MODELING EXTREME INSURANCE LOSSES USING TRANSMUTATION AND COPULA

by

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To my family

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ABSTRACT

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Adviser: Theophilus Ogunyemi, Ph.D.

In this dissertation, we apply transmutation to the theoretical work in insurance. From our extensive literature search, this seems to be a novel piece of work with regards to the transmutation, we particularly focus on the theoretical application of the exponential, Pareto and Weibull distributions. By shedding light on this unexplored area, our findings contribute valuable insights into the broader domain of insurance studies. We also do some exploratory work with regard to future research pursuit on a combined application of copula and transmutation to insurance data.

This work starts with the application of transmutation to the exponential distribution as one of the common distributions for modeling insurance data with large losses. For example, when a hurricane strikes a coastal area, it brings a number of serious hazards. These hazards include heavy rains, high winds, a storm surge, and even tornadoes. All these hazards cumulatively cause tremendous amounts of losses. In most cases, no one insurance company can handle covering these humongous losses. Due to climate change, large losses do occur from hurricanes, tornados, fires, floods, and other serious natural hazards. These catastrophes are becoming less rare due to climate change according to meteorologists.

Because of their heavier tails than the Exponential distribution, we also look into the theory and application of transmutation to Pareto and Weibull distributions. The two distributions are some of the distributions most commonly used in modeling losses.

Risk exposure is often investigated via probability-based models. Some of these models or tools are often called risk indicators. Such risk indicators, which are key, inform actuaries about the level to which their companies are exposed to particular aspects of risks, particularly in this study, with emphasis on large losses. Accordingly, we look into the theory and application of key risk indicators, specifically Value-at-Risk (VaR), Tail Value-at-Risk ($TVaR$), and Loss Elimination Ratio (LER), under transmuted Exponential, Pareto, and Weibull distributions.

There are several actuarial concepts we consider theoretically under transmutation and different types of deductibles, policy limits, inflation, and co-insurance.

Lastly, we lay the theoretical groundwork for future research in Bayesian theory, incorporating transmutation and copula. Our experience so far in this area is that this triple combination may not be an easy undertaking for several distributions. For now, we only work on the triple combination on the exponential distribution with regard to actuarial modeling.

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CHAPTER ONE

INTRODUCTION

1.1 Background and Objectives

In actuarial science, various probability distributions are used to model insurance losses' severity and frequency and to estimate the likelihood of various loss scenarios. The frequency of losses is modeled with discrete distributions such as Poisson, Binomial, and Negative Binomial [15]. In practice, continuous distributions such as Burr, Pareto, Weibull, Exponential, Gamma, and Log-Normal are used to model insurance loss severity. There are also several composite models that combine elements of multiple distributions to provide a more flexible and accurate representation of insurance losses. For example, the compound Poisson model is a composite model that combines a Pareto distribution with a gamma distribution to model the severity of insurance claims and the Poisson model and logarithm distribution to model the frequency of claims [15]. Deductibles, policy limits, and coinsurance are all important factors to consider when modeling insurance losses. Actuaries must be aware of the impact of these factors on the distribution of losses and on the expected frequency and size of claims. They estimate the parameters of selected models considering the presence of these factors. By considering the impact of these factors, actuaries can create more accurate and reliable models, which are essential for managing risk and making informed decisions in the insurance and financial industries. One of the key actuaries' responsibilities is to evaluate the likelihood and potential impact of extreme events, such as natural disasters, terrorist attacks, and pandemics, that can cause significant financial losses for individuals, businesses, and organizations. These events are heavily skewed, hence introducing skewness and flexibility into the existing loss distributions will help to model these events adequately [13]. In statistics and actuarial science, a transmuted distribution introduced in [17] is a transformation of a probability distribution that allows for the incorporation of additional information or assumptions about the data. Sometimes a standard probability distribution may not adequately describe the data being analyzed, particularly where large losses are involved. By transforming the distribution, additional parameters can be introduced that better capture the characteristics of the data. One of such parameters is the transmutation parameter introduced by Shaw and Buckley (2007) [17]. By adjusting the parameters of a standard probability distribution, a transmuted distribution can be obtained which can better fit the data with large values. This can lead to more accurate predictions and better risk management. A transmuted distribution can be used to address situations where the data has significant skewness [7]. Thus, a transmuted distribution can be used to handle situations with extreme values in the data. By transforming the distribution, the tails of the distribution can be adjusted to better capture these extreme values, which can be important for risk management and decision-making. Researchers have applied transmuted distributions to different aspects of life. In 2009, Aryal and Tsokos conducted a study on the transmuted generalized extreme value distribution, where they explored the applications and properties of the transmuted distributions [5]. They also introduced the transmuted Weibull distribution [6] and the transmuted log-logistic distribution [4], examining their structural properties and utility in modeling diverse datasets. Khan, King, and Hudson (2017) [14] applied a transmuted generalized exponential distribution to survival data. In a study by Hassan, Wani, Shafi, and Sheikh (2020), they introduced the Lindley-Quasi Xgamma Distribution (LQXD) and analyzed its properties and applications [11].

Chhetri et. al (2017) conducted research on transmuted Pareto [8], which is extended by Aslam, Yousaf, and Ali (2020) to include mixture and censoring under Bayesian framework [7]. While these studies have contributed to the understanding and exploration of transmuted distributions, they did not explore the application of transmutation in the insurance industry. The aim of this study is to fill that gap.

1.2 Thesis Outline

The focus of this research is to apply transmutation to insurance loss distributions. Extreme insurance loss data will be used to compare the efficiency of these transmuted distributions in modeling these losses to that of the original loss distributions.

