POINT ESTIMATORS AND CONFIDENCE INTERVALS UNDER SEQUENTIAL SAMPLING STRATEGIES WITH APPLICATIONS

by

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This work is lovingly dedicated to my late mother, whose memory inspires me every day, to my husband for his support and belief in my dreams, and to my daughters, the brightest stars in my sky, who motivate me to reach for new heights.

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ABSTRACT

POINT ESTIMATORS AND CONFIDENCE INTERVALS UNDER SEQUENTIAL SAMPLING STRATEGIES WITH APPLICATIONS

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Statistical inference is the process of making informed decisions about a larger population by analyzing a smaller group of data collected with some form of sampling. In many statistical inference problems, where some prescribed accuracy is desired, the required sample size often depends on unknown population parameters and thus remains unknown. Then, it is necessary to conduct a sequential sampling procedure, where an experimenter takes one observation at a time successively until a predefined stopping rule is satisfied. This thesis involves sequential sampling procedures dealing with three statistical inference problems. These are (i) *bounded variance point estimation* (BVPE) of a function of the scale parameter in a gamma distribution with known shape parameter; (ii) *fixed-width confidence interval* (FWCI) estimation for comparing two independent Bernoulli proportions; and (iii) *fixed-accuracy confidence interval* (FACI) estimation for the shape parameter of a Weibull distribution based on record data.

In the first research problem, given a gamma population with known shape parameter α, we develop a general theory for estimating a function $g(\cdot)$ of the scale parameter β with bounded variance. We begin by defining a sequential sampling procedure with $g(\cdot)$ satisfying some desired condition in proposing the stopping rule, and show the procedure enjoys appealing asymptotic properties. After these general conditions, we substitute $g(\cdot)$ with specific functions including the gamma mean, the gamma variance, the

gamma rate parameter, and a gamma survival probability as four possible illustrations. For each illustration, Monte Carlo simulations are carried out to justify the remarkable performance of our proposed sequential sampling procedure. This is further substantiated with a real data study on the weights of newborn babies.

In the second research problem, we are interested in the proportions of a common characteristic possessed by two independent dichotomous populations, denoted by p_1 and p_2. We propose sequential sampling procedures for constructing FWCIs to compare the magnitude of p_1 and p_2 based on the log transformation and the logit transformation, respectively, which are followed by Monte Carlo simulations. We then implement these sequential sampling procedures to solve a real-world problem of mobile games A/B testing.

In the third research problem, we focus on utilizing the record data to estimate the shape parameter of a two-parameter Weibull population, which is widely used in lifetime data analysis. A sequential sampling procedure is developed for constructing a FACI for the Weibull shape parameter β, no matter whether the scale parameter α is known or unknown.

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CHAPTER ONE

INTRODUCTION

1.1 What Is Sequential Analysis?

Sequential analysis is a concept that has been around for a very long time. In 1940, Mahalanobis highlighted how crucial it is to collect data in stages and came up with a method to estimate the size of jute crops in Bengal, Mahalanobis (1940) . This pioneering work is seen as an early example of sequential analysis, a method later expanded upon by Abraham Wald and his team in the 1940s. They created a systematic approach and methodology for sequential testing to minimize the number of samples needed while ensuring the decisions made at the end were still reliable. Wald's work, published in 1947, is considered a landmark in the field, Wald (1947).

The development of sequential analysis was initially driven by practical needs, particularly for the efficient evaluation of anti-aircraft guns and other weapons during World War II. These methodologies proved to be extremely critical, enabling accurate results with fewer sample inspections, which was crucial for gaining an upper hand in frontline engagements. The innovations in this area were designated as "classified" during the early to mid-1940s. As methodological researchers recognized its potential, sequential analysis started being applied to a broad array of practical issues, including inventory management, queueing theory, reliability, life testing, quality control, experimental design, and multiple comparisons, among others. The list below includes significant contributions and is composed of books and monographs by Armitage (1975), Ghosh and Sen (1991a), Jennison and Turnbull (1999), Rosenberger and Lachin (2002), and Whitehead (1997).

There are many well-known books and studies in this field. We've noted works by Wald (1947) and Mukhopadhyay et al. (2004) before. In addition, further volumes such as Kiefer and Sobel (1968), Chernoff (1972), Chow et al. (1971), Ghosh (1970), Ghosh et al. (1997a), Ghosh and Sen (1991b), Gibbons et al. (1977), Govindarajulu (1981), Gupta and Panchapakesan (2002), Gut (1988), Mukhopadhyay and Solanky (1994), Schmitz (1972), Sen (1981), Shiryaev (1978), Siegmund (1985a), and Woodroofe (1982) can be found. Notably, some texts, such as Gibbons et al. (1977) and Gut (1988), only partially address sequential analysis. Govindarajulu (2004) stands out by providing codes for certain selected computer programs, and Mukhopadhyay and De Silva (2008) covered all major topics by employing sequential methodologies and integrated software, indeed covering plenty.

1.2 Why Sequential Analysis Is Important

Sequential analysis is a statistical methodology that revolutionizes decision-making processes by allowing continual refinement of conclusions as data becomes available. Unlike traditional statistical approaches that require the collection of an entire dataset before analysis, sequential analysis permits interim evaluations at predefined intervals. Sequential analysis holds paramount importance due to its efficiency and necessity in decision-making processes. The efficiency of this statistical methodology stems from its ability to adaptively analyze data as it becomes available, optimizing resource usage and saving time. Rather than waiting for the completion of data collection, sequential analysis allows for timely adjustments and informed decision-making based on accumulating evidence. This efficiency is particularly crucial in fields like clinical trials and quality control scenario in manufacturing, where adaptive designs can lead to early trial terminations or sample size adjustments, resulting in significant time and cost savings.

To illustrate the efficiency of sequential analysis methodology in quality detection through a more detailed example, let's see the following example paraphrased from Chen (1981). We are presented with a scenario where we need to decide on the acceptance or

rejection of a batch consisting of 20 products. The criterion for decision-making is straightforward: if the batch contains 3 or more defective products, it must be rejected. Traditionally, this would involve inspecting all 20 products to find out the total number of defects, comparing this figure against the threshold of 3 defective products to reach a verdict. However, the sequential analysis methodology offers a more efficient approach. Initially, we randomly select and inspect a subset of 3 products from the batch. If this initial inspection reveals 3 defective items, we can immediately decide to reject the batch without further inspection, saving time and resources. Conversely, if fewer than 3 defects are found, let's denote the number of defects discovered at this stage as i the process continues in a tailored manner. We then select an additional $3-i$ products from the remaining ones for inspection. This process of inspecting $3-i$ additional products and updating the cumulative count of defects, i, continues iteratively. The process concludes under one of two conditions: either the cumulative count of defective products reaches 3, prompting an immediate rejection of the batch, or all 20 products are inspected without identifying 3 defective items, in which case the batch is accepted. This sequential method's advantage lies in its potential to make an early rejection decision if the defect rate is high, thus demonstrating its efficiency. By possibly avoiding the inspection of all 20 products, the sequential analysis not only saves time but also optimizes resource utilization, highlighting its adaptability and effectiveness in quality control procedures.

The necessity of sequential analysis is evident in scenarios where continuous monitoring and real-time adaptation are paramount. To appreciate the value of sequential estimation,we adopt a well-known toy example put forward by Hu and Hong (2022). Let's consider a practical example. Suppose we have a sample $X_1, \ldots, X_n$ drawn from a normal distribution $N(\theta, \sigma^2)$ with both θ and σ^2 unknown. A data scientist wishes to estimate θ using the sample mean $\bar{X}_n$, while ensuring the variance of this estimator does not exceed a pre-specified level $b > 0$. Based on the Lehmann-Scheffé theorem and the Minimum

Variance Unbiased Estimator (MVUE) principle, $\bar{X}_n$ is the minimum variance unbiased estimator for θ (see "Theory of Statistics" by Mark J. Schervish, Theorem 5.5). However, this is a challenging task because the exact variance of $\bar{X}_n$, which is $\frac{\sigma^2}{n}$, must be less than b but σ^2 is unknown.

Stein (1945, 1949) proposed a two-stage sequential learning scheme to address this. The data analyst begins by taking a pilot sample $\boldsymbol{X}_m = (X_1, \ldots, X_m)$ where the pilot sample size m satisfies $m > \max\{2, 1/b+3\}$. The sample variance $S_m^2 = \frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X}_m)^2$ is then calculated, and the size of an additional sample k is determined by $k = \left\lfloor 1 + mS_m^2 \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, indicating the minimum additional samples required to achieve the variance threshold. The final sample size n is thus $m+k$, and the estimator $\bar{X}_n$ is the mean of the combined sample. The rationale behind this sequential method is to adjust the sample size based on an initial estimate of variance. If S_m^2 is large, it suggests a larger σ^2, and thus more samples are required. Conversely, if S_m^2 is small, fewer additional samples are needed. Stein's method ensures that $\bar{X}_n$, produced by this algorithm, is an unbiased estimator of θ and its variance $V(\bar{X}_n)$ is less than b. To see why the estimator is unbiased and the variance is constrained, we can refer to the following expectations, which are derived under the assumption of independent and identically distributed (i.i.d.) normal variables:

$$\begin{aligned}
\mathsf{E}(\bar{X}_n) &= \mathsf{E}\left(\mathsf{E}(\bar{X}_n | X_m)\right) \\
&= \mathsf{E}\left(\theta + \frac{\sum_{i=1}^{m}(X_i - m\theta)}{m+k}\right) \\
&= \theta + \mathsf{E}\left(\frac{\sum_{i=1}^{m}(X_i - m\theta)}{m+k}\right) \\
&= \theta.
\end{aligned}$$

This confirms that $\bar{X}_n$ is unbiased, and

$$\begin{aligned} \mathsf{V}(\bar{X}_n) &= \mathsf{E}\left(\frac{\sigma^2}{m+k}\right) \\ &\leq \mathsf{E}\left(\frac{1}{S_m^2/\sigma^2}\right) \\ &= \frac{1}{m-3} \\ &< b. \end{aligned}$$

This equation shows that $V(\bar{X}_n)$ is less than b, as desired. Furthermore, since S_m^2 is an unbiased estimator of σ^2 , $(m-1)S_m^2/\sigma^2$ follows a chi-square distribution with $m-1$ degrees of freedom, denoted χ^2_{m-1}. This relationship is used to establish the inequality relating the variance of $\bar{X}_n$ to b. In conclusion, the sequential learning procedure allows for the adaptive estimation ofθ with a controlled variance, demonstrating its necessity in statistical analysis when dealing with unknown population parameters.

1.3 Thesis Outline

In this thesis, we introduce the concept and utility of sequential sampling procedures, highlighting the critical role of sequential analysis in effective and expedient decision-making. Our examination is concentrated on the methodologies of purely sequential sampling, addressing three distinct statistical inference challenges, namely

- Bounded variance point estimation (BVPE) of a function of the scale parameter in a gamma distribution with known shape parameter,
- Fixed-width confidence interval (FWCI) estimation for comparing two independent Bernoulli proportions, and
- Fixed-accuracy confidence interval (FACI) estimation for the shape parameter of a Weibull distribution based on record data,

which will be thoroughly discussed in the subsequent chapters.

Chapter 2 presents the first research problem, focusing on a gamma population with a known shape parameter α, we develop a general theory for estimating a function $g(\cdot)$ of the scale parameterβ with bounded variance. First, we establish a sequential sampling procedure with a function $g(\cdot)$ satisfying some desired condition in proposing the stopping rule, and show the procedure enjoys appealing asymptotic properties. We then apply $g(\cdot)$ to specific cases such as the gamma distribution's mean, variance, rate parameter, and survival probability, illustrating the method's versatility.Our sequential sampling method's effectiveness is confirmed through Monte Carlo simulations for each illustration and further supported by real data analysis, focusing on the weights of newborn babies.

In Chapter 3, we address a second research problem concerning the comparison of proportions, p_1 and p_2, representing a shared trait across two independent dichotomous populations. We suggest sequential sampling methods to construct Fixed Width Confidence Intervals (FWCIs) for assessing the difference between p_1 and p_2 using both log and logit transformations. These methods are evaluated via Monte Carlo simulations and applied to a practical scenario of A/B testing in mobile gaming.

Chapter 4 delves into the third research problem, focusing on the use of record data for estimating the shape parameter of a two-parameter Weibull population, a common model in lifetime data analysis. We develop a sequential sampling method to establish a Fixed Accuracy Confidence Interval (FACI) for the Weibull shape parameter β, regardless of the known status of the scale parameter α.

