\delta_{rc}$. Lastly, we considered the case when the Middle class's fraction of the wealth threshold was ten-fold bigger than the Commoners and the Elites, with depletion factor $\delta_{rc} = 1.732 \cdot 10^{-5}$ and $\delta_{wc} = 2.2\delta_{rc}$, Fig. 9.2. We note that in this simulation, the system eventually approached a limit cycle with periodic oscillations of a period of about 450 'years.' The limit cycle was clearly seen in Fig. 9.3(b), where it has the unusual form of almost a triangle. This was related to the sharp spikes in the



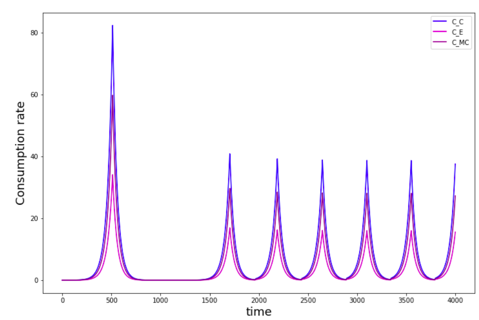
(a) Long time period showing the periodic behavior.



(b) Death rates for (a)



(c) η, w and w_{th}

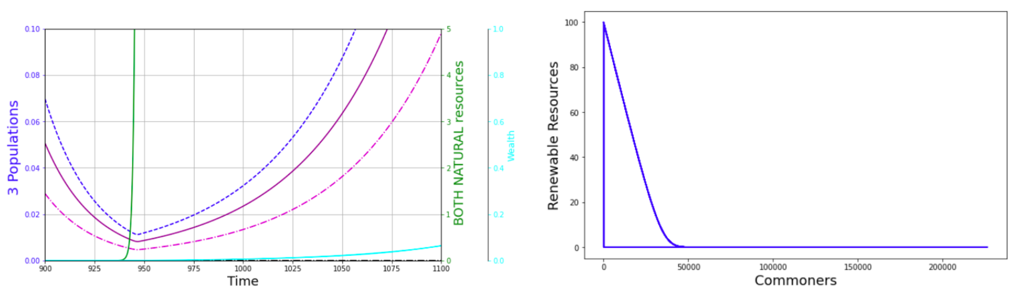


(d) Consumption rates for (c)

Figure 9.2: HANDY-MC-II model with detailed graphs of the other variable factors. The parameters listed in table 9.4 and a fixed depletion factor of $\delta_{wc} = 2.2\delta_{rc}$ with $\delta_{wm} = 1.05\delta_{wc}$. The Middle class' fraction $\kappa_1 = 10$ of the wealth threshold, and the pay factor is $\delta_{wm} = 2.31 \times \delta_{rc}$. Eventually, the system settles into periodic oscillations, which are a limit cycle. The sharp spikes in the solutions are a surprising feature usually not found in ODEs systems.

solutions that were very interesting, as they were uncharacteristic of ODEs system. Then, periods of steady behavior between the spikes could be seen. As was done in

Chapter 8, we depict in the sub-figures the plots of the three populations' other factors. The death rates $(\alpha_c, \alpha_{mc}, \alpha_e)$ were shown in Fig. 9.2(b). Fig. 9.2(c) showed the plots of the wealth, wealth threshold, and η. Next, Fig. 9.2(d) plots the three consumption factors (C_c, C_{mc}, C_e). The periodic behavior could be clearly seen in all the plots. Finally, some details in Fig.9.2 were shown in Fig. 9.3 for deeper insight. Sub-figure 9.3(a) provided a close look at the point in time where the system was closest to full collapse.



(a) Zoom of the collapse 9.2(a).

(b) Periodic oscillations in 9.2(a).

Figure 9.3: HANDY-MC-II model - details. Zoom at the collapse in 9.2(a), with very slow recovery of the wealth, and a fast recovery of the renewables (L). A plot of the Commoners vs. the renewable resources, indicating a limit cycle (R).

CHAPTER TEN

SUMMARY AND FUTURE WORK

The main contribution of this dissertation is the introduction and extensions of models for societal long-term well-being, which exhibit a wide range of possible behaviors. The simulations presented showed only the tip of the iceberg, and the choices of the parameter values and the simulations here, were strongly influenced by the original HANDY model. We believe that further studies of these models will allow societies and their leaders to obtain deeper insight into how to choose development trajectories that avoid collapse, and lead to stable, equitable and high quality life for all.