In Chapter 2, we consider the exponential and transmuted exponential distribution. We explore the application of transmuted exponential distribution to insurance loss modeling. Also, we do a comparative analysis between the exponential and transmuted exponential in assessing risks, using measures like Value-at-Risk and Tails-Value-at-Risk.

Similar to Chapter 2, Chapter 3 considers the Pareto and the transmuted Pareto distributions. We do a similar risk assessment comparison as in Chapter 2, but with Pareto distribution.

In Chapter 4, we focus on the Weibull and transmuted Weibull distributions and do a similar analysis as we did in Chapter 2 and 3.

In Chapter 5, we introduce an innovative approach that combines copula functions and transmuted distributions. We derive the likelihood function from using different deductibles and policy limits, and utilize Bayesian methodologies to find posterior distributions with both informative and non-informative priors.

Finally, Chapter 6 concludes our work and present key findings. It also proposes the direction for future research, including further exploration of Bayesian and copula-driven approaches, and extend the use of MCMC methods for parameter estimation.

CHAPTER TWO

EXPONENTIAL AND TRANSMUTED EXPONENTIAL DISTRIBUTIONS

2.1 Effects of Transmutation on Modeling with Exponential Distributions

Let X be an exponential random variable with parameter θ. The Probability Density Function (PDF) of X is given by:

$$g(x) = \frac{1}{\theta}\mathrm{e}^{-\frac{x}{\theta}}, \quad x > 0, \quad \theta > 0. \tag{2.1}$$

The Cumulative Distribution Function (CDF) is also given by:

$$G(x) = 1 - \mathrm{e}^{-\frac{x}{\theta}}, \quad x > 0. \tag{2.2}$$

Definition: The cumulative distribution function (cdf) of a transmuted random variable proposed by [17] is given by:

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad -1 \leq \lambda \leq 1 \tag{2.3}$$

where $G(x)$ is the cdf of the original distribution.

Definition: The pdf that corresponds to this transmuted random variable is also given by:

$$f(x) = g(x)\{1 + \lambda - 2\lambda G(x)\}, \tag{2.4}$$

where λ is the transmutation parameter and $g(x)$ and $G(x)$ are the pdf and cdf of the original distribution, respectively. Hence substituting (2.2) into (2.3), the CDF of the transmuted exponential is given by:

$$\begin{aligned} F(x) &= (1+\lambda)\{1 - \mathrm{e}^{-\frac{x}{\theta}}\} - \lambda(1 - \mathrm{e}^{-\frac{x}{\theta}})^2 \\ &= (1 - \mathrm{e}^{-\frac{x}{\theta}})(1 + \lambda\mathrm{e}^{-\frac{x}{\theta}}) \end{aligned} \tag{2.5}$$

and the corresponding survival function is given by

$$S(x) = 1 - F(x) = \mathrm{e}^{-\frac{x}{\theta}}\left(1 - \lambda + \lambda \mathrm{e}^{-\frac{x}{\theta}}\right) \tag{2.6}$$

Therefore, from (2.1) and (2.4), the PDF of transmuted exponential distribution is given by:

$$f(x;\theta,\lambda) = \frac{1}{\theta}\mathrm{e}^{-\frac{x}{\theta}}\left(1 - \lambda + 2\lambda \mathrm{e}^{-\frac{x}{\theta}}\right), \quad x > 0, \quad \theta > 0, \quad -1 \leq \lambda \leq 1. \tag{2.7}$$

Notice that when $\lambda = 0$, both distributions in (2.5) and (2.7) reduce to the cdf and pdf of the ordinary exponential distribution, respectively.

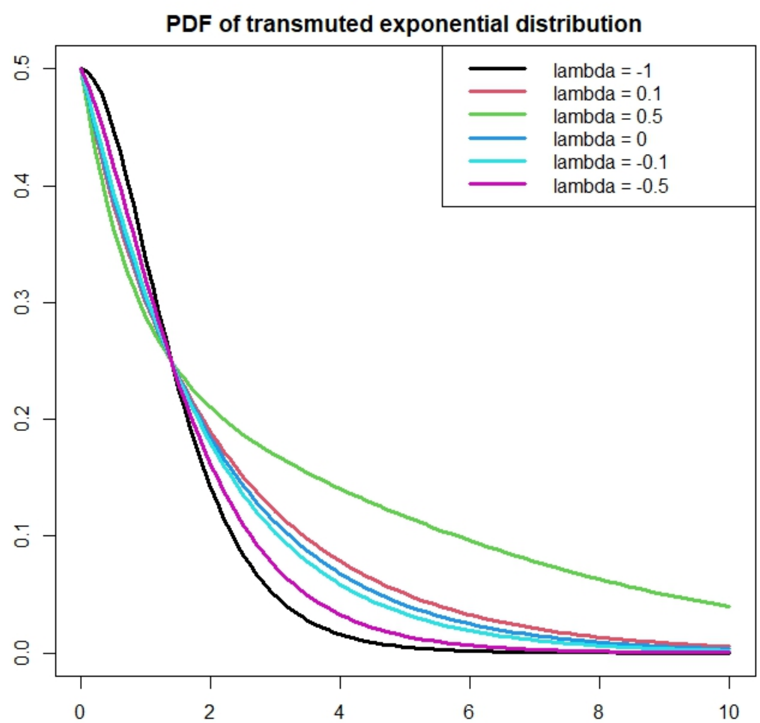

Figure 2.1: Plot of the pdf of transmuted exponential distribution with $\theta = 2$.