Chapter 5 offers a brief overview of our work. Finally, Chapter 6 provides the ideas for future work.

CHAPTER TWO

ESTIMATING A FUNCTION OF THE SCALE PARAMETER IN A GAMMA DISTRIBUTION WITH BOUNDED VARIANCE

2.1 Introduction

The gamma distribution, as one of the most commonly used probability distributions for modeling data, plays an important role in many areas such as business management, environmental science, reliability engineering, and medical science. For example, Nenes et al. (2010) considered modeling the demand for commercial items employing a gamma distribution with a probability mass at zero; Robbins (1959) investigated the use of the gamma distribution to determine the half-life of rotenone applied in freshwater; Ibrahim et al. (2019) applied the gamma distribution to evaluate and predict the reliability and the projected lifetime of phosphor-converted white LEDs based on luminous flux degradation; and Cui et al. (2023) proposed a three-layer feature selection combined with a gamma distribution-based generalized linear model for anticancer drug response prediction.

It is safe to say that there is a large volume of articles on the estimation of the gamma distribution parameters, most of which are based on fixed-sample-size procedures. In a lot of statistical inference problems, however, requirements on some predetermined accuracy (e.g., confidence interval estimation with a fixed width, point estimation with a bounded risk, testing hypotheses with controlled type-I and type-II errors, etc.) leads to the nonexistence of such fixed-sample-size procedures. As a consequence, sequential or multistage sampling procedures are necessary. In the context of gamma parameters estimation, Takada and Nagata (1995) constructed the fixed-width sequential confidence interval for the mean of a gamma population; Isogai and Uno (1995) considered the

minimum risk point estimation problem for a gamma mean, and proposed a sequential sampling procedure shown to be asymptotically better than the one given by Woodroofe (1977); Zacks and Khan (2011) developed both two-stage and purely sequential estimation procedures for fixed-width confidence interval of the scale parameter of a gamma distribution when the shape parameter is known; Mahmoudi and Roughani Shahraki (2015) and Roughani and Mahmoudi (2015) subsequently investigated the bounded risk estimation problem for the gamma scale parameter in a two-stage sampling procedure; Mahmoudi et al. (2019) further studied the same problem in a purely sequential sampling procedure; Zhuang et al. (2020) established sequential fixed-accuracy confidence intervals for the survival function of a gamma distribution; and most recently, Bapat (2023) obtained novel two-stage sampling procedures to estimate the ratio and the sum of shape parameters coming from two independent gamma populations.

In a parallel direction, there is also interest in estimating a function of the parameter(s) sequentially to make the inference problem more practical. To list a few, Uno et al. (2004) put forward sequential point estimation of a function of the exponential scale parameter subject to squared error loss plus a linear cost; Mukhopadhyay and Wang (2019) developed a general theory of sequential minimum risk point estimation of a function of a normal mean; and Banerjee and Mukhopadhyay (2021) designed multistage minimum risk point estimation strategies for a function of a normal mean. For a broad-ranging review of sequential analysis as a powerful tool, one may refer to many resources including the following monographs: Anscombe(1952, 1953), Robbins (1959), Chow and Robbins (1965), Siegmund (1985b), Ghosh et al. (1997b), and Mukhopadhyay and De Silva (2008).

In this chapter, we examine a sequential approach to tackling the so-called *bounded variance point estimation* (BVPE) problem for a function $g(\cdot)$ of the scale parameter β of a gamma distribution, assuming the shape parameter α is known to us. That is, the sequential methodology is devised to constrain the variance of the estimator to a prescribed bound

$b(>0)$. BVPE can be applied in many fields such as actuarial science and medical science, showing its practical applicability. One may see Hu and Zhuang (2022) and Hu and Hong (2022) for more details.

The remainder of the chapter is organized as follows. In Section 2.2, we lay down some useful preliminaries, establish a general theory, and demonstrate the appealing properties of sequential BVPE for a function of the gamma scale parameter. In Section 2.3, we exhibit four illustrations where the general framework developed in Section 2.2 is readily applied, namely, the gamma mean, the gamma variance, the gamma rate parameter, and the survival probability. Extensive sets of Monte Carlo simulations are also carried out to verify the theoretical findings empirically. Section 2.4 includes real data analysis to showcase the possibility of incorporating our procedure in real life problems. The chapter is wrapped up with some brief overall thoughts in Section 2.5.

2.2 Preliminaries, Sequential Estimation, and Properties

Suppose that $X_1, X_2, \ldots$ are independent observations from a gamma population $\Gamma(\alpha, \beta)$ with the associated probability density function (pdf) given by

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \text{ for } x > 0.$$

Here, the scale parameter $\beta(>0)$ is of interest and remains unknown, while the shape parameter $\alpha(>0)$ is *known* to us. We desire to estimate $g(\beta)$, a *monotone* function of the scale parameter. Since β takes only positive values, we let the function $g(\cdot)$ be defined on the positive real line for simplicity; and assume that for all $x \in \mathscr{R}^{+}$, $g(x)$ is twice continuously differentiable. Denote the first two derivatives by $g'(x)$ and $g''(x)$, respectively. In the spirits of Mukhopadhyay and Wang (2019) and Mukhopadhyay (2021), we adopt a similar but relatively relaxed condition as follows:

$$\left|g''(x)\right| \leq \sum_{j=1}^{d} a_j x^{k_j}, \tag{2.2.1}$$

where $d(\geq 1)$ is a fixed integer, $a_j(\geq 0)$ and k_j (not necessarily nonnegative) are appropriate fixed real numbers, for $j = 1, ..., d$. One may wonder if the condition (2.2.1) is too strong to be practical. In fact, there are a lot of common and meaningful functions of β regarding a gamma distribution that satisfy this condition. Here are some typical examples.

(i) The mean of a gamma distribution $g(\beta) = \alpha\beta$:

$$g'(\beta) = \alpha > 0,\ |g''(\beta)| = 0,$$

where $d = 1, a_1 = \alpha$, and $k_1 = 0$.

(ii) The variance of a gamma distribution $g(\beta) = \alpha\beta^2$:

$$g'(\beta) = 2\alpha\beta > 0,\ |g''(\beta)| = 2\alpha,$$

where $d = 1, a_1 = 2\alpha$, and $k_1 = 0$.

(iii) The rate parameter of a gamma distribution $g(\beta) = \beta^{-1}$:

$$g'(\beta) = -\beta^{-2} < 0,\ |g''(\beta)| = 2\beta^{-3},$$

where $d = 1, a_1 = 2$, and $k_1 = -3$.

(iv) A survival probability of a gamma distribution $g(\beta) = \Pr(X > c) = \int_{c/\beta}^{\infty} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$ with $c > 0$ being an appropriate constant:

$$\begin{aligned} g'(\beta) &= -\frac{c^\alpha}{\Gamma(\alpha)} \beta^{-(\alpha+1)} e^{-c/\beta} < 0, \\ g''(\beta) &= \frac{c^\alpha}{\Gamma(\alpha)} \beta^{-(\alpha+2)} e^{-c/\beta} (\alpha + 1 + c\beta^{-1}) \\ &\Rightarrow |g''(\beta)| \leq \frac{c^\alpha(\alpha+1)}{\Gamma(\alpha)} \beta^{-(\alpha+2)} + \frac{c^{\alpha+1}}{\Gamma(\alpha)} \beta^{-(\alpha+3)}, \end{aligned}$$

where $d = 2, a_1 = \frac{c^\alpha(\alpha+1)}{\Gamma(\alpha)}, k_1 = -(\alpha+2), a_2 = \frac{c^{\alpha+1}}{\Gamma(\alpha)}$, and $k_2 = -(\alpha+3)$.

Having recorded $X_1, ..., X_n, n \geq 1$, a maximum likelihood estimator (MLE) of β is given by $\hat{\beta}_n = \alpha^{-1}\bar{X}_n$, where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ denotes the sample mean. According to the invariant property, a natural MLE of $g(\beta)$ is given by $g(\hat{\beta}_n)$. As for its variance denoted by $\mathsf{V}[g(\hat{\beta}_n)]$, in general, it may not be easy to find an explicit expression. Hence, we employ the delta method to obtain an approximation, which is valid as $g(\cdot)$ has a continuous second derivative. Applying Taylor's theorem, we have

$$g(\hat{\beta}_n) - g(\beta) = g'(\beta)(\hat{\beta}_n - \beta) + \frac{1}{2}g''(\xi_n)(\hat{\beta}_n - \beta)^2,$$

where ξ_n is a random variance lying between $\hat{\beta}_n$ and β. Then,

$$\mathsf{V}[g(\hat{\beta}_n)] = \frac{\{g'(\beta)\}^2\beta^2}{\alpha n} + \frac{1}{4}\mathsf{V}[g''(\xi_n)(\hat{\beta}_n - \beta)^2] + g'(\beta)\mathsf{Cov}(\hat{\beta}_n - \beta, g''(\xi_n)(\hat{\beta}_n - \beta)^2). \tag{2.2.2}$$

Under the condition (2.2.1),

$$|g''(\xi_n)| \leq \sum_{j=1}^{d} a_j \xi_n^{k_j} \leq \sum_{j=1}^{d} a_j (\alpha^{-1}\bar{X}_n + \beta)^{k_j}.$$

As $\bar{X}_n \sim \Gamma(n\alpha, \beta/n)$, it is clear that $\bar{X}_n^{pk_j}$ is integrable given $pk_j + n\alpha > 0$. Therefore, $\{g''(\xi_n)\}^p$ is integrable when

$$p \min_{j \in \{1,...,d\}} k_j + n\alpha > 0. \tag{2.2.3}$$

Suppose there exists a $p = 2p_0 > 2$ satisfying (2.2.3). Then, by Hölder's inequality,

$$\begin{aligned} \mathsf{V}[g''(\xi_n)(\hat{\beta}_n - \beta)^2] &\leq \mathsf{E}^{1/p_0}[\{g''(\xi_n)\}^{2p_0}] \cdot \mathsf{E}^{1/q_0}[(\hat{\beta}_n - \beta)^{4q_0}] \\ &= O(1)O(n^{-2}) = O(n^{-2}), \end{aligned} \tag{2.2.4}$$

where $1/p_0 + 1/q_0 = 1$. In a similar fashion,

$$\begin{aligned} \mathsf{Cov}(\hat{\beta}_n - \beta, g''(\xi_n)(\hat{\beta}_n - \beta)^2) &\leq \mathsf{V}^{1/2}[\hat{\beta}_n - \beta] \cdot \mathsf{V}^{1/2}[g''(\xi_n)(\hat{\beta}_n - \beta)^2] \\ &= O(n^{-1/2})O(n^{-1}) = O(n^{-3/2}), \end{aligned} \tag{2.2.5}$$

Putting together (2.2.2), (2.2.4) and (2.2.5), we arrive at

$$\mathsf{V}[g(\hat{\beta}_n)] = \frac{\{g'(\beta)\}^2\beta^2}{\alpha n} + O(n^{-3/2}). \tag{2.2.6}$$

This suggests that the delta method gives a good approximation of $\mathsf{V}[g(\hat{\beta}_n)]$ with an error up to the order $O(n^{-3/2})$, assuming the condition (2.2.1), and that (2.2.3) holds for some $p > 2$. Now, our goal is to control the variance of the estimator $g(\hat{\beta}_n)$ under a predetermined small level $b(> 0)$, that is,

$$\mathsf{V}[g(\hat{\beta}_n)] \leq b. \tag{2.2.7}$$

Omitting the small quantity $O(n^{-3/2})$ in (2.2.6), we have that

$$\mathsf{V}[g(\hat{\beta}_n)] \approx \frac{\{g'(\beta)\}^2\beta^2}{\alpha n} \leq b \Rightarrow n \geq \frac{\{g'(\beta)\}^2\beta^2}{\alpha b} = n^*, \text{ say.} \tag{2.2.8}$$

Here, n^* is defined as the *optimal sample size* whose magnitude remains unknown, though, since it involves the unknown scale parameter β. A sequential sampling procedure can be implemented to solve this estimation problem, where one estimates β by updating its MLE at every stage as needed.