This dissertation presented three extensions of the HANDY mathematical model, constructed as tools to study the dynamics of large stratified societies. It focused on the interactions among the populations of Elites (rich) $x_e(t)$, Commoners (workers) $x_c(t)$ and Middle Class (professionals, managers, small business owners) $x_{mc}(t)$, their use of Natural Resources, renewable $y_r(t)$ and non-renewable $y_n(t)$, and the generation of human Wealth $w(t)$. This provided a tools kit for the study of various aspects of such societies, especially the potential conditions for their prosperity and flourishing, their long-term sustainability, and possible collapse. The models, which are increasingly more complex, and more realistic, allow us to conduct 'thought experiments' and study the effects of each one of the model assumptions on society's long-time behavior, which cannot be done with real or historical civilizations. In particular, it allows to investigate the effects of inequality, over-depletion of resources, as well as the concepts of individual's 'quality of life,' and how these affect the societies' well being. Moreover, it may provide insights into how to avoid collapse and potentially to increase the overall societal wealth and well being. We note that the models here do not represent any particular society or civilization, current or past. As was mentioned in the Introduction, the main interest is in finding the common potential causes that affect such societies.

The step-wise increase in the model complexity was done in Chapters 3, 5, 7 and 9. The introduction of the basic HANDY model can be found in Chapter 3, and it consisted of the basic system of four ODEs. Adding class mobility between the rich and the workers was done in Chapter 5, leading to the HANDY-SM model. Chapter 7 described the addition of the middle class, which added the Middle class rate of growth ODE. The consumption rates, wealth threshold and death rate coefficient functions were appropriately updated, resulting in the HANDY-MC-I model.

Finally, Chapter 9 introduced the HANDY-MC-II model, in which the natural resources $y(t)$ were split into the renewables $y_r(t)$ and non-renewables $y_n(t)$. This added another rate equation to the model, and also, the various rate functions and coefficients were appropriately modified. Thus, the resulting HANDY-MC-II model was the most complex and realistic.

The analysis of all the models, especially their steady states and stability, was very similar in each model. However, the existence of solutions to the original HANDY model on a finite time interval was established by constructing three domains S_1, S_2 and S_0 in $\mathbb{R}^4_+$ and showing that when the initial conditions are in either S_1 or S_2, the right-hand side function $\boldsymbol{F}$ was Lipschitz continuous and, hence, as long as the solution remained in each domain, it existed and was unique. On the other hand, to prove the existence of solutions to the HANDY-SM model and the HANDY-MC-I, we used Theorem 2.4, which didn't require Lipschitz continuity. To that end, we established the nonnegativity, when the initial conditions are nonnegative (i.e. $\mathbb{R}^4_+$ is invariant under the system), and the boundedness of the solutions. This allowed us to use Theorem 2.4 and prove the existence. But, the theorem does not guarantee the uniqueness of the solution, and this issue remained open. It seems that the same approach will work for the HANDY-MC-II model and we plan to address this topic in a future work. The study of the steady states and their stability was very similar in all three models: HANDY, HANDY-SM, and HANDY-MC-I. The stability was deduced from the eigenvalues of the system's Jacobian matrix, the derivation of which took considerable effort. In all three models, the zero steady state $(0, 0, 0, 0)$ in each of HANDY and HANDY-SM, and $(0, 0, 0, 0, 0)$ in HANDY-MC-I, was found to be unstable, and solutions that were initially in its neighborhood escaped from it. The steady state $(0, 0, \lambda, 0)$, where λ is the carrying capacity of nature, in the HANDY and HANDY-SM, and $(0, 0, 0, \lambda, 0)$ in HANDY-MC-I, was found to be stable, and numerically (in the simulations) also attracting (asymptotically stable). The stability of the steady states in the simulations were determined from the computation of the eigenvalues in each case and, indeed, in the presentations of the simulations we provided in each case the steady states and the related eigenvalues. We found that in most of the simulations, all the eigenvalues had nonpositive real parts, hence the steady states were stable and mostly attracting, too. In other simulations we found that the system had a limit cycle, which was seen in the simulations. The question of mathematically establishing the existence of limit cycles is unresolved, yet. It is known that such systems can have chaotic behavior, which we did not observe in our simulations.