2.2 The Effect of Transmutation on Insurance Coverage Modifications Using the Exponential Distribution

Using the survival function of the transmuted distribution derived in (2.6), the expected value of X is given by:

$$\begin{aligned} E_\lambda[X] &= \int_0^\infty S(x;\theta,\lambda)dx \\ &= \int_0^\infty \mathrm{e}^{-\frac{x}{\theta}}\left(1-\lambda+\lambda\mathrm{e}^{-\frac{x}{\theta}}\right)dx \\ &= \left(\frac{2-\lambda}{2}\right)\theta. \end{aligned} \tag{2.8}$$

The above result in (2.8) indicates that, with transmutation, the mean of the exponential distribution gets scaled by a factor of $\left(\frac{2-\lambda}{2}\right)$.

Table 2.1: Numerical Analysis of the Expected Values of the Transmuted Exponential Distribution with different Lambdas

θ	λ	Expected Value
	-1	$15,000$
	-0.5	$12,500$
$10,000$	0	$10,000$
	0.5	$7,500$
	1	$5,000$

From 2.1 above, we observe that the transmuted exponential distributions with positive and negative λ values yield expected values that are generally different from that of the exponential distribution. The specific λ values in the transmuted exponential distribution lead to variations in the expected values, resulting in higher or lower averages compared to the exponential distribution. When $\lambda > 0$, the transmuted exponential distribution has a lower expected value, compared with that

of the exponential distribution. On the other hand, when $\lambda < 0$, the transmuted exponential distribution relatively has a higher expected value.

2.2.1 Payment Per Loss Under Ordinary Deductible

In the insurance industry, payment per loss refers to the amount that an insurance company pays out to an individual or organization when they suffer a covered loss [15]. When someone purchases an insurance policy, he or she pays a premium to the insurance company in exchange for the promise that the insurer will cover the cost of certain types of losses that they may experience. The insurance payment is typically determined by the terms of the insurance policy, which outlines the types of losses that are covered, the deductible that the policyholder is responsible for paying, and the maximum amount that the insurer will pay out for a particular type of loss which is referred to as a policy limit.

For example, let's say that someone has a car (auto) insurance policy with a \$300 deductible and a maximum payout of \$15,000 per accident. If they are involved in an accident and the cost of the damage to their car is \$5,000, they would be responsible for paying the \$300 deductible, and the insurance company would pay the remaining \$4,700 to cover the cost of the damage. It's important to note that insurance payment per loss is subject to the terms and conditions of the policy. Some policies may have limits on the amount that an insurance company will pay out for certain types of losses, or may exclude certain types of losses altogether. The deductible may be ordinary or franchise. With an ordinary deductible as explained in the example above, the insurance company pays the loss less the deductible once the loss exceeds the deductible. In a franchise deductible policy, the company pays the full loss amount once the loss exceeds the deductible. Mathematically, with an ordinary deductible of d, the payment per loss variable is defined as follows:

$$Y^L = (X-d)_+ = \begin{cases} 0, & X \leq d \\ X-d, & X > d \end{cases} \tag{2.9}$$

In general, the expected payment per loss with an ordinary deductible of d is computed as follows:

$$E\left[Y^L\right] = E[(X-d)_+] = E[X] - E[X \wedge d]. \tag{2.10}$$

Under the original exponential distribution without transmutation, (2.10) is

$$\begin{aligned} E\left[Y^L\right] &= \theta - \int_0^d S(x)dx \\ &= \theta - \int_0^d \mathrm{e}^{\frac{-x}{\theta}} dx \\ &= \theta \mathrm{e}^{\frac{-d}{\theta}}. \end{aligned} \tag{2.11}$$

Theorem 1. The k^{th} moment for a payment per loss random variable with an ordinary deductible d under the transmuted exponential distribution is given by

$$E[(X-d)^k] = 2^{-k}\theta^k \Gamma(k+1)\mathrm{e}^{\frac{-2d}{\theta}}\left(\lambda + (1-\lambda)2^k \mathrm{e}^{\frac{d}{\theta}}\right), \quad k > -1.$$

Proof.

$$\begin{aligned} E[(X-d)^k] &= \int_d^\infty (x-d)^k f(x)dx \\ &= \int_d^\infty (x-d)^k \frac{1}{\theta}\mathrm{e}^{-\frac{x}{\theta}}\left(1-\lambda+2\lambda \mathrm{e}^{-\frac{x}{\theta}}\right)dx \end{aligned}$$

Let: $x - d = t \implies dx = dt$

Therefore,

$$\begin{aligned}
E[(X-d)^k] &= \frac{1}{\theta}\int_0^\infty t^k e^{-\frac{(t+d)}{\theta}}\big(1-\lambda+2\lambda e^{-\frac{(t+d)}{\theta}}\big)dt \\
&= \frac{1}{\theta}e^{-\frac{2d}{\theta}}\int_0^\infty t^k e^{-\frac{t}{\theta}}\left(2\lambda e^{-\frac{t}{\theta}}+(1-\lambda)e^{\frac{d}{\theta}}\right)dt \\
&= \frac{\Gamma(k+1)\theta^k e^{-\frac{2d}{\theta}}\left((1-\lambda)2^k e^{\frac{d}{\theta}}+\lambda\right)}{2^k} \\
&= 2^{-k}\theta^k\Gamma(k+1)e^{\frac{-2d}{\theta}}\left(\lambda+(1-\lambda)2^k e^{\frac{d}{\theta}}\right)
\end{aligned}$$

□

Theorem 2. If $k=1$, under the transmuted exponential distribution, the expected payment per loss is $\theta e^{\frac{-d}{\theta}}\left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2}\right)$.