Beginning with a pilot sample of size $m(\geq 1)$, $X_1, ..., X_m$, we propose the following sequential procedure which leads to a final sample size required for bounding the variance of the estimator. The associated stopping rule is given by

$$N = N(b) = \inf\left\{ n \geq m : n \geq \frac{\{g'(\hat{\beta}_n)\}^2\hat{\beta}_n^2}{\alpha b} \right\}, \tag{2.2.9}$$

where $\hat{\beta}_n = \bar{X}_n/\alpha$. Concerning the implementation of (2.2.9), if $m \geq \frac{\{g'(\hat{\beta}_m)\}^2\hat{\beta}_m^2}{\alpha b}$ is already satisfied, we do not take any additional observations, and the final sample size is $N = m$. Otherwise, we collect one more observation X_{m+1} and update $\hat{\beta}_{m+1}$ to check with the stopping rule (2.2.9). Sampling is then terminated at the first time $N = n(\geq m)$ such that $n \geq \frac{\{g'(\hat{\beta}_n)\}^2\hat{\beta}_n^2}{\alpha b}$ occurs. At last, with the fully accrued data $X_1, ..., X_m, ..., X_N$, we construct

the bounded variance point estimator for $g(\beta)$ by

$$g(\hat{\beta}_N) = g(\bar{X}_N/\alpha). \tag{2.2.10}$$

Obviously, we can claim that $\Pr\{N < \infty\} = 1$ and $N \to \infty$ with probability 1 (w.p.1) as $b \to 0$.

The sequential estimation procedure (2.2.9) is efficient in terms of the final sample size required, as its expected value will be close to the optimal sample size n^* defined in (2.2.8) in some certain manner. In, we conclude the so-called asymptotic first-order efficiency in Theorem 2.2.1 generally, and second-order analysis may be available according to the specific function of $g(\cdot)$ under consideration. A few typical illustrations will be provided in Section 2.3.

Theorem 2.2.1. *Assume the condition* (2.2.1)*, and that* $p\min_{j\in\{1,\ldots,d\}} k_j + m\alpha > 0$ *for some* $p > 2$*. For the sequential estimation procedure* (2.2.9)*, we have that as* $b \to 0$*,*

$$\mathsf{E}[N/n^*] = 1 + o(1). \tag{2.2.11}$$

Proof. For $n = m, m+1, \ldots$, define

$$y_n = \left[\frac{g'(\hat{\beta}_n)\hat{\beta}_n}{g'(\beta)\beta}\right]^2.$$

Since $\hat{\beta}_n$ is the MLE of β and $g(\cdot)$ is a continuous function, then $\lim_{n\to\infty} y_n = 1$ w.p.1. Let $f(n) = n$ so it is trivial that $\lim_{n\to\infty} f(n) = \infty$ and $\lim_{n\to\infty} f(n)/f(n-1) = \infty$. The stopping rule (2.2.9) can be rewritten as

$$N = \inf\{n \geq m : y_n \leq f(n)/n^*\},$$

which matches that of Chow and Robbins (1965). Hence, $\lim_{n\to\infty} n^{*-1}N = 1$ w.p.1 by their Lemma 1. It suffices to show the uniform integrability of $\{n^{*-1}N : 0 < b \leq 1\}$ along the lines of Ghosh and Mukhopadhyay (1980) and Hu and Mukhopadhyay (2019).

Write for any $c > 0$,

$$\mathsf{E}\left[\{N/n^*\}\mathbb{I}(N > cn^*)\right] = c\Pr(N > cn^*) + \int_c^\infty \Pr(N > xn^*)dx, \tag{2.2.12}$$

where $\mathbb{I}(A)$ represents the indicator function of an event A. Let $t = \lfloor xn^* \rfloor$, where $\lfloor u \rfloor$ indicates the largest integer that is strictly smaller than $u(> 0)$. Select $b_0 \leq 1$ such that $c > c_0(\geq 2)$ for small $b \leq b_0$ and $g'(\hat{\beta}_n)/g'(\beta) < 2$ w.p.1 for $n \geq \lfloor cn^* \rfloor$. Then,

$$\begin{aligned}
\Pr(N > xn^*) \leq \Pr(N > t) &\leq \Pr\left(\left[\frac{g'(\hat{\beta}_t)\hat{\beta}_t}{g'(\beta)\beta}\right]^2 \geq t\right) \\
&\leq \Pr\left(|\hat{\beta}_t - \beta| \geq (t^{1/2}/2 - 1)\beta\right) \\
&\leq \frac{\mathsf{E}[|\hat{\beta}_t - \beta|^s]}{[(t^{1/2}/2 - 1)\beta]^s} = O(x^{-s}),
\end{aligned} \tag{2.2.13}$$

for some appropriate $s > 1$. It leads to $c\Pr(N > cn^*) = O(c^{1-s})$ and $\int_c^\infty \Pr(N > xn^*)dx = O(c^{1-s})$ for sufficiently large c. From (2.2.12), we can claim that $\{n^{*-1}N : 0 < b \leq 1\}$ is uniformly integrable, and therefore (2.2.11) stands. □

As our ultimate target is to bound the variance of the point estimator of $g(\beta)$, it is anticipated that $\mathsf{V}[g(\hat{\beta}_N)]$ remains below the prescribed small level $b > 0$ or approximately b. The next theorem demonstrates that the sequential estimation procedure (2.2.9) yields an estimator whose variance is close to b in some certain way.

Theorem 2.2.2. *Assume the condition* (2.2.1)*, and that* $p\min_{j\in\{1,\ldots,d\}} k_j + m\alpha > 0$ *for some* $p > 2$*. For the sequential estimation procedure* (2.2.9)*, we have that as* $b \to 0$*,*

$$\mathsf{V}[g(\hat{\beta}_N)] = b + O(b^{3/2}). \tag{2.2.14}$$

Proof. We set out to prove (2.2.14) by appealing to Yu (1989). Note that for a random sample of the gamma population $\Gamma(\alpha, \beta)$, $X_1, \ldots, X_n$, the Fisher information about β is given by

$$I_{\beta,n}(\beta) = \mathsf{E}\left[-\frac{d^2}{d\beta^2}\log\prod_i^n f(X_i;\beta)\right] = \frac{n\alpha}{\beta^2}.$$

Using $\lambda = g(\beta)$ to reparameterize the gamma population, the corresponding Fisher information with respect to λ is

$$I_{\lambda,n}(\lambda) = I_{\beta,n}(\beta)\left(\frac{d\beta}{d\lambda}\right)^2 = \frac{n\alpha}{\{g'(\beta)\}^{-2}\beta^2}.$$

Since the magnitude of β is veiled, we use the observed Fisher information at $\hat{\beta}_n$ to estimate $I_{\lambda,n}(\lambda)$. Then, one has

$$I_{\hat{\lambda}_n,n}(\hat{\lambda}_n) = \frac{n\alpha}{\{g'(\hat{\beta}_n)\}^{-2}\hat{\beta}_n^2}.$$

The stopping rule (2.2.9) can be rewritten as

$$N = \inf\{n \geq m : I_{\hat{\lambda}_n,n}(\hat{\lambda}_n) \leq b\}, \tag{2.2.15}$$

which matches the stopping rule proposed in Yu (1989). Therefore, one can immediately obtain that as $b \to 0$,

$$\sqrt{N}(g(\hat{\beta}_N) - g(\beta)) \xrightarrow{d} N(0, \{g'(\beta)\}^2\beta^2/\alpha), \tag{2.2.16}$$

and

$$\sqrt{N}(\hat{\beta}_N - \beta) \xrightarrow{d} N(0, \beta^2/\alpha). \tag{2.2.17}$$

Observe that for any $q \geq 1$, $\hat{\beta}_N^q \leq \sup_{n\geq m}\hat{\beta}_n^q$, and $\mathsf{E}(\hat{\beta}_n^q) \sim \beta^q < \infty$. The uniform integrability of $\{\hat{\beta}_N^q, 0 < b \leq 1\}$ can be established through Wiener's (1939) ergodic theorem. Apply Taylor's theorem, and we express

$$g(\hat{\beta}_N) - g(\beta) = g'(\beta)(\hat{\beta}_N - \beta) + \frac{1}{2}g''(\kappa_N)(\hat{\beta}_N - \beta)^2, \tag{2.2.18}$$

where κ_N is a random variable lying between $\hat{\beta}_N$ and β. Then, following our previously used arguments leading to (2.2.6), we can claim that

$$\mathsf{V}[g(\hat{\beta}_N)] = b + O(b^{3/2}).$$

The proof is now complete. □

2.3 Illustrations

In this section, we substitute the general $g(\beta)$ with the aforementioned practically useful functions for illustrative purposes. For each illustration, we specify the stopping rule and carry out Monte Carlo simulations to validate the theoretical findings empirically.

2.3.1 The gamma mean

To estimate the gamma mean $g(\beta) = \alpha\beta$, we follow the general framework of sequential estimation established in Section 2.2, and propose the procedure $\mathscr{P}_1$ associated with the stopping rule given by

$$N_1 = \inf\left\{ n \geq m : n \geq \frac{\alpha \hat{\beta}_n^2}{b} \right\}. \tag{2.3.1}$$

As was pointed out, $d = 1, a_1 = \alpha$ and $k_1 = 0$ in this case so that any pilot sample size $m \geq 1$ will satisfy the condition (2.2.3). Next, note that the stopping rule can be alternatively represented by

$$N_1 = \inf\left\{ n \geq m : \sum_{i=1}^{n} W_i \leq (\alpha b)^{1/2} \beta^{-1} n^{3/2} \right\}, \tag{2.3.2}$$

where $W_1, ..., W_n$ are independent and identically distributed (i.i.d.) $\Gamma(\alpha, 1)$ random variables. It then matches the stopping rule of Woodroofe (1977). Therefore, in addition to the asymptotic first-order efficiency as per (2.2.11), we have the following second-order approximation by appealing to the nonlinear renewal theory: for $m > 4/\alpha$,

$$\mathsf{E}[N_1 - n_1^*] = \frac{1}{2} - \frac{1}{\alpha} - \frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n} \mathsf{E}\left(\sum_{i=1}^{n} W_i - \frac{3}{2}\alpha n \right)^{+} + o(1) \tag{2.3.3}$$

as $b \to 0$, where $n_1^* = \alpha\beta^2/b$ is the corresponding optimal sample size. One may refer to Lai and Siegmund (1977, 1979) and Mukhopadhyay and De Silva (2008) for more background details.

An extensive set of Monte Carlo simulations have been conducted to investigate the performance of the sequential estimation procedure $\mathscr{P}_1$. We generated pseudorandom

samples from a gamma population with $\alpha = 2$ and $\beta = 2$, but pretended that β was unknown. In this case, the second-order expansion in (2.3.3) is computable, and a separate R program was written to approximate it numerically. In the spirits of Mukhopadhyay and Solanky (1994, Table 3.8.1), we excluded any term smaller than 10^{-15} in magnitude in the infinite sum, and obtained that $\mathsf{E}[N_1 - n_1^*] \approx -0.5680$.

Then, we fixed the pilot sample size $m = 20$. A wide range of values of b including $0.160, 0.080, 0.040, 0.016, 0.008$ and 0.004 were considered, so that n_1^* turned out 50, 100 (small), 200, 500 (moderate), and 1000, 2000 (large) accordingly. The procedure was repeated for a total of 10,000 times. In Table 2.1, we record the average final sample size $\bar{n}_1$ with the associated standard deviation $s(n_1)$, the first-order approximation $\bar{n}_1/n_1^*$, the second-order approximation $\bar{n}_1 - n_1^*$, the variance of the estimates $\mathsf{V}[g(\hat{\beta})_{n_1}]$, and the ratio of $\mathsf{V}[g(\hat{\beta}_{n_1})]$ to b.

Table 2.1: Simulations from $\Gamma(2,2)$ with $m = 20$ under $10,000$ runs implementing the sequential estimation procedure $\mathscr{P}_1$

b	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	$\bar{n}_1 - n_1^*$	$\mathsf{V}(g(\hat{\beta}_{n_1}))$	$\mathsf{V}(g(\hat{\beta}_{n_1}))/b$
0.160	50	49.2343	10.3368	0.9847	−0.7657	0.187391	1.171194
0.080	100	99.4338	14.4372	0.9943	−0.5662	0.087127	1.089093
0.040	200	199.6266	20.1623	0.9981	−0.3734	0.041503	1.037576
0.016	500	499.3923	31.8152	0.9988	−0.6077	0.016356	1.022236
0.008	1000	999.0571	44.8184	0.9991	−0.9429	0.008074	1.009238
0.004	2000	1999.6929	62.8701	0.9998	−0.3071	0.003958	0.989538

From Table 2.1, it is clear that as $b(n_1^*)$ gets smaller (larger), the first-order efficiency term $\bar{n}_1/n_1^*$ approaches 1, and the second-order efficiency term $\bar{n}_1 - n_1^*$ is comparable to -0.5680. Across the board, the variance estimate $\mathsf{V}(g(\hat{\beta}_{n_1}))$ is close to the

target b across the board, and the ratio $\mathsf{V}(g(\hat{\beta}_{n_1}))/b$ is comparable to 1. These results show that the developed sequential estimation procedure $\mathscr{P}_1$ has performed remarkably well.