We wrote a basic computer code, described in Chapter 4, for the original HANDY model, based on the explicit Euler time marching scheme, which was used in the computer algorithm that generated the simulations. The code was appropriately modified for each one of the HANDY, HANDY-SM, HANDY-MC-I and HANDY-MC-II models. The code was fast and stable, and the results compared very well with the simulations in [12] and also in Motesharrei et. al. [13, 14] in the case of the HANDY model.

We turn to a more detailed description of each chapter, especially those with the computer simulations. There, we provide a more detailed description of the simulations and what they can teach us. We note that the coefficients and the initial conditions are on the 'thought experiments' side, i.e., do not represent any society or civilization, current or historical. Some were chosen solely because they led to different and interesting qualitative behaviors of the solutions.

Chapter provided a detailed mathematical analysis of the HANDY model. A minor modification was introduced into the model as the original inequality factor κ was branched into two factors κ_0–wealth threshold constant, and κ_1– inequality pay factor, and the effects of the split were further investigated. Then, two ratios were defined; the rate constant of wealth to wealth threshold, $\eta = w/w_{th}$ and the ratio of the populations at steady state as $\psi = x_c/x_e$. The dependence of the consumption and death rates on η, and the dependence of w_{th} on the two inequality factors led to the need for the three domains in $\mathbb{R}^4_+$, as described above, such that $\boldsymbol{F}$ was Lipschitz in two of them. Moreover, nonzero steady states where found and inspected for stability, and some were shown to be stable and attracting.

Chapter provided computer simulations of the HANDY model, which supported the analysis in the previous chapter. We found that both inequality factors had significant effects on the solutions' behavior, and on the stability of the system. This showed in addressing such inequalities in real societies. The study of κ_0 showed that even a small increase in the Elites' contribution to the wealth threshold made a difference in the system's approach to the steady states. Moreover, increases in the consumption factor κ_1 seemed to cause the collapse of the workers population first, followed by that of the Elites.

Chapter described an extension of the model into the HANDY-SM model, by allowing mobility between the rich becoming workers (by going bankrupt), and the workers 'making it,' and becoming rich. Also, the depletion factor δ was split into two, to distinguish between the depletion factor of the natural resources δ_y and the wealth growth factor δ_w. The existence of the solutions was established by using Theorem 2.4, based on their boundedness and positivity. The stability of the steady states was studied in three different cases depending on the values of η and κ_1. Each case led to a different set of steady states and different Jacobin matrices. The simulations of the HANDY-SM model were presented in Chapter . First, they dealt with the case when the Elites share of the wealth was ten times bigger than that of the Commoners. The increased overall wealth, which included food reserves, prolonged the time for the society to collapse. Then, we found that increasing the Elites consumption rate forced the workers to supply the increased demand, which made the approach to the steady state smoother and faster. With a larger difference in consumption rates, for instance with $\kappa_1 = 8$, the simulations showed decreased societal wealth. As the Elites enjoyed their increased access to resources, a larger consumption rate caused both populations to deplete the natural resources faster, which led to slower population and wealth growth. We found that in the case of consumption rate of $\kappa_1 = 2$, the society accumulated the highest wealth value. Next, adding the mobility factors to the system produced solutions in which the levels of the populations changed as a fraction of the rich became workers, and a fraction of the workers became rich. In the case of larger difference between the two mobility factor γ, we found that in a short time period the populations decreased and then approached quickly to the steady state. By increasing δ_y, the system had undergone a damped oscillatory approach to the steady state.