Proof.

$$\begin{aligned}
E\big[Y^L\big] &= E[X]-E[X\wedge d] \\
&= \left(\frac{2-\lambda}{2}\right)\theta-\int_0^d S(x)dx \\
&= \left(\frac{2-\lambda}{2}\right)\theta-\int_0^d e^{\frac{-x}{\theta}}\big(1-\lambda+\lambda e^{\frac{-x}{\theta}}\big)dx \\
&= \theta e^{\frac{-d}{\theta}}\left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2}\right)
\end{aligned} \tag{2.12}$$

□

We observe from (2.11) and (2.12) that, with transmutation, the expected payment per loss of the original distribution gets scaled by a factor of $\left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2}\right)=1-\lambda\left(1-\frac{1}{2}e^{\frac{-d}{\theta}}\right)$. For $-1\le\lambda<0$, $E[Y^L]$ for the ordinary exponential distribution gets scaled up.

Table 2.2: Expected Payment Per Loss for the Transmuted Exponential and Exponential Distribution with Different Lambdas at $d=500$

θ	λ	Transmuted Exponential	Exponential
10,000	-1	14,500	
	-0.5	12,006	
	0	9,512	9,512
	0.5	7,018	
	1	4,524	

From Table (2.2), we observe that with $\theta = 10,000$, the expected payment per loss for the transmuted exponential distribution is greater than that of exponential for negative $\lambda's$.

2.2.2 Payment Per Loss under Franchise Deductible

The expected payment per loss with a franchise deductible d is defined as follows:

$$Y^L = \begin{cases} 0, & X \leq d \\ X, & X > d. \end{cases} \tag{2.13}$$

With a franchise deductible of d, the expected payment per loss is calculated as

$$\begin{aligned} E[Y^L] &= E[X|X > d] \\ &= E[(X - d + d)|X > d] \\ &= E[(X - d)_+] + d[1 - F_X(d)]. \end{aligned} \tag{2.14}$$

Under exponential distribution, (2.14) is

$$E[Y^L] = \theta e^{\frac{-d}{\theta}} + d[e^{\frac{-d}{\theta}}] = e^{\frac{-d}{\theta}}(\theta + d).$$

Theorem 3. The expected payment per loss for the transmuted exponential distribution for a franchise deductible policy is $e^{\frac{-d}{\theta}}\left(\theta(\frac{2-\lambda}{2}) + d(1 - \lambda + \lambda e^{\frac{-d}{\theta}})\right)$.

Proof. Applying (2.14) and the result in Theorem 1,

$$\begin{aligned} E\big[Y^L\big] &= E[(X-d)_+] + d[1-F_X(d)] \\ &= \theta e^{\frac{-d}{\theta}} \left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2} \right) + d \left[e^{\frac{-d}{\theta}} \big(1-\lambda+\lambda e^{\frac{-d}{\theta}}\big) \right] \\ &= e^{\frac{-d}{\theta}} \left[\frac{\theta(2-\lambda) + 2d(1-\lambda+\lambda e^{\frac{-d}{\theta}})}{2} \right] \\ &= e^{\frac{-d}{\theta}} \left[\theta\big(\frac{2-\lambda}{2}\big) + d(1-\lambda+\lambda e^{\frac{-d}{\theta}}) \right]. \end{aligned} \tag{2.15}$$

□

Comparing $E[Y^L]$ for the exponential distribution under a franchise deductible policy and (2.15), we observe that with transmutation, both θ and d get scaled up by a factor $\frac{2-\lambda}{2}$ and $(1-\lambda+\lambda e^{\frac{d}{\theta}})$, respectively, for $-1 \leq \lambda < 0$.

2.2.3 <u>Payment Per Payment (Y^P)</u>

2.2.3.1 <u>Payment Per Payment under Ordinary Deductible:</u> The payment per payment variable is defined as $Y^P = Y^L | Y^L > 0$. Thus

$$Y^P = (X-d)|X > d = \begin{cases} undefined, & X \leq d \\ X-d, & X > d. \end{cases} \tag{2.16}$$

The expected payment per payment from the insurer's point is calculated as follows:

$$\begin{aligned} E[Y^P] &= \frac{E[X] - E[X \wedge d]}{1-F_X(d)} \\ &= \frac{E[(X-d)_+]}{1-F_X(d)}. \end{aligned} \tag{2.17}$$

From actuarial notes, the expected payment per payment under exponential distribution is given by $E[Y^P] = \theta$,

Theorem 4. The expected payment per payment by an insurer under a transmuted exponential distribution is $\theta \left[\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})} \right]$.

Proof. From (2.6), (2.12) and (2.17)

$$
\begin{aligned}
E[Y^P] &= \frac{E[(X-d)_+]}{1-F_X(d)} \\
&= \frac{\theta e^{\frac{-d}{\theta}}\left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2}\right)}{e^{-\frac{x}{\theta}}\left(1-\lambda+\lambda e^{-\frac{x}{\theta}}\right)} \\
&= \frac{\theta\left(2\theta e^{\frac{-d}{\theta}}(1-\lambda)+\lambda\theta e^{\frac{-2d}{\theta}}\right)}{2e^{\frac{-2d}{\theta}}(1-\lambda+\lambda e^{\frac{-d}{\theta}})} \\
&= \theta\left[\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right].
\end{aligned}
\tag{2.18}
$$

□

We observe that, for an exponential distribution, given that a loss has exceeded the deductible, the expected payment per payment in (2.17) for the exponential distribution is the mean of the original distribution. This is due to the memoryless property of the exponential distribution. The result in (2.18) shows that the transmuted distribution overcomes the memoryless property of the original distribution, so given that a loss amount has exceeded the deductible, the expected payment per payment is the mean of the original distribution multiplied by a factor of $\left\{\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right\}$. Again, notice that for $-1 \leq \lambda < 0$ results in (2.18) being greater than θ.