2.3.2 The gamma variance

In a parallel fashion, we propose the sequential procedure $\mathscr{P}_2$ as follows to estimate the gamma variance $g(\beta) = \alpha\beta^2$.

$$N_2 = \inf\left\{ n \geq m : n \geq \frac{4\alpha\hat{\beta}_n^4}{b} \right\}. \tag{2.3.4}$$

As was pointed out, $d = 1, a_1 = 2\alpha$ and $k_1 = 1$ in this instance so that any pilot sample size $m \geq 1$ will satisfy the condition (2.2.3). Similarly, a rewrite of the stopping rule (2.3.4) will bring it in line with that of Woodroofe (1977).

$$N_1 = \inf\left\{ n \geq m : \sum_{i=1}^{n} W_i \leq (\alpha^3 b/4)^{1/4}\beta^{-1}n^{5/4} \right\}, \tag{2.3.5}$$

where $W_1, ..., W_n$ are i.i.d. $\Gamma(\alpha, 1)$ random variables. Therefore, we can claim the second-order approximation herein: for $m > 4/\alpha$,

$$\mathsf{E}[N_2 - n_2^*] = \frac{1}{2} - \frac{2}{\alpha} - \frac{4}{\alpha}\sum_{n=1}^{\infty}\frac{1}{n}\mathsf{E}\left(\sum_{i=1}^{n} W_i - \frac{5}{4}\alpha n\right)^{+} + o(1) \tag{2.3.6}$$

as $b \to 0$, where $n_2^* = 4\alpha\beta^4/b$ is the corresponding optimal sample size.

To conduct Monte Carlo simulations, we generated pseudorandom samples from a gamma population with $\alpha = 2$ and $\beta = 1$. In this situation, a numerical approximation of the second-order expansion in (2.3.6) yields $\mathsf{E}[N_2 - n_2^*] \approx -3.5258$. Again, we fixed the pilot sample size $m = 20$, and considered a wide range of values of b including $0.160, 0.080, 0.040, 0.016, 0.008$ and 0.004 so that n_2^* turned out 50, 100 (small), 200, 500 (moderate), and 1000, 2000 (large) accordingly. After running the procedure 10,000 times independently, we summarize the results in Table 2.2.

Table 2.2: Simulations from $\Gamma(2,1)$ with $m = 20$ under $10,000$ runs implementing the sequential estimation procedure $\mathscr{P}_2$

b	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	$\bar{n}_2 - n_2^*$	$\mathsf{V}(g(\hat{\beta}_{n_2}))$	$\mathsf{V}(g(\hat{\beta}_{n_2}))/b$
0.160	50	46.9295	18.8175	0.9386	−3.0705	0.178606	1.116289
0.080	100	95.2952	30.3808	0.9530	−4.7048	0.114880	1.435996
0.040	200	195.9255	42.3965	0.9796	−4.0745	0.051403	1.285075
0.016	500	496.2600	63.8226	0.9925	−3.7400	0.016814	1.050885
0.008	1000	995.5808	90.1453	0.9956	−4.4192	0.008248	1.031040
0.004	2000	1996.9680	126.3538	0.9985	−3.0320	0.004017	1.004328

2.3.3 The rate parameter

To estimate the rate parameter of a gamma distribution given as $g(\beta) = \beta^{-1}$, we propose the following sequential procedure $\mathscr{P}_3$:

$$N_3 = \inf\left\{n \geq m : n \geq (\alpha b)^{-1}\hat{\beta}_n^{-2}\right\}. \tag{2.3.7}$$

As was pointed out, $d = 1, a_1 = 2$ and $k_1 = -3$ so that any pilot sample size $m > 6/\alpha$ will satisfy the condition (2.2.3). In this case, the optimal sample size is $n_3^* = (\alpha b)^{-1}\beta^{-2}$. However, the stopping rule (2.3.7) can hardly be rewritten following that of Woodroofe (1977) due to the negative moment of $\hat{\beta}_n$. As a consequence, we leave out the second-order analysis here.

Monte Carlo simulations have been carried out to demonstrate the remarkable performance of the sequential estimation procedure $\mathscr{P}_3$. We generated pseudorandom observations from a gamma distribution with $\alpha = 2$ and $\beta = 1$. We set the pilot sample size $m = 20$ so that it was large enough. We took into consideration the values of b including 0.010, 0.005, 0.0025, 0.0010, 0.0005, and 0.00025, respectively, so that again the optimal sample size n_3^* could be determined as 50, 100 (small), 200, 500 (moderate), and 1000, 2000 (large) accordingly. The summaries from 10,000 independent runs are recorded in Table 2.3.

Table 2.3: Simulations from $\Gamma(2,1)$ with $m = 20$ under $10,000$ runs implementing the sequential estimation procedure $\mathscr{P}_3$

b	n_3^*	$\bar{n}_3$	$s(n_3)$	$\bar{n}_3/n_3^*$	$\bar{n}_3 - n_3^*$	$\mathsf{V}(g(\hat{\beta}_{n_3}))$	$\mathsf{V}(g(\hat{\beta}_{n_3}))/b$
0.010	50	50.8874	9.9734	1.0177	0.8874	0.010076	1.007606
0.005	100	100.9992	14.0009	1.0010	0.9992	0.004965	0.992902
0.0025	200	200.0173	19.8177	1.0051	0.0173	0.002467	0.986840
0.0010	500	500.9509	31.2984	1.0019	0.9509	0.000982	0.981624
0.0005	1000	1001.2121	44.5421	1.0012	1.2121	0.000497	0.993567
0.00025	2000	2000.2164	63.4808	1.0001	0.2164	0.000252	1.007719

2.3.4 The survival probability

One may also be interested in survival probability as a function of β, say $g(\beta) = \Pr(X > c)$, where c is a positive constant. The corresponding sequential estimation procedure $\mathscr{P}_4$ can be then constructed by

$$N_4 = \inf\left\{ n \geq m : n \geq \frac{c^{2\alpha}\hat{\beta}_n^{-2\alpha}e^{-2c/\hat{\beta}_n}}{\{\Gamma(\alpha)\}^2\alpha b} \right\}. \tag{2.3.8}$$

As was pointed out, $d = 2$ and $\min\{k_1, k_2\} = -(\alpha+3)$ so that any pilot sample size $m > 2 + 6/\alpha$ will satisfy the condition (2.2.3). In this scenario, the optimal sample size is $n_4^* = \frac{c^{2\alpha}\beta^{-2\alpha}e^{-2c/\beta}}{\{\Gamma(\alpha)\}^2\alpha b}$. Similarly, we are not going to obtain the asymptotic second-order efficiency as the stopping rule (2.3.7) can hardly be rewritten in the spirit of Woodroofe (1977).

For Monte Carlo simulations, we generated pseudorandom observations from a gamma distribution with $\alpha = 2$ and $\beta = 2$ to estimate the probability $\Pr(X > 3)$, that is, the constant $c = 3$. A pilot sample size of $m = 20$ would be large enough in this situation. To make the optimal sample size n_4^* range among the values including 50, 100 (small), 200, 500 (moderate), and 1000, 2000 (large), we computed the corresponding values of b in turn.

That is, $b = 0.00252, 0.00126, 0.00063, 0.000252, 0.000126$ and 0.000063. The simulated findings from 10,000 independent runs are summarized in Table 2.4.

Table 2.4: Simulations from $\Gamma(2,2)$ with $c = 3$ and $m = 20$ under $10,000$ runs implementing the sequential estimation procedure $\mathscr{P}_4$

b	n_4^*	$\bar{n}_4$	$s(n_4)$	$\bar{n}_4/n_4^*$	$\bar{n}_4 - n_4^*$	$\mathsf{V}(g(\hat{\beta}_{n_4}))$	$\mathsf{V}(g(\hat{\beta}_{n_4}))/b$
0.00252	50	50.0294	4.9539	1.0006	0.0294	0.002606	1.033928
0.00126	100	100.0929	7.0023	1.0009	0.0929	0.001265	1.003583
0.00063	200	200.1026	9.9768	1.0005	0.1026	0.000635	1.007231
0.000252	500	500.0495	15.7865	1.0001	0.0495	0.000252	1.000772
0.000126	1000	1000.1743	22.3803	1.0002	0.1743	0.000127	1.006751
0.000063	2000	1999.7089	31.6515	0.9999	−0.2911	0.000063	1.003911

2.4 Real Data Analysis

In this section, we implemented the developed sequential techniques in a real-world study. Along the lines of Roughani and Mahmoudi (2015) and Mahmoudi and Roughani Shahraki (2015), we further analyzed the dataset containing weights of 346 babies born between September 23, 2014, and November 21, 2014, in Imam Ali Hospital of Shahrekord, Iran, and obtained more estimates in addition to that of the gamma scale parameter in those papers. It has been justified that these weights can be considered as a random sample from some normal population. The full data with summary information can be found in Roughani and Mahmoudi (2015).

Based on the normal-gamma transformation proposed in Zacks and Khan (2011), if $X_1, ..., X_n$ are (i.i.d.) normal random variables with mean μ and variance σ^2, then

$$Y_i = \frac{i}{i+1}(X_{i+1} - \bar{X}_i)^2,\ 1 \leq i \leq n-1$$

are independent random variables from a gamma distribution with a known shape $\alpha = 1/2$ and unknown scale $\beta = 2\sigma^2$. We performed such transformation onto the data to get a random sample of 345 observations from a gamma population with $\alpha = 1/2$ and some unknown β. From here, we proceeded the sequential stopping rules (2.3.1), (2.3.4), (2.3.7) and (2.3.8) to estimate the mean, variance, rate parameter, and survival probability. The estimation results are summarized in Table 2.5.

N_1 : estimate the mean from (2.3.1);

N_2 : estimate the variance from (2.3.4);

N_3 : estimate the rate parameter from (2.3.7);

N_4 : estimate the survival probability with $c = 1$ from (2.3.8).

Table 2.5: Illustration with data on weights of babies

	N_1			N_2			N_3			N_4		
m	b	N	est.	b	N	est.	b	N	est.	b	N	est.
5	0.001	62	0.174	0.001	70	0.092	0.1	130	2.564	0.0001	89	0.034
	0.0005	89	0.145	0.0005	88	0.072	0.05	286	2.703	0.00005	161	0.025
10	0.001	68	0.185	0.001	69	0.092	0.1	136	2.632	0.0001	88	0.034
	0.0005	95	0.155	0.0005	87	0.072	0.05	281	2.631	0.00005	159	0.025
15	0.001	69	0.185	0.001	30	0.058	0.1	130	2.564	0.0001	74	0.027
	0.0005	96	0.155	0.0005	73	0.068	0.05	283	2.632	0.00005	158	0.025
20	0.001	73	0.190	0.001	31	0.061	0.1	139	2.632	0.0001	76	0.026
	0.0005	114	0.170	0.0005	75	0.068	0.05	281	2.631	0.00005	158	0.025

Having fixed choices for $m = 5, 10, 15, 20$ and varied the value of b, we incorporated the four estimation situations. The terminal sample sizes N are reported along with the corresponding final estimates.

2.5 Conclusions

In this chapter, we propose a sequential sampling procedure for estimating $g(\beta)$, a general function of the scale parameter β in a gamma distribution, with its variance bounded by a prefixed small level, given that the shape parameter α is known. On the one hand, it has extended the work of Mahmoudi et al. (2019) which focused on sequential point estimation of β alone; and on the other hand, we have relaxed the conditions imposed on the general function $g(\cdot)$ compared with Mukhopadhyay and Wang (2019) and Mukhopadhyay (2021), making the procedure more practically applicable.

We follow up with four interesting illustrations with the function $g(\beta)$ substituted with the gamma mean, the gamma variance, the gamma rate parameter, and a survival probability, respectively. Appealing first-order and/or second-order properties are exhibited, and Monte Carlo simulations are conducted to demonstrate the remarkable performance of the procedures. Finally, a real data study further shows the potential utilization of our newly developed sequential estimation.