The addition of the Middle class in Chapter , the HANDY-MC-I model, expanded the model and increased the number of the parameters to 25. This, as noted above, made the model more realistic, however at the expense of complexity and longer runs of the simulations. The analysis was done as explained above, with the added equations and parameters. The uniqueness of the solutions and further study of the possible system behavior remain tasks to be done. Chapter simulations reinforced the analysis done in Chapter . Moreover, we performed a more thorough study of the behavior of other factors, specifically $\eta, w_{th}, C_c, C_m, C_e$ and the three populations' death rate functions. We found that the consumption rates of both the Middle class and the Elites considerably influenced the life quality of the workers, in that they make the Commoners work harder to supply the needs of the societies. We simulated this through a comparison among the death rate functions α_c, α_{mc} and α_e, when the consumption rates were increased. When the consumption rate increased, the death rate dropped since in the model it increased η, which in turn, increased the populations growth. The simulations also showed two different kinds of periodic behaviors. One with overshoot and rapid collapse and the other was a faster approach to a periodic steady state, and Figs. 8.3 and 8.4 indicated that the system had limit cycles, which is a topic for future study. However, in our limited simulations, adding the Middle class to the system still showed similar oscillatory behavior, and the decaying oscillations were very similar to those in the basic HANDY model. There is a need to perform a wider range of simulations to study the other types of behavior that the addition of the Middle class made possible. Finally, Chapter introduced the HANDY-MC-II model, in which the natural resources were split into renewables and nonrenewables, and this was the most complex model in this study. It consisted of six ODEs for the rates of growth of the three populations, the two natural resources, and wealth. One of its aims was to investigate the long-term behavior once the nonrenewable resources were completely depleted. The analysis of the model was left open, and two sets of simulations were described. The simulations included different values of δ_{wc}, which led to different levels of the wealth. With more wealth, the populations reached higher numbers, especially at the time period when the two reserves coexisted. However, when the nonrenewable stock drained out, the model solutions resembled rapid collapse, as in the HANDY-MC-I model, with longer times to recovery. This may indicate that societies may find it harder to adapt to the loss of the nonrenewable resources. Since the origin, $(0, 0, 0, 0, 0, 0)$, was assumed (as in HANDY-MC-I) to be an unstable steady state, we presented in the simulations zooms of the time periods close to collapse to clarify that the system's collapse was close but above zero, and from which it recovered. Moreover, by using larger depletion factors, the system exhibited an approach to a limit cycle. Another observation was about experimenting with a bigger δ_{wc}, which showed longer collapse periods and smaller population values, and then bouncing back again to another overshoot that was followed by another almost complete collapse. We conclude that by changing the values of the parameters in the HANDY-MC-II model, we can study their effects on the system's approach to the steady states, which was one of our main goals. Furthermore, this last model seems to be complex and inclusive enough so it may be of interest to apply it to real societies.

10.1 Future work

This dissertation opens the ways for the further study of the models, and we pose some additional unresolved questions that may be of interest to explore. Mathematical questions of interest, among others, are: the further analysis of the steady states and to establish the existence of periodic solutions, limit cycles and possible chaotic solutions. The models' parameter spaces have high dimensions and it seems that in different regions in such a parameter space different types of behavior are possible. Indeed, the existence of periodic solutions and limit cycles are suggested by the simulations, especially those in Chapter 9. It may be of interest to prove that the computer solutions converges to the mathematical solutions when the time step tends to zero. Moreover, the main open question here is the uniqueness of the solutions, which need further and deeper analysis, since the function $\boldsymbol{F}$ is not globally Lipschitz. Furthermore, the analysis of the HANDY-MC-II model needs to be done, including finding the steady states and their stability. It may be of interest to modify the social mobility coefficients and other parameters and make them dependent on ratio of the wealth to wealth threshold η, similarly to what has been done with the consumption and death rate coefficient functions.

As was noted above, the HANDY-MC-II model may be already applicable to some societies. However, to make the models even more realistic, at the expense of increased complexity, one may:

- Rethink the wealth and the wealth threshold, to make them more realistic by not freezing at $\eta = 1$.
- Add a separate deforestation component into the rate equation for the renewables.
- Study a case where a health care component is added, which depends on the accumulated wealth.
- Make other coefficients and the the mobility factors dependent on η.
- Explore how to add over-population into the models by adding logistic-type terms.

- Explore how to add cost-of-living and inflation into the models.
- Add environmental functions such as pollution, greenhouse gases and deforestation resulting from over-population.

Some of these ideas will be investigated in the near future.

These models allow us, following Motesharrei et. al. [13, 14], to study in depth the effects of inequality in resources and pay on the well being of the society. Also, it is of importance to use them to study the contribution of the depletion of nonrenewables on the societies dynamics.

In terms of major applications, it seems that adapting one of the models to a historically known event of society collapse, and exploring the conditions under which the model would 'predict' the collapse, and what could have been done to avoid it, might provide deep insight into such events. Furthermore, it will be an effective proof that these models are capable of useful predictions. We conclude that the field of their applications is wide open and there are many directions in which we can proceed. It goes without saying that there is a considerable need, and plenty room for interdisciplinary cooperation.

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