Table 2.3: Numerical Analysis of Payment per Payment for a Transmuted Exponential Distribution with $d = 500$

θ	λ	PMT per PMT (Y^P)
	-1	$14,535$
	-0.5	$12,321$
$10,000$	0	$10,000$
	0.5	$7,562$
	1	$5,000$

As mentioned earlier, $E[Y^P]$ for the standard exponential distribution is the same as θ due to its memoryless property. However, the numerical results in Table (2.3) show that the expected payment per payment for the transmuted exponential distribution varies, depending on the choice of λ.

2.2.3.2 Payment Per Payment (Y^P) under Franchise Deductible: For a franchise deductible policy, the payment per payment of an exponential distribution is given by $(\theta + d)$.

Theorem 5. For a franchise deductible of d, the expected payment per payment of a transmuted exponential distribution is $\theta\left(\frac{2-\lambda}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right) + d$.

Proof. Applying (2.15) and Theorem 2,

$$
\begin{aligned}
E[Y^P] &= \frac{E[(X-d)_+]}{1-F_X(d)} \\
&= \frac{\theta\left(2e^{\frac{-d}{\theta}}(1-\lambda) + \lambda e^{\frac{-d}{\theta}}\right) + 2de^{\frac{-d}{\theta}}\left(1-\lambda+\lambda e^{\frac{-d}{\theta}}\right)}{2e^{\frac{-d}{\theta}}\left(1-\lambda+\lambda e^{\frac{-d}{\theta}}\right)} \qquad (2.19) \\
&= \theta\left(\frac{2-\lambda}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right) + d.
\end{aligned}
$$

$\square$

From (2.19), we observe that with transmutation, θ of the payment per payment under the exponential distribution gets scaled by a factor of $\left(\frac{2-\lambda}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right)$, while the deductible doesn't change.

2.2.4 Payment per Loss Under Policy Limit:

A policy limit is the opposite of a deductible. It is a contractual arrangement where the insurance company pays the entire loss for damages below a certain threshold, but for damages above that threshold, it only pays up to a maximum amount. This creates a right-censored random variable, which has a mixed distribution (see [15]). With a policy limit of u, the payment per loss random variable is given by

$$Y^L = X \wedge u = \begin{cases} X, & X \le u \\ u, & X > u. \end{cases} \tag{2.20}$$

The expected payment per loss under this setting is given by

$$\begin{aligned} E[Y^L] &= \int_0^u x f(x) dx + u[1 - F_X(u)] \\ &= \int_0^u S_X(x) dx. \end{aligned} \tag{2.21}$$

For an exponential distribution, (2.21) yields

$$E[Y^L] = \int_0^u e^{-\frac{x}{\theta}} dx = \theta[1 - e^{\frac{-u}{\theta}}] = \theta F(u) = \theta\big(1 - S_X(u)\big)$$

Theorem 6. The k^{th} moment for payment per loss random variable with a policy limit u under the transmuted exponential distribution is given by:

$$2^{-k}\theta^k\left[\Gamma\Big((k+1),0\Big)\big(2^k(1-\lambda)+\lambda\big) - 2^k\Gamma\Big((k+1),\frac{u}{\theta}\Big)(1-\lambda) - \lambda\Gamma\Big((k+1),\frac{2u}{\theta}\Big)\right]$$
$$+ u^k e^{-\frac{u}{\theta}}\big(1-\lambda+\lambda e^{-\frac{u}{\theta}}\big).$$

Proof.

$$
\begin{aligned}
E[(X \wedge u)^k] &= \int_0^u x^k f(x)dx + \int_u^\infty u^k f(x)dx \\
&= \int_0^u x^k \frac{1}{\theta} \mathrm{e}^{-\frac{x}{\theta}} \big(1-\lambda+2\lambda \mathrm{e}^{-\frac{x}{\theta}}\big)dx + u^k[1-F_X(u)] \\
&= 2^{-k}\theta^k \Bigg[\Gamma\Bigg((k+1),0\Bigg)\big(2^k(1-\lambda)+\lambda\big) - 2^k\Gamma\Bigg((k+1),\frac{u}{\theta}\Bigg)(1-\lambda) \\
&- \lambda\Gamma\Bigg((k+1),\frac{2u}{\theta}\Bigg)\Bigg] + u^k \mathrm{e}^{-\frac{u}{\theta}}\big(1-\lambda+\lambda \mathrm{e}^{-\frac{u}{\theta}}\big)
\end{aligned}
$$

□

Theorem 7. If $k = 1$, with a policy limit u, the expected payment per loss of a transmuted exponential distribution is

$$
(1-\lambda)\Bigg(1-\mathrm{e}^{\frac{-u}{\theta}}\Bigg)\theta + \frac{\lambda\theta}{2}\Bigg(1-\mathrm{e}^{\frac{-2u}{\theta}}\Bigg)
$$

Proof. Applying (2.6) and (2.21),

$$
\begin{aligned}
E(Y^L) &= \int_0^u S_X(x)dx \\
&= \int_0^u \mathrm{e}^{\frac{-x}{\theta}}\big(1-\lambda+\lambda \mathrm{e}^{\frac{-x}{\theta}}\big)dx \\
&= (1-\lambda)\int_0^u \mathrm{e}^{\frac{-x}{\theta}}dx + \lambda\int_0^u \mathrm{e}^{\frac{-2x}{\theta}}dx \\
&= (1-\lambda)\Bigg(1-\mathrm{e}^{\frac{-u}{\theta}}\Bigg)\theta + \frac{\lambda\theta}{2}\Bigg(1-\mathrm{e}^{\frac{-2u}{\theta}}\Bigg).
\end{aligned}
\tag{2.22}
$$

□

Observe that, when $\lambda = 0$, (2.22) reduces to (2.21).