CHAPTER THREE

SEQUENTIAL CONFIDENCE INTERVALS FOR COMPARING TWO PROPORTIONS WITH APPLICATIONS IN A/B TESTING

3.1 Introduction

A/B testing is a statistical strategy for comparing two or more versions of a variable to see which one performs better. Participants are assigned to distinct groups at random, are exposed to varying versions (A and B), and the outcomes are analyzed to determine whether any of the differences are statistically significant. This technique is an example of statistical hypothesis testing, which is the process of forming a hypothesis regarding the relationship that exists between two sets of data. After then, the data sets are compared to figure out whether there is a relationship that is statistically significant. A/B testing helps companies and researchers make choices about product features, marketing strategies, or user experiences based on data, which leads to better results based on real-world evidence. For example, Brata and Brata (2020) used A/B testing alongside evaluation of users' mental models to improve the user experience of a Japanese language mobile learning application. Kohavi et al. (2009) provide insights into how Microsoft utilizes online controlled experiments (A/B tests) to make data-driven decisions in product development. Additionally, many companies regularly share case studies and blog posts detailing how A/B testing has helped them make decisions and get better results.

A/B testing is a powerful methodology applied to estimate two proportions in diverse domains, including applications in the mobile gaming industry and credit card banking services. In the context of mobile games, A/B testing can be employed to compare different game features or user interfaces, aiming to identify the version that leads to a

higher proportion of player engagement, retention, or in-app purchases. For example, in our study, we will use the Cookie Cats to apply A/B testing. In Cookie Cats game, as players move through the game's stages, they will periodically meet gates that demand them to either wait for an extended length of time or make an in-app purchase in order to continue. These gates aren't just a way to drive in-app sales; they also provide players a much-needed break, increasing and extending their pleasure of the game. These gates must be appropriately placed. An A/B test can assist in determining the best positions for these gates. We'll look at an A/B test in Cookie Cats where we changed the first gate from level 30 to level 40. We will specifically look at how this modification affects player retention. Similarly, in the realm of credit card banking, A/B testing can be used to assess the effectiveness of different marketing strategies or user interfaces for online banking platforms. For instance, one might compare the proportion of users who complete a credit card application process or engage in specific financial activities. This approach enables financial institutions to refine their services based on statistical evidence, ultimately enhancing user satisfaction and the overall performance of their credit card offerings.

In both mobile gaming and credit card banking, where A/B testing serves as a pivotal tool for refining user experiences and optimizing strategies, the subsequent step of calculating confidence intervals for the estimated proportions becomes equally significant. Once A/B testing identifies the version that yields a superior outcome, the application of confidence intervals provides a quantitative measure of the precision and reliability of those estimates.

Confidence intervals play a fundamental role in the interpretation of proportions in statistical analysis. A confidence interval provides a range of numbers within which the true proportion of a population having a specific attribute is expected to fall, together with a set level of confidence. Estimating the percentage of website users who click a link is an example of one proportion. CIs measure uncertainty about observed differences between

groups in two proportions, such as A/B testing. These intervals help analysts evaluate their estimates and decide on the statistical significance and practical relevance of observed effects, improving research and experimental interpretation.

It can be stated that there are many articles on confidence intervals for one or two proportion such as Tango (1998) considers a model for the difference of two proportions in a paired or matched design of clinical trials, case-control studies and also sensitivity comparison studies of two laboratory tests; Biggerstaff (2008) present confidence intervals for the difference of two binomial proportions estimated from pooled samples with unequal pool sizes, Chan and Zhang (1999) proposed test-based methods of constructing exact confidence intervals for the difference in two binomial proportions; Tang et al. (2010) construct several explicit asymptotic two-sided confidence intervals for the difference between two correlated proportions using the method of variance of estimates recovery; Soms (1989) apply some results to obtain exact unconditional confidence intervals for the difference between two proportions, Zhou and Qin (2005) derive an Edgeworth expansion for the studentized difference between the two correlated sample proportions, Carlin and Doyle (2001) explain how CI are used for making comparisons and discuss the connection between the calculations and the corresponding hypothesis tests, Reiczigel et al. (2008) propose a method to construct exact two-dimensional joint confidence sets (CS) for the two unknown probabilities,based on two independent samples, Morisette and Khorram (1998) describe the calculation of an exact equal-tail confidence interval for the proportions of correctly classified pixels, revisited the problem of interval estimation of a binomial proportion by Brown et al. (2001), Brown et al. (2002) address the classic problem of interval estimation of a binomial proportion, Pan (2002) first discuss interval estimation for a binomial proportion based on one observed sample, then for the difference of two binomial proportions based on two independent samples, Wang (2010) construct the

optimal one-sided $1-\alpha$ confidence interval of form $[L(X,Y),1]$ for p_1-p_0 and to discuss its application and a generalization to other discrete sample spaces.

In many statistical problems, predetermined accuracy requirements (like fixed-width confidence intervals) make fixed-sample-size methods unfeasible. Therefore, sequential, or multistage sampling, which adjust sample size based on data, becomes essential for achieving desired accuracy. This is especially useful in areas needing quick decisions, like clinical trials, manufacturing quality control, etc, leading to timely, accurate estimates with reduced resource and time expenditure. In the context of Sequential Confidence Intervals for one or two Proportions, Frey (2010) developed sequential methods for obtaining fixed-width confidence intervals for proportion; Erazo and Goldsman (2023) achieved a confidence interval of a predetermined width while optimizing sampling expenses, considering that the cost of observations may vary between the two distributions; Yaacoub et al. (2019) investigated a sequential CI for a Bernoulli proportion p but with the new twist that the CI is tandem-width; Shan (2020) proposed using importance sampling to calculate confidence intervals that almost always guarantee the coverage. For each confidence interval method, Erazo et al. (2023) analyzed the achieved coverage and explore the balance between the number of observations and the stages needed to achieve the desired width of the confidence interval.

In this chapter, we examine sequential confidence intervals for comparing two proportions, with a focus on A/B testing applications. We discuss the importance of confidence intervals in interpreting A/B testing results within mobile gaming sector. The chapter is structured as follows: Section 3.2 introduces fixed-width confidence intervals for the ratio of two proportions and presents simulated study results to validate these methods. Section 3.3 extends this discussion to fixed-width confidence intervals for the odds ratio, again supported by simulated studies. In Section 3.4, we apply these statistical tools to A/B testing in mobile games, using a case study of the game Cookie Cats to illustrate the

practical implications of gate placement on player retention. Finally, Section 3.5 provides conclusions, summarizing the findings and highlighting the significance of sequential confidence intervals in making data-driven decisions through A/B testing.

3.2 Fixed-Width Confidence Intervals for the Ratio of Two Proportions

Suppose we are interested in some common characteristic, referred to as *success*, possessed by two independent dichotomous populations, say X and Y. The success probabilities are given by p_1 and p_2, respectively, where $0 < p_i < 1, i = 1,2$. Our goal is to compare their magnitudes and determine if one is significantly greater than the other.

Assume that we have collected random samples $X_1,...,X_{n_1}$ and $Y_1,...,Y_{n_2}$ from X and Y, respectively, where the sample sizes n_1 and n_2 are not necessarily the same. Then, X_i's are *independent and identically distributed* (i.i.d.) Bernoulli(p_1) random variables, and the sample proportion $\bar{X}_{n_1} = n_1^{-1}\sum_{i=1}^{n_1} X_i$ serves as an unbiased estimator of p_1. However, although $0 < p_1 < 1$, there is a positive probability that $\bar{X}_{n_1}$ equals 0 or 1 in a given sample. To avoid this, we adopt the "plus four" idea put forward by Agresti and Caffo (2000), and propose the following biased but consistent estimator

$$\hat{p}_{1,n_1} = \frac{\sum_{i=1}^{n_1} X_i + 1}{n_1 + 2}. \tag{3.2.1}$$

Observe that $\hat{p}_{1,n_1}$ can be treated as a weighted average of $\bar{X}_{n_1}$ (the sample proportion) and $1/2$ (a naïve estimator of p_1), and is therefore always strictly between 0 and 1. Similarly, we estimate p_2 by

$$\hat{p}_{2,n_2} = \frac{\sum_{j=1}^{n_2} Y_j + 1}{n_2 + 2}. \tag{3.2.2}$$

To compare the magnitudes of p_1 and p_2, we construct a confidence interval for p_1/p_2 (or a monotone function of p_1/p_2) with some prescribed accuracy. As p_1/p_2 is always positive, we apply the log transformation on it and the resulting quantity $\log(p_1/p_2)$ takes values on $(-\infty,\infty)$. According to the the central limit theorem and the delta method,

for $i = 1, 2$,

$$\sqrt{n_i}\left(\log \hat{p}_{i,n_i} - \log p_i\right) \xrightarrow{d} N\left(0, \sigma_i^2\right), \tag{3.2.3}$$

as $n_i \to \infty$, where $\xrightarrow{d}$ represents convergence in distribution and $\sigma_i^2 = (1-p_i)/p_i$. For large enough n_1 and n_2, we have the following approximate normality of the difference in $\log \hat{p}_{1,n_1}$ and $\log \hat{p}_{2,n_2}$:

$$\log \hat{p}_{1,n_1} - \log \hat{p}_{2,n_2} - (\log p_1 - \log p_2) \dot{\sim} N\left(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right), \tag{3.2.4}$$

where $W \dot{\sim} F$ represents that the random variable W is approximately distributed as F. This can be used to construct a large-sample approximate confidence interval for $\log p_1 - \log p_2$ to compare p_1 and p_2. For the sake of estimation precision, we predetermine both the confidence level and the width of the interval, which is therefore called a *fixed-width confidence interval* (FWCI). That is, with fixed $\alpha(>0)$ and $d(>0)$, we consider the confidence interval of the form given by

$$I_{n_1,n_2} = \left[\log \hat{p}_{1,n_1} - \log \hat{p}_{2,n_2} \pm d\right], \tag{3.2.5}$$

which further satisfies that

$$\Pr\left(\log p_1 - \log p_2 \in I_{n_1,n_2}\right) \approx 1 - \alpha. \tag{3.2.6}$$

Next, we set out to determine the minimum sample sizes needed to the meet the fixed width and coverage probability requirements. Define

$$\Delta = d^2/z^2, \tag{3.2.7}$$

where $z = z_{\alpha/2}$, the upper $100(\alpha/2)\%$ point of a standard normal distribution. From (3.2.6), the required sample size in total, $n_1 + n_2$, must satisfy that

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \leq \Delta, \tag{3.2.8}$$

To minimize the total sample size, the Cauchy-Schwarz inequality leads to

$$n_1 + n_2 \geq (\sigma_1 + \sigma_2)^2/\Delta, \tag{3.2.9}$$

with equality when $n_1/n_2 = \sigma_1/\sigma_2$. In this sense, we can specify the *optimal sample sizes* of n_1^*, n_2^*, and n^* as follows:

$$n_1^* = \sigma_1(\sigma_1+\sigma_2)/\Delta,\ n_2^* = \sigma_2(\sigma_1+\sigma_2)/\Delta,\ \text{ and }\ n^* = n_1^* + n_2^*. \tag{3.2.10}$$

We tacitly disregard the fact that n_1^*, n_2^*, or n^* may not be an integer.