2.2.5 Summary of Results

Table 2.4: Summary of Results for Y^L and Y^P Under Different Deductibles and Policy Limit

Policy	$E[Y_\lambda^L]$	$E[Y_\lambda^P]$
Ordinary Deductible	$\theta e^{\frac{-d}{\theta}}\left(\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2}\right)$	$\theta\left[\frac{2(1-\lambda)+\lambda e^{\frac{-d}{\theta}}}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right]$
Franchise Deductible	$e^{\frac{-d}{\theta}}\left(\theta(\frac{2-\lambda}{2}) + d(1-\lambda+\lambda e^{\frac{-d}{\theta}})\right)$	$\theta\left(\frac{2-\lambda}{2(1-\lambda+\lambda e^{\frac{-d}{\theta}})}\right)+d$
Policy Limit	$(1-\lambda)\left(1-e^{\frac{-u}{\theta}}\right)\theta+\frac{\lambda\theta}{2}\left(1-e^{\frac{-2u}{\theta}}\right)$	

2.2.6 Loss Elimination Ratio (LER) and Increased Limit Factor (ILF)

The Loss Elimination Ratio (LER) compares the expected payment when an ordinary deductible is applied to the expected payment when no deductible is applied. It is calculated as the ratio of the decrease in the expected payment with the deductible to the expected payment without the deductible (see [15]). LER is a measure used to evaluate the effectiveness of loss control measures. It represents the percentage reduction in losses that can be achieved by implementing a particular loss control measure. For example, if a company expects to have \$1,000 in losses without loss control measures, but with loss control measures, the expected losses drop to \$500, then LER would be: $(\$1,000 - \$500)/\$1,000 = 50\%$ This means that the implementation of the loss control measures is expected to reduce losses by 50%. Mathematically,

$$LER = \frac{E[X \wedge d]}{E[X]}. \tag{2.23}$$

In insurance, the Increased limit factor (ILF) is a measure used to adjust the premium for an insurance policy when the insured value exceeds the policy limit. This can occur in situations where the insured value of a property, such as a building, increases over time due to inflation, appreciation, or other factors. The purpose of the ILF is to ensure that the insured value is adequately covered by the policy limit and that the insurer is not exposed to excessive risk in the event of a loss. Without the ILF, the insured value could exceed the policy limit, leaving the insured underinsured and potentially facing financial losses in the event of a claim. Mathematically,

$$ILF = \frac{E[X \wedge u]}{E[X \wedge d]}. \tag{2.24}$$

The LER and ILF of an exponential distribution are $F_X(d)$ and $\frac{F_X(u)}{F_X(d)}$, respectively.

Theorem 8. The LER of a transmuted exponential distribution is
$LER_\lambda = \frac{S_X(d)\left(2(\lambda-1)-\lambda S_X(d)\right)}{2-\lambda} + 1.$

Proof. Applying the results in (2.22) and (2.23)

$$\begin{aligned} LER_\lambda &= \frac{\left\{\frac{\theta\left(2e^{\frac{-d}{\theta}}(\lambda-1)-\lambda(1+e^{\frac{-2d}{\theta}})+2\right)}{2}\right\}}{\frac{\theta(2-\lambda)}{2}} \\ &= \frac{e^{\frac{-d}{\theta}}\left(2(\lambda-1)-\lambda e^{\frac{-d}{\theta}}\right)}{2-\lambda} + 1 \\ &= \frac{S_X(d)\left(2(\lambda-1)-\lambda S_X(d)\right)}{2-\lambda} + 1. \end{aligned} \tag{2.25}$$

□

Theorem 9. The ILF of a transmuted exponential distribution is

$$ILF_\lambda = \frac{S_X(u)\{2(\lambda-1)-\lambda S_X(u)\}+2-\lambda}{S_X(d)\{2(\lambda-1)-\lambda S_X(d)\}+2-\lambda}$$

.

Proof. Applying the results in (2.22) and (2.24),

$$\begin{aligned}
ILF_\lambda &= \frac{\theta\{2\mathrm{e}^{\frac{-u}{\theta}}(\lambda-1)-\lambda(1+\mathrm{e}^{\frac{-2u}{\theta}})+2\}}{\theta\{2\mathrm{e}^{\frac{-d}{\theta}}(\lambda-1)-\lambda(1+\mathrm{e}^{\frac{-2d}{\theta}})+2\}} \\
&= \frac{\mathrm{e}^{\frac{-u}{\theta}}\{2(\lambda-1)-\lambda\mathrm{e}^{\frac{-u}{\theta}}\}+2-\lambda}{\mathrm{e}^{\frac{-d}{\theta}}\{2(\lambda-1)-\lambda\mathrm{e}^{\frac{-d}{\theta}}\}+2-\lambda} \\
&= \frac{S_X(u)\{2(\lambda-1)-\lambda S_X(u)\}+2-\lambda}{S_X(d)\{2(\lambda-1)-\lambda S_X(d)\}+2-\lambda}.
\end{aligned} \tag{2.26}$$

□

2.2.7 Effects of Inflation and Coinsurance

2.2.7.1 Inflation Factor: Let d be an ordinary deductible and $(1+r)$ be a uniform inflation factor. Then the expected payment per loss is given by

$$E[Y^L] = (1+r)\left\{E[X] - E\left[X \wedge \frac{d}{1+r}\right]\right\} \tag{2.27}$$

For an exponential distribution, (2.20) gives

$$E[Y^L] = (1+r)\theta\mathrm{e}^{-\frac{d}{\theta(1+r)}}. \tag{2.28}$$

Theorem 10. Given an ordinary deductible d and a uniform inflation factor $(1+r)$, the expected payment per loss of a transmuted exponential distribution is $(1+r)\theta\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left[\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right]$.