As p_1 and p_2 are two unknown parameters, it is essential to estimate σ_1^2 and σ_2^2 by updating their estimators at every stage as necessary. Beginning with pilot samples $X_1, ..., X_{m_1}$ from X and $Y_1, ..., Y_{m_2}$ from Y, we propose the following sequential estimation procedure with the associated stopping rule given by

$$N = N_1 + N_2 = \inf\{n_1 + n_2 \geq m_1 + m_2 : n_1^{-1}\hat{\sigma}_{1,n_1}^2 + n_2^{-1}\hat{\sigma}_{2,n_2}^2 \leq \Delta\}, \tag{3.2.11}$$

where n_1 and n_2 indicate the numbers of observations that are taken from X and Y, respectively, and for $i = 1, 2$, $\hat{\sigma}_{i,n_i}^2 = (1 - \hat{p}_{i,n_i})/\hat{p}_{i,n_i}$ with $\hat{p}_{i,n_i}$ defined in (3.2.1)-(3.2.2). By utilizing the "plus four" modification, $\Pr\left(0 < \hat{p}_{i,n_i} < 1\right) = 1$ so that $\hat{\sigma}_{i,n_i}^2$ is well-defined with probability one (w.p.1). Suppose that at some point, we have gathered n_1 and n_2 observations from X and Y, respectively, but the stopping rule (3.2.11) is not satisfied, which implies that we should continue sampling. The question is from which population we are going to take the next observation. According to the equality condition for (3.2.9), we propose the following allocation scheme:

$$\text{If } n_1/n_2 > (\leq)\hat{\sigma}_{1,n_1}/\hat{\sigma}_{2,n_2}, \text{ collect one additional observation from } Y(X). \tag{3.2.12}$$

The sequential estimation procedure (3.2.11)-(3.2.12) is implemented as follows. With the pilot samples, if $m_1^{-1}\hat{\sigma}_{1,m_1}^2 + m_2^{-1}\hat{\sigma}_{2,m_2}^2 \leq \Delta$ has already been satisfied, we do not take any additional observations, and the final sample size is $N = m_1 + m_2$. Otherwise, we

compare m_1/m_2 with $\hat{\sigma}_{1,m_1}/\hat{\sigma}_{2,m_2}$, and pick the next observation as per (3.2.12). After obtaining the updated $\hat{\sigma}^2_{1,n_1}$ or $\hat{\sigma}^2_{2,n_2}$, we check with the boundary crossing condition (3.2.11). This process is repeated until $n_1^{-1}\hat{\sigma}^2_{1,n_1} + n_2^{-1}\hat{\sigma}^2_{2,n_2} \leq \Delta$ happens for the first time. By referring to Chow and Robbins (1965), we can claim that
$\Pr(N_1 < \infty, N_2 < \infty | p_1, p_2) = 1$, which shows that the procedure will stop w.p.1. Finally, with the fully accrued data $\{X_1, ..., X_{m_1}, ..., X_{N_1}; Y_1, ..., Y_{m_2}, ..., Y_{N_2}\}$, we construct the FWCI

$$I_{N_1,N_2} = \left[\log \hat{p}_{1,N_1} - \log \hat{p}_{2,N_2} \pm d\right] \tag{3.2.13}$$

for $\log p_1 - \log p_2$. If the interval I_{N_1,N_2} contains zero, we conclude that there is no significant difference in p_1 and p_2 at a pre-specified level of α; and if I_{N_1,N_2} contains only positive (negative) values, we conclude that $p_1 > (<) p_2$ at level $\alpha/2$.

The sequential estimation procedure (3.2.11)-(3.2.12) enjoys the following efficiency properties as summarized in Theorem 3.2.1.

Theorem 3.2.1. *Under the sequential sampling strategy* (3.2.11) *and the allocation scheme* (3.2.12)*, with* p_1, p_2, d*, and* α *fixed, as* $d \to 0$*, we have:*

$$(i) \qquad \mathsf{E}[N_1/n_1^*] \to 1, \mathsf{E}[N_2/n_2^*] \to 1, \text{ and } \mathsf{E}[N/n^*] \to 1, \tag{3.2.14}$$

$$(ii) \qquad \Pr\left(\log p_1 - \log p_2 \in I_{N_1,N_2}\right) \to 1 - \alpha, \tag{3.2.15}$$

where n_1^**,* n_2^**, and* n^* *come from* (3.2.10)*, and* I_{N_1,N_2} *comes from* (3.2.13)*.*

Proof. One can easily find that the associated stopping rule (3.2.11) and allocation scheme (3.2.12) are similar with the rule R_1 of Robbins et al. (1967). Their techniques can be applied here to justify both (3.3.11) and (3.3.12), so we omit the proof for brevity. One may refer to Srivastava (1970), Chapter 13 of Mukhopadhyay and De Silva (2008), and other sources for many details. □

3.2.1 Simulated studies

To investigate the performance of our proposed sequential estimation procedure (3.2.11)-(3.2.12), we have conducted an extensive set of Monte Carlo simulations. For illustrative purposes, we first present the results under the following settings: X and Y are Bernoulli populations with success probabilities $p_1 = 0.3$ and $p_2 = 0.2$, respectively; the level α is fixed to be 0.05 so that the confidence level is $1-\alpha = 0.95$; the pilot sample sizes are set to both 20; and a wide range of d (half width) from 0.6 to 0.1 with increment -0.1 is taken into account. For each configuration, we have run the simulation for 10,000 times, and summarize the findings In Table 3.2. We have recorded the three optimal sample sizes (n_1^*, n_2^*, n^*), the three average final sample sizes $(\bar{n}_1, \bar{n}_2, \bar{n})$ along with the standard deviations, and the three ratios $(\bar{n}_1/n_1^*, \bar{n}_2/n_2^*, \bar{n}/n^*)$ accordingly. In the last but one column, $\bar{cp}$ is the proportion of confidence intervals that successfully capture the parameter under estimation, which is to be compared with the confidence level. And in the last column, Power is referred to as the proportion of confidence intervals that successfully identify $p_1 > p_2$, that is, the proportion of confidence intervals containing positive values alone. Note that this "power" only provides a conservative estimate, because we are using a two-sided confidence interval to help make a one-sided conclusion.

Table 3.1: Simulated results with $p_1 = 0.3$, $p_2 = 0.2$, $\alpha = 0.05$, and $m_1 = m_2 = 20$ implementing the sequential estimation procedure (3.2.11)-(3.2.12) under $10,000$ runs.

d	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	n^*	$\bar{n}$	$s(n)$	$\bar{n}/n^*$	$\bar{cp}$	Power
0.6	57.50	55.63	12.52	0.9676	75.28	72.60	17.57	0.9643	132.78	128.23	27.00	0.9657	0.9515	0.2303
0.5	82.80	80.94	15.17	0.9775	108.41	105.81	21.24	0.9761	191.20	186.75	32.75	0.9767	0.9536	0.3302
0.4	129.37	127.31	18.99	0.9841	169.39	166.59	26.45	0.9835	298.76	293.91	40.89	0.9838	0.9514	0.4883
0.3	229.99	227.92	25.31	0.9910	301.13	298.53	35.57	0.9914	531.12	526.45	54.69	0.9912	0.9522	0.7460
0.2	517.48	514.83	38.15	0.9948	677.54	674.26	53.27	0.9952	1195.02	1189.09	82.33	0.9950	0.9502	0.9775
0.1	2069.93	2066.66	76.96	0.9984	2710.17	2706.13	105.80	0.9985	4780.09	4772.79	164.08	0.9985	0.9469	1.0000

From Table 3.1, we find that the three ratios $\bar{n}_1/n_1^*$, $\bar{n}_2/n_2^*$, and $\bar{n}/n^*$ are all slightly below 1. However, as d decreases, the three ratios get closer and closer to 1, which empirically verifies (3.3.11) in Theorem 3.2.1. The coverage probability averages $\bar{cp}$ are all around $1-\alpha = 0.95$, verifying (3.3.12). We also observe that Power increases rapidly as d decreases. When $d = 0.2$, one is able to conclude $p_1 > p_2$ at least 97.75% of the time; and when $d = 0.1$, this rate increases to 100%. This indicates that our proposed sequential estimation procedure (3.2.11)-(3.2.12) can help identify which proportion is larger when there does exist a difference in the two proportions for small d values.

In Table 3.1, we have considered the scenario where $p_2 < p_1 < 1/2$. Since the optimal sample size n_1^* or n_2^* is not symmetric about 1/2, we have also carried out a set of simulations when $p_1 > p_2 > 1/2$. In particular, $p_1 = 0.8$ and $p_2 = 0.7$, and a wide range of d from 0.20 to 0.05 with increment -0.05 has been considered. The findings are displayed in Table 3.2. There is little to no difference in the performance compared to that summarized in Table 3.1.

Table 3.2: Simulated results with $p_1 = 0.8$, $p_2 = 0.7$, $\alpha = 0.05$, and $m_1 = m_2 = 20$ implementing the sequential estimation procedure (3.2.11)-(3.2.12) under $10,000$ runs.

d	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	n^*	$\bar{n}$	$s(n)$	$\bar{n}/n^*$	$\bar{cp}$	Power
0.20	55.44	55.59	14.00	1.0026	72.59	71.96	15.93	0.9913	128.04	127.55	26.89	0.9962	0.9476	0.2313
0.15	98.57	98.52	19.35	0.9995	129.06	128.53	21.43	0.9959	227.62	227.05	36.75	0.9975	0.9447	0.4016
0.10	221.78	221.46	28.26	0.9986	290.38	290.01	31.56	0.9987	512.15	511.47	53.63	0.9987	0.9473	0.7455
0.05	887.11	888.64	55.87	1.0017	1161.50	1162.35	61.93	1.0007	2048.61	2050.99	105.58	1.0012	0.9503	0.9989

At last, we investigate the performance of the proposed sequential estimation procedure (3.2.11)-(3.2.12) when the two proportions are identical. In particular, we have conducted simulations under $p_1 = p_2 = 0.2$, $\alpha = 0.05$, $m_1 = m_2 = 20$ with d varying from 0.7 to 0.1. This time, note that Power has the same definition as the coverage probability, so

we combine the last two columns $\bar{cp}$ and Power in Table 3.1 and Table 3.2, and rename it "$\bar{cp}$/Power." The findings are summarized in Table 3.3, which again validate Theorem 3.2.1. We leave out many details for brevity.

3.3 Fixed-Width Confidence Intervals for the Odds Ratio

In Section 3.2, the log transformation yielded a FWCI for $\log(p_1/p_2)$, the log of the ratio of two proportions. In this section, we consider the logit transformation, which helps to construct a FWCI for $\log \frac{p_1/(1-p_1)}{p_2/(1-p_2)} = \log \frac{p_1}{1-p_1} - \log \frac{p_2}{1-p_2}$, the log of the odds ratio.

Table 3.3: Simulated results with $p_1 = p_2 = 0.2$, $\alpha = 0.05$, and $m_1 = m_2 = 20$ implementing the sequential estimation procedure (3.2.11)-(3.2.12) under $10,000$ runs.

d	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	n^*	$\bar{n}$	$s(n)$	$\bar{n}/n^*$	$\bar{cp}$/Power
0.7	62.72	59.83	15.66	0.9540	62.72	59.91	15.53	0.9552	125.44	119.74	27.93	0.9546	0.9582
0.6	85.37	82.61	18.15	0.9677	85.37	82.49	18.14	0.9663	170.73	165.10	32.55	0.9670	0.9573
0.5	122.93	120.04	21.86	0.9765	122.93	120.00	21.82	0.9762	245.85	240.04	39.10	0.9764	0.9541
0.4	192.07	189.46	27.65	0.9864	192.07	188.96	27.29	0.9838	384.15	378.42	49.39	0.9851	0.9573
0.3	341.46	338.70	36.59	0.9919	341.46	338.45	36.60	0.9912	682.93	677.15	65.49	0.9915	0.9528
0.2	768.29	764.50	54.84	0.9951	768.29	765.13	55.06	0.9959	1536.58	1529.64	98.31	0.9955	0.9478
0.1	3073.17	3067.66	109.34	0.9982	3073.17	3068.90	110.25	0.9986	6164.33	6136.55	196.40	0.9984	0.9493

We continue to use $\hat{p}_{1,n_1}$ and $\hat{p}_{2,n_2}$ defined in (3.2.1)-(3.2.2) as the point estimators of p_1 and p_2, respectively. According to the central limit theorem and the delta method, for $i = 1, 2$,

$$\sqrt{n_i}\left(\log \frac{\hat{p}_{i,n_i}}{1-\hat{p}_{i,n_i}} - \log \frac{p_i}{1-p_i}\right) \xrightarrow{d} N\left(0, \frac{1}{p_i(1-p_i)}\right), \tag{3.3.1}$$

as $n_i \to \infty$. Then, for large enough n_1 and n_2, we have

$$\begin{aligned} \log \frac{\hat{p}_{1,n_1}}{1-\hat{p}_{1,n_1}} - \log \frac{\hat{p}_{2,n_2}}{1-\hat{p}_{2,n_2}} - \left(\log \frac{p_1}{1-p_1} - \log \frac{p_2}{1-p_2}\right) \\ \dot{\sim} N\left(0, \frac{1}{n_1 p_1(1-p_1)} + \frac{1}{n_2 p_2(1-p_2)}\right), \end{aligned} \tag{3.3.2}$$

which can be used to construct a large-sample approximate confidence interval for $\log \frac{p_1}{1-p_1} - \log \frac{p_2}{1-p_2}$ to compare p_1 and p_2. With prescribed $\alpha > 0$ and $d > 0$, we consider the FWCI given by

$$J_{n_1,n_2} = \left[\log \frac{\hat{p}_{1,n_1}}{1-\hat{p}_{1,n_1}} - \log \frac{\hat{p}_{2,n_2}}{1-\hat{p}_{2,n_2}} \pm d \right], \tag{3.3.3}$$

which further satisfies that

$$\Pr\left(\log \frac{p_1}{1-p_1} - \log \frac{p_2}{1-p_2} \in J_{n_1,n_2} \right) \approx 1-\alpha. \tag{3.3.4}$$

Define $\delta_1^2 = [p_1(1-p_1)]^{-1}$ and $\delta_2^2 = [p_2(1-p_2)]^{-1}$. From (3.3.4), the required sample size in total, n_1+n_2, must satisfy that

$$\frac{\delta_1^2}{n_1} + \frac{\delta_2^2}{n_2} \le \Delta, \tag{3.3.5}$$

where Δ is define in (3.2.7). Apply the Cauchy-Schwarz inequality, and we obtain

$$n_1+n_2 \ge (\delta_1+\delta_2)^2/\Delta, \tag{3.3.6}$$

with equality when $n_1/n_2 = \delta_1/\delta_2$. In the same fashion of (3.2.10), the optimal sample sizes are

$$n_1^* = \delta_1(\delta_1+\delta_2)/\Delta,\ n_2^* = \delta_2(\delta_1+\delta_2)/\Delta,\ \text{ and }\ n^* = n_1^*+n_2^*. \tag{3.3.7}$$

Again, it is essential to estimate the unknown δ_1^2 and δ_2^2 by updating their estimators at every stage as necessary. In the spirits of (3.2.11)-(3.2.12), we propose the following sequential estimation procedure with the associated stopping rule and allocation scheme given by

$$T = N_1+N_2 = \inf\{n_1+n_2 \ge m_1+m_2 : n_1^{-1}\hat{\delta}_{1,n_1}^2 + n_2^{-1}\hat{\delta}_{2,n_2}^2 \le \Delta\}, \tag{3.3.8}$$

and

$$\text{if } n_1/n_2 > (\le)\hat{\delta}_{1,n_1}/\hat{\delta}_{2,n_2}, \text{ collect one additional observation from } Y(X), \tag{3.3.9}$$

where $\hat{\delta}^2_{i,n_i} = [\hat{p}_{i,n_i}(1-\hat{p}_{i,n_i})]^{-1}$, $i = 1,2$. Since $\Pr\left(0 < \hat{p}_{i,n_i} < 1\right) = 1$, $\hat{\delta}^2_{i,n_i}$ is well-defined w.p.1. The implementation of the sequential estimation procedure (3.3.8)-(3.3.9) is analogous with that of the procedure (3.2.11)-(3.2.12), and sampling will terminate w.p.1. After having collected the full data $\{X_1,...,X_{m_1},...,X_{N_1};Y_1,...,Y_{m_2},...,Y_{N_2}\}$, we construct the FWCI

$$J_{N_1,N_2} = \left[\log\frac{\hat{p}_{1,N_1}}{1-\hat{p}_{1,N_1}} - \log\frac{\hat{p}_{2,N_2}}{1-\hat{p}_{2,N_2}} \pm d\right] \tag{3.3.10}$$

for $\log\frac{p_1}{1-p_1} - \log\frac{p_2}{1-p_2}$. If the interval J_{N_1,N_2} contains zero, we conclude that there is no significant difference in p_1 and p_2 at level α; and if J_{N_1,N_2} contains only positive (negative) values, we conclude that $p_1 > (<) p_2$ at level $\alpha/2$.

In the spirit of Theorem 3.2.1, we state the the efficiency properties enjoyed by the sequential estimation procedure in the following theorem.

Theorem 3.3.1. *Under the sequential sampling strategy* (3.3.8) *and the allocation scheme* (3.3.9)*, with* p_1, p_2, d*, and* α *fixed, as* $d \to 0$*, we have:*

$$(i) \qquad \mathsf{E}[N_1/n_1^*] \to 1, \mathsf{E}[N_2/n_2^*] \to 1, \text{ and } \mathsf{E}[N/n^*] \to 1, \tag{3.3.11}$$

$$(ii) \qquad \Pr\left(\log\frac{p_1}{1-p_1} - \log\frac{p_2}{1-p_2} \in J_{N_1,N_2}\right) \to 1-\alpha, \tag{3.3.12}$$

where n_1^**,* n_2^**, and* n^* *come from* (3.3.7)*, and* J_{N_1,N_2} *comes from* (3.3.10)*.*

Proof. The proof will be the same with that of Theorem 3.2.1, and thus is omitted for brevity. □

3.3.1 Simulated studies

To investigate the performance of the sequential estimation procedure (3.3.8)-(3.3.9), we have conducted an extensive set of Monte Carlo simulations in the same fashion of Section 3.2.1. With the confidence level $1-\alpha = 0.95$ and pilot sample sizes $m_1 = m_2 = 20$, we have considered the following two scenarios: (i) X and Y are Bernoulli

populations with success probabilities $p_1 = 0.3$ and $p_2 = 0.2$, respectively; and (ii) X and Y are Bernoulli populations with identical success probability $p_1 = p_2 = 0.2$. We exclude the case in which X and Y are Bernoulli populations with success probabilities $p_1 > p_2 > 1/2$ since the optimal sample sizes n_1^* and n_2^* are both symmetric about 1/2. The corresponding simulated results are summarized in Tables 3.4-3.5.

Comparing the simulated results implementing the sequential estimation procedures (3.2.11)-(3.2.12) under the log transformation and (3.3.8)-(3.3.9) under the logit transformation, we find little to no difference in the performance of the coverage probability or the power. When it comes to the sample size needed, for a same d value, the former procedure requires a smaller one indicating that it can be more efficient; however, the latter has a smaller standard deviation of the sample size and thus is less variable, suggesting that it is more "robust."

Table 3.4: Simulated results with $p_1 = 0.3$, $p_2 = 0.2$, $\alpha = 0.05$, and $m_1 = m_2 = 20$ implementing the sequential estimation procedure (3.3.8)-(3.3.9) under $10,000$ runs.

d	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	n^*	$\bar{n}$	$s(n)$	$\bar{n}/n^*$	$\bar{cp}$	Power
0.8	61.33	61.75	5.70	1.0068	70.26	70.54	9.68	1.0040	131.59	132.29	13.95	1.0053	0.9596	0.2280
0.7	80.10	80.43	6.51	1.0041	91.77	92.07	11.12	1.0033	171.87	172.50	16.08	1.0036	0.9622	0.3067
0.6	109.03	109.37	7.63	1.0031	124.91	125.10	12.93	1.0015	233.93	234.46	18.74	1.0023	0.9588	0.3944
0.5	157.00	157.31	9.19	1.0020	179.86	179.88	15.44	1.0001	336.86	337.19	22.44	1.0010	0.9553	0.5365
0.4	245.31	245.59	11.54	1.0011	281.04	281.52	19.57	1.0017	526.35	527.11	28.40	1.0014	0.9521	0.7572
0.3	436.11	436.41	15.59	1.0007	499.62	499.97	26.18	1.0007	935.73	936.38	38.28	1.0007	0.9534	0.9398
0.2	981.24	981.00	23.00	0.9998	1124.15	1123.92	38.96	0.9998	2105.39	2104.93	56.68	0.9998	0.9512	0.9993
0.1	3924.95	3924.65	46.71	0.9999	4496.60	4496.08	78.45	0.9999	8421.55	8420.72	114.69	0.9999	0.9492	1.0000

Table 3.5: Simulated results with $p_1 = p_2 = 0.2$, $\alpha = 0.05$, and $m_1 = m_2 = 20$ implementing the sequential estimation procedure (3.3.8)-(3.3.9) under $10,000$ runs.

d	n_1^*	$\bar{n}_1$	$s(n_1)$	$\bar{n}_1/n_1^*$	n_2^*	$\bar{n}_2$	$s(n_2)$	$\bar{n}_2/n_2^*$	n^*	$\bar{n}$	$s(n)$	$\bar{n}/n^*$	$\bar{cp}$/Power
0.8	75.03	74.93	10.08	0.9987	75.03	74.90	10.08	0.9983	150.06	149.83	17.93	0.9985	0.9591
0.7	98.00	97.96	11.71	0.9996	98.00	97.80	11.38	0.9980	195.99	195.76	20.57	0.9988	0.9615
0.6	133.38	133.26	13.63	0.9991	133.38	133.26	13.52	0.9991	266.77	266.52	24.17	0.9991	0.9567
0.5	192.07	191.96	16.34	0.9994	192.07	192.02	16.59	0.9997	384.15	383.98	29.43	0.9996	0.9522
0.4	300.11	300.11	20.56	1.0000	300.11	299.99	20.75	0.9996	600.23	600.10	36.92	0.9998	0.9537
0.3	533.54	533.24	27.47	0.9995	533.54	533.49	27.63	0.9999	1067.07	1066.73	49.24	0.9997	0.9510
0.2	1200.46	1199.64	41.11	0.9993	1200.46	1199.51	41.07	0.9992	2400.91	2399.15	73.45	0.9993	0.9505
0.1	4801.82	4800.14	82.81	0.9996	4801.82	4800.63	82.44	0.9998	9603.65	9600.77	148.06	0.9997	0.9484

3.4 Mobile Games A/B Testing

In this section, we revisit the aforementioned mobile games A/B testing problem. We collect a data set from the Kaggle platform (`https://www.kaggle.com/code/yufengsui/datacamp-project-mobile-games-a-b-testing/notebook`), referred to as the Cookie Cats data, which consists of information on more than 90,000 users of the mobile puzzle game “Cookie Cats" made by Tactile Entertainment. The following variables are included:

- userid: a unique label that identifies each user;
- version: whether the first gate is at level 30 (gate_30, version A) or at level 40 (gate_40, version B);
- sum_gamerounds: the number of game rounds played by the user during the first week after installation;
- retention_1: whether a user came back to play the game 1 day after installation (True) or not (False);
- retention_7: whether a user came back to play the game 7 days after installation (True) or not (False).

We are interested in the measure retention_7 and use it to determine which version of the game is more successful in attracting users with a focus on player retention.

The two Cookie Cats versions can be treated as two independent Bernoulli populations. We use p_1 and p_2 to denote the 7-day retention rates of version A and version B, respectively. With fixed $\alpha = 0.05$ and $d = 0.1$, we implement both sequential estimation procedures (3.2.11)-(3.2.12) and (3.3.8)-(3.3.9) to collect the data needed for constructing FWCIs to compare the magnitudes of p_1 and p_2, which are further used to determine which version is better. Pilot samples of size 50 are taken to initiate the process. The summary of the analyses is displayed in Table 3.6.

Table 3.6: A/B testing for Cookie Cats

Procedure	N_1	N_2	$\hat{p}_1$	$\hat{p}_2$	FWCI
Sequential estimation procedure (3.2.11)-(3.2.12)	384	427	0.199	0.170	$[0.059, 0.259]$
Sequential estimation procedure (3.3.8)-(3.3.9)	581	632	0.206	0.166	$[0.167, 0.367]$

The sequential estimation procedure (3.2.11)-(3.2.12) terminates with 384 observations from version A and 427 observations from version B. The resulting FWCI for $\log p_1 - \log p_2$ is $[0.059, 0.259]$ indicating that version A with the first gate at level 30 has a significantly higher 7-day retention rate.

The sequential estimation procedure (3.3.8)-(3.3.9) requires larger sample sizes and terminates with 581 observations from version A and 632 observations from version B. The resulting FWCI for $\log \frac{p_1}{1-p_1} - \log \frac{p_2}{1-p_2}$ is $[0.167, 0.367]$ indicating that version A with the first gate at level 30 has a significantly higher 7-day retention rate, as well. Both FWCIs lead to the same conclusion that assigning the first gate in Cookie Cats at level 30 is more appealing than at level 40.

3.5 Conclusions

In this chapter, we propose a sequential sampling procedures for constructing Fixed Width Confidence Intervals FWCIs to compare the magnitude of p_1 and p_2 based on the log transformation and the logit transformation, respectively. On the hand, we adopt Agresti and Caffo (2000) "plus four" approach, resulting in a biased yet consistent estimator. This adjustment is meant to smooth out the extremes of the probabilities, making the calculation method more reliable for real-world use.

We follow up on our findings with Monte Carlo simulations, which are conducted to demonstrate the remarkable performance of the procedures. Finally, we apply these sequential sampling procedures in a practical scenario, demonstrating their utility in the A/B testing of mobile games.