Proof. From (2.22) and (2.27)

$$\begin{aligned}
E[Y^L] &= (1+r)\left\{E[X] - E\left[X \wedge \frac{d}{1+r}\right]\right\} \\
&= (1+r)\left[\frac{\theta(2-\lambda)}{2} - \int_0^{\frac{d}{1+r}} \mathrm{e}^{\frac{-x}{\theta}}(1-\lambda+\lambda\mathrm{e}^{\frac{-x}{\theta}})dx\right] \\
&= (1+r)\theta\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left[\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right].
\end{aligned} \tag{2.29}$$

$\square$

Remark: When $\lambda = 0$, (2.29) reduces to (2.28).

2.2.7.2 Coinsurance, Inflation Factor, Deductible, and Policy Limit: Let c be a coinsurance factor. With the coinsurance modification, the insurance company pays a fraction c of the loss and the policyholder covers the remainder. When all the coverage modifications are put together, the payment per loss random variable Y^L is defined as

$$Y^L = \begin{cases} 0, & X < \frac{d}{1+r} \\ c[(1+r)X - d], & \frac{d}{1+r} \leq X < \frac{u}{1+r} \\ c(u-d), & X \geq \frac{u}{1+r}, \end{cases} . \tag{2.30}$$

where d is an ordinary deductible, u is the policy limit, and r is the inflation rate.

The expected payment per loss of the random variable defined in (2.30) is

$$E[Y^L] = c(1+r)\left[E\big(X \wedge \frac{u}{1+r}\big) - E\big(X \wedge \frac{d}{1+r}\big)\right]. \tag{2.31}$$

The expected payment per payment is also given by

$$E[Y^P] = \frac{E[Y^L]}{1 - F_X(\frac{d}{1+r})}. \tag{2.32}$$

Applying (2.31), the expected payment per loss is given by

$$c(1+r)\theta\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}} - \mathrm{e}^{-\frac{d}{\theta(1+r)}}\right], \quad \theta > 0, \quad u > d, \quad 0 < r < 1. \tag{2.33}$$

Also, the expected payment per payment as defined in (2.32) is

$$c(1+r)\theta\mathrm{e}^{\frac{d}{\theta(1+r)}}\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}} - \mathrm{e}^{-\frac{d}{\theta(1+r)}}\right] \quad \theta > 0, \quad u > d, \quad 0 < r < 1. \tag{2.34}$$

Theorem 11. For a transmuted exponential distribution, the expected payment per loss with a deductible, coinsurance, policy limit, and inflation factor is $c(1+r)\theta\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{u}{\theta(1+r)}}}{2}\right)-\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right)\right]$.

Proof. Applying (2.31) to the transmuted exponential distribution,

$$\begin{aligned}
E[Y^L] &= c(1+r)\left[E\big(X\wedge\frac{u}{1+r}\big)-E\big(X\wedge\frac{d}{1+r}\big)\right]\\
&= c(1+r)\left[\int_0^{\frac{u}{1+r}}\mathrm{e}^{\frac{-x}{\theta}}(1-\lambda+\lambda\mathrm{e}^{\frac{-x}{\theta}})dx-\int_0^{\frac{d}{1+r}}\mathrm{e}^{\frac{-x}{\theta}}(1-\lambda+\lambda\mathrm{e}^{\frac{-x}{\theta}})dx\right]\\
&= c(1+r)\theta\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{u}{\theta(1+r)}}}{2}\right)-\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right)\right].
\end{aligned} \tag{2.35}$$

□

Remark: With $\lambda = 0$, (2.35) reduces to (2.33).

Theorem 12. For a transmuted exponential distribution, the expected payment per payment with a deductible, coinsurance, policy limit and inflation factor is $\dfrac{c(1+r)\theta\mathrm{e}^{\frac{d}{\theta(1+r)}}\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{u}{\theta(1+r)}}}{2}\right)-\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right)\right]}{\left(1-\lambda+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}\right)}$.

Proof. Applying (2.32),

$$\begin{aligned}
E[Y^P] &= \frac{E[Y^L]}{1-F_X(\frac{d}{1+r})}\\
&= \frac{c(1+r)\theta\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{u}{\theta(1+r)}}}{2}\right)-\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right)\right]}{\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(1-\lambda+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}\right)}\\
&= \frac{c(1+r)\theta\mathrm{e}^{\frac{d}{\theta(1+r)}}\left[\mathrm{e}^{-\frac{u}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{u}{\theta(1+r)}}}{2}\right)-\mathrm{e}^{-\frac{d}{\theta(1+r)}}\left(\frac{2(1-\lambda)+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}}{2}\right)\right]}{\left(1-\lambda+\lambda\mathrm{e}^{-\frac{d}{\theta(1+r)}}\right)}.
\end{aligned} \tag{2.36}$$

□

Remark: With $\lambda = 0$, (2.36) reduces to (2.34).