CHAPTER FOUR

FIXED-ACCURACY CONFIDENCE INTERVALS FOR THE SHAPE PARAMETER OF A WEIBULL DISTRIBUTION USING RECORD DATA

4.1 Introduction

The Weibull distribution is widely used in reliability analysis and lifetime data analysis. It has a variety of shapes which allows it to model many different forms of failure mechanisms and hazard functions. For example, Keshavan et al. (1980) applied the Weibull distribution to analyze the reliability of Pyrex glass under stress, showcasing its utility in material science; Newell et al. (2002) explored the recoil compressive failure in high-performance polymers through the application of both two and four-parameter Weibull models, highlighting their effectiveness in understanding material reliability; Qureshi and Sheikh (1997) applied the Weibull distribution to model adhesive wear in metals, highlighting its usefulness in reliability engineering; Almeida (1999) applies Weibull statistics to predict coating failures, showing increased accuracy with larger system sizes and application to non-brittle materials; Li et al. (2003) applied the Weibull distribution to predict the failure probability of concrete components, using cracks of different shapes to model imperfections and verifying their approach with experimental tests. The Weibull distribution is used in many studies across different fields. We only show a few examples out of thousands of papers such as Carroll (2003) points out the Weibull model's unique usefulness in analyzing survival data by estimating event rates and survival time in clinical trials; Dahm et al. (2002) applied the Weibull distribution to model the distribution of radiocesium (137Cs) in Luxembourg's pasture soils, offering insights into its vertical and lateral spread; Heo et al. (2001) investigate parameter estimation for regional flood

frequency using a Weibull model, comparing estimators' asymptotic variances to the Cramér–Rao Lower Bound.

In statistics, a record value is often used to describe the largest value obtained from a sequence of random variables. When it comes to data collection, in some cases, researchers may only be able to collect record values or may only be interested in record values. For example, audiences are more excited to see athletes break world records; and climatologists desire to predict a rainfall that is greater than any previous ones. Chandler (1952) initiated a statistical study on record values. Ahsanullah (2004) and Arnold et al. (2011) provided thorough information on the theories and statistical inferences of record values. In the light of Chandler (1952) and Zakerzadeh and Jafari (2015), let us define the record values of a Weibull distribution.

Suppose that $X_1, X_2, ...$ is a series of independent and identically distributed (i.i.d.) random variables from a Weibull distribution, denoted by $W(\alpha, \beta)$, with the common probability density function given by

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\},\ x > 0. \tag{4.1.1}$$

The cumulative distribution function is accordingly given by

$$F(x;\alpha,\beta) = 1 - \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}. \tag{4.1.2}$$

Here, α is the scale parameter, and β is the shape parameter. Throughout the chapter, we are interested in estimating β and α is regarded as a nuisance parameter.

Then, the record value will start with the first random variable in the series, X_1. If $X_2 > X_1$, then it is the second record value; and if $X_2 \leq X_1$ but $X_3 > X_1$, then X_3 is the second record value. Moving further, for $j \geq 2$, X_j is a record if and only if its value is greater than all of the previous observations, $X_1, ..., X_{j-1}$. Hence, we can define the

sequence of the record values $\{R_n\}$ as follows:

$$R_n = X_{T_n},\ n = 1,2,..., \tag{4.1.3}$$

where T_n is the record time of the n-th record, i.e., the n-th record R_n is the T_n-th observation in the sequence of $\{X_1, X_2, ...\}$. Arnold et al. (2011) further studied the properties of record values, which is very helpful in our derivations in the next section.

4.2 Fixed-Accuracy Confidence Intervals for the Weibull Shape Parameter

With a fixed $n \geq 2$, let $\boldsymbol{R}_n = \{R_1, ..., R_n\}$ be a set of records from the Weibull distribution $W(\alpha, \beta)$. Then, the likelihood function can be expressed as

$$L(\alpha, \beta | \boldsymbol{R}_n) = f(r_n) \prod_{i=1}^{n-1} \frac{f(r_i)}{1 - F(r_i)} = \left(\frac{\beta}{\alpha}\right)^n \exp\left\{-\left(\frac{r_n}{\alpha}\right)^\beta\right\} \prod_{i=1}^{n} \left(\frac{r_i}{\alpha}\right)^{\beta-1}. \tag{4.2.1}$$

The log-likelihood function $l(\alpha, \beta | \boldsymbol{R}_n) = \log L(\alpha, \beta | \boldsymbol{R}_n)$ is therefore

$$l(\alpha, \beta | \boldsymbol{R}_n) = n \log \beta - \left(\frac{r_n}{\alpha}\right)^\beta + \sum_{i=1}^{n} (\beta - 1) \log r_i - n\beta \log \alpha. \tag{4.2.2}$$

We take the partial derivatives of the log-likelihood function (4.2.2) to derive the maximum likelihood estimators (MLEs) of α and β, which leads to

$$\hat{\alpha}_n = \frac{r_n}{n^{1/\hat{\beta}_n}} \ \text{ and } \ \hat{\beta}_n = \frac{n}{\sum_{i=1}^{n} \log(r_n / r_i)}. \tag{4.2.3}$$

In the light of Banerjee and Mukhopadhyay (2016), we consider a *fixed-accuracy confidence interval* (FACI) for the Weibull shape β based on the sequence of records $\boldsymbol{R}_n$, since β is a positive parameter. A $1 - \gamma$ FACI is defined as follows:

$$Z_n = \left[\hat{\beta}_n / d, \hat{\beta}_n d\right], \tag{4.2.4}$$

where $\hat{\beta}_n$ is the MLE of β, $0 < \gamma < 1$ is a pregiven confidence level, and $d > 1$ is a prefixed accuracy measure. To study the distribution of $\hat{\beta}_n$, we cite a useful result from Theorem 1 of Zakerzadeh and Jafari (2015) below as our Theorem 4.2.1.

Theorem 4.2.1. *Given the sequence of records $\{\boldsymbol{R}_n\}$, define $M_n = 2n\beta/\hat{\beta}_n$. Then, M_n follows a chi-squared distribution with $2(n-1)$ degrees of freedom.*

In many real life scenarios, data collection is required before one is able to do any statistical inference. And it is often a problem or even a confusion with regards to the sample size in need. That is, how many record data one should collect to get a reliable inference result? Here, we provide an exact solution. As the FACI (4.2.4) has a confidence level $1-\gamma$, applying Theorem 4.2.1 we have

$$\begin{aligned} \Pr(\beta \in Z_n) &= \Pr(\hat{\beta}_n/d < \beta < \hat{\beta}_n d) = \Pr\left(d^{-1} < \beta/\hat{\beta}_n < d\right) \\ &= \Pr\left(2n/d < \chi^2_{2(n-1)} < 2nd\right) \geq 1-\gamma, \end{aligned} \tag{4.2.5}$$

where $\chi^2_{2(n-1)}$ represents a chi-squared random variable with $2(n-1)$ degrees of freedom. We then propose a formula based on (4.2.5) as follows, which can be used to determine the smallest number of record values needed directly:

$$N = \inf\left\{n \geq 2 : \Pr\left(2n/d < \chi^2_{2(n-1)} < 2nd\right) \geq 1-\gamma\right\}. \tag{4.2.6}$$

With the acquired sample data $\{R_1, R_2, ..., R_N\}$, we compute the MLE of β and construct

$$Z_N = \left[\hat{\beta}_N/d, \hat{\beta}_N d\right]. \tag{4.2.7}$$

The major advantages of this new procedure (4.2.6) are: (i) the final confidence interval will reside within $(0,\infty)$ which is the parameter space of the Weibull shape β; (ii) the coverage probability is close to the preassigned confidence level $1-\gamma$; (iii) the sample size can be determined exactly with preassigned confidence level $1-\gamma$ and prescribed accuracy level d; and (iv) the procedure works no matter whether the Weibull scale α is known or unknown, and in fact, the estimation does not involve α.

4.3 Simulation Ideas

Simulating record values is not trivial. Nevzorov (2000) demonstrated that the expected time to observe a new record is infinite. Doostparast (2009) mentioned that one

could only expect to see about $\log n$ record observations from a sample of size n. Luckett (2013) pointed out that generating a random sample of n upper record values can be challenging because it may need to take many repeated sample points even for relatively small n.

Many researchers have turned to different algorithms for simulating record values. Teimouri and Gupta (2012) illustrated an algorithm for simulating the nth upper record from Weibull distribution through simulations of observations from gamma distributions. Luckett (2013) brought up an algorithm based on the conditional survivor function which works for distributions with a closed form inverse survivor function. Here, we focus on the simulation algorithm in the light of Wu and Tseng (2007), which discussed the relationships of upper record values from Weibull distributions with observations from standard exponential distributions. The major results are summarized as follows. Let $X_{u(1)},...,X_{u(n)}$ be a sequence of the observed upper record values from a Weibull distribution. Define $Y_{u(i)} = (X_{u(i)}/\alpha)^{\beta}$, $i = 1,...,n$. Then, $Y_{u(1)},...,Y_{u(n)}$ is a sequence of observed upper record values from a standard exponential distribution. In addition, the differences between two adjacent upper record values, $Y_{u(1)}, Y_{u(2)} - Y_{u(1)},..., Y_{u(n)} - Y_{u(n-1)}$ are i.i.d. from a standard exponential distribution.

Therefore, our simulation for a sequence of n upper record values from a Weibull(α,β) distribution can be designed according to the following algorithm:

Step 1: Generate n random observations from a standard exponential distribution, and denote them by $d_1, d_2, ..., d_n$;

Step 2: Calculate the upper record values of a standard exponential distribution according to the formulas: $Y_{u(1)} = d_1$, and $Y_{u(i)} = d_i + Y_{u(i-1)}$ for $i \geq 2$;

Step 3: Obtain the sequence of upper record values from Weibull(α,β) by computing $X_{u(i)} = \alpha Y_{u(i)}^{1/\beta}$, $i = 1,2,...,n$.

Using this proposed algorithm for generating record values, Monte Carlo simulations can be conducted to investigate the performance of the sampling procedure (4.2.6). We leave out simulated results for brevity alone.

CHAPTER FIVE

SUMMARY

5.1 Summary of Findings

This dissertation has explored the field of sequential sampling processes, addressing three crucial statistical inference problems. This research significantly contributes to the statistical inference literature by thoroughly investigating bounded variance point estimation (BVPE) for a function of the scale parameter in a Gamma distribution, fixed-width confidence interval (FWCI) estimation for comparing two independent Bernoulli proportions, and fixed-accuracy confidence interval (FACI) estimation for the shape parameter of a Weibull distribution based on record data.

Some of the most important results from each part of the research are:

Gamma Distribution with Bounded Variance: A sequential sampling procedure was proposed for estimating a function of the scale parameter in a Gamma distribution. This approach, emphasizing bounded variance, underscores the utility and efficiency of sequential methodologies in achieving desired statistical properties with minimal sample sizes.

Comparing Two Proportions in A/B Testing: The development of sequential sampling procedures for constructing FWCIs to compare two proportions offers substantial improvements in assessing outcomes in A/B testing scenarios, particularly in mobile gaming applications. This segment highlighted the practical implications of sequential analysis in real-world decision-making.

Weibull Distribution Shape Parameter Estimation: Addressing the estimation of the Weibull shape parameter using record data, the research established a sequential

sampling strategy that efficiently constructs FACIs, showcasing the versatility of sequential approaches in handling diverse statistical challenges.

5.2 Impact on the Field

This dissertation not only makes theoretical improvements, but also provides practical methods for statistical inference. The suggested approaches not only enhance the statistical literature but also offer viable frameworks for addressing estimation difficulties in many disciplines by connecting theory and practice.

CHAPTER SIX

FUTURE WORK

This dissertation opens up a wide range of study directions that could be investigated in the future. Some of these include:

Extension to Multivariate Distributions and to Other Distributions: Investigating the application of sequential estimation procedures to multivariate distributions could expand the scope of these methodologies, addressing more complex data analysis scenarios. While our focus has been on gamma and Weibull distributions, future work could explore the extension of our sequential sampling methodologies to other distributions. This could include distributions that are commonly encountered in other applied domains.

Real-World Applications: We have applied our methodologies to simulated data and some real-world scenarios. Future research could further validate these techniques through a wide range of applications, especially in industries where sequential decision-making is critical, such as healthcare, finance, and manufacturing.

Machine Learning Integration: With the rise of AI and machine learning, combining our sequential estimate methods with predictive modeling could lead to new insights and better results. In the future, researchers might look into how sequential sampling methods can be used with machine learning algorithms to make predictions more accurate and make the best use of resources when analyzing real-time data.

Finally, future study could combine ideas from different fields, like computer science, statistics, and domain-specific knowledge, to find solutions to hard problems in the real world. In the fields of environmental science or public health, for example, better

monitoring and intervention plans could be made by learning about the statistical properties of sequential analysis.

By following these paths, future research can expand upon the groundwork established by this dissertation, advancing statistical methodology and its applicability in a variety of contexts.

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