2.2.8 Value at Risk (VaR) and Tail Value at Risk (TVaR)

2.2.8.1 Value at Risk (VaR) Value-at-risk is one of the widely used risk measurement tools in actuarial science and finance. Its fundamental definition revolves around determining the highest anticipated loss within a specified probability [10]. For example, suppose we have an insurance portfolio and we want to estimate the potential loss that the portfolio may experience over a one-year period with $\alpha = 95\%$ confidence level. Then, if we compute VaR and get a value $\$Y$ amount, it means there is a 5%, that is, $(1-\alpha)\%$ probability that the amount of the claims will exceed Y during a one-year period. Due to its pragmatic nature, VaR serves as a highly useful risk measure in numerous real-world risk management applications [10]. Formally, let X be a loss random variable, which could be an aggregate loss, such that $F_X(x)$ is the cumulative distribution function of X. Then, at $100\alpha\%$ level, $VaR_\alpha(X)$ or π_α is the $100\alpha th$ percentile of the distribution of X [15]. Mathematically,

$$VaR_\alpha(X) = \inf_{x \geq 0} \{x | F_X(x) \geq \alpha\}, \quad 0 < \alpha < 1. \tag{2.37}$$

For continuous distributions, the $VaR_\alpha(X)$ can be simplified as:

$$P[X > \pi_\alpha] = 1 - \alpha, \tag{2.38}$$

implying

$$VaR_\alpha(X) = \pi_\alpha = F^{-1}(\alpha). \tag{2.39}$$

It is important to note that $VaR_\alpha(X)$ is both left-continuous and non-decreasing with respect to α [10].

The $VaR_\alpha(X)$ of an exponential distribution at α level is given by

$$\pi_\alpha = VaR_\alpha(X) = -\theta \ln(1-\alpha)$$

.

Theorem 13. The Value-at-Risk of a transmuted exponential distribution is $\pi_{\alpha,\lambda} = \theta\Bigg(-\ln\left\{\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-\alpha)}}{2\lambda}\right\}\Bigg)$, where $0<\alpha<1$ is the confidence level, $\lambda \neq 0$ is the transmutation parameter and θ is the scale parameter.

Proof. Applying the definition of $VaR_\alpha(X)$ in (2.5) and (2.38),

$$\mathrm{e}^{-\frac{\pi_\alpha}{\theta}}\left(1-\lambda+\lambda\mathrm{e}^{-\frac{\pi_\alpha}{\theta}}\right) = 1-\alpha$$

Solving the quadratic equation above for π_α, We have

$$\pi_{\alpha,\lambda} = \theta\Bigg(-\ln\left\{\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-\alpha)}}{2\lambda}\right\}\Bigg) \tag{2.40}$$

□

Note: $\lim_{\lambda\to 0}\pi_{\alpha,\lambda} = \pi_\alpha = -\theta\ln(1-\alpha)$.

Consider the limit as $\lambda \to 0$:

$$\lim_{\lambda\to 0}\pi_{\alpha,\lambda} = -\theta\ln\lim_{\lambda\to 0}\left\{\frac{(\lambda-1)+\left((1-\lambda)^2+4\lambda(1-\alpha)^{\frac{1}{2}}\right)}{2\lambda}\right\} = \frac{0}{0}$$

Applying L'Hopital's Rule,

$$\begin{aligned}
&-\theta \ln \lim_{\lambda\to 0}\left\{\frac{\frac{d}{d\lambda}\left[(\lambda-1)+\left((1-\lambda)^2+4\lambda(1-\alpha)^{\frac{1}{2}}\right)\right]}{\frac{d}{d\lambda}(2\lambda)}\right\}\\
&=-\theta \ln \lim_{\lambda\to 0}\left\{\frac{1+\frac{1}{2}\left(-2(1-\lambda)+4(1-\alpha)\right)\left((1-\lambda)^2+4\lambda(1-\alpha)^{\frac{-1}{2}}\right)}{2}\right\}\\
&=-\theta \ln \frac{1+\frac{1}{2}\left((-2)+4(1-\alpha)\right)}{2}\\
&=-\theta \ln \frac{1-1+2(1-\alpha)}{2}\\
&=-\theta \ln(1-\alpha)=\pi_\alpha.
\end{aligned} \tag{2.41}$$

2.2.8.2 Tail Value at Risk (TVaR) Tail Value at Risk ($TVaR$), also known as Conditional Value at Risk ($CVaR$), is a risk measure that extends the concept of Value at Risk (VaR). While VaR quantifies the maximum loss with a certain level of confidence, $TVaR$ focuses on the expected loss beyond the VaR threshold [1]. In essence, $TVaR$ provides an estimate of the average loss that could occur if the portfolio experienced losses exceeding the VaR level. Mathematically,

$$TVaR_\alpha(X)=E[X|X>\pi_\alpha]=\frac{\int_{\pi_\alpha}^{\infty} xf(x)dx}{1-\alpha},\quad 0<\alpha<1. \tag{2.42}$$

Embrechts and Wang [9] derived an alternative formula for $TVaR$. From $1-\alpha=1-F_X[VaR_\alpha(X)]$, we can write $TVaR$ as

$$TVaR_\alpha(X)=VaR_\alpha(X)+\frac{1-F_X[VaR_\alpha(X)]}{1-\alpha}\{E[X|X>VaR_\alpha(X)]-VaR_\alpha(X)\}. \tag{2.43}$$

If X is continuous at $VaR_\alpha(X)$, then according to [15], (2.43) reduces to

$$TVaR=VaR_\alpha(X)+\frac{E[X]-E[X\wedge VaR_\alpha(X)]}{1-\alpha}. \tag{2.44}$$

For an exponential distribution, $TVaR_\alpha(X) = -\theta \ln(1-\alpha) + \theta$.

Theorem 14. At an α level, the $TVaR_\alpha(X)$ of a transmuted exponential distribution is

$$TVaR_{\alpha,\lambda}(X) = VaR_{\alpha,\lambda}(X) + \theta e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}} \left\{ \frac{2(1-\lambda) + \lambda e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}}}{2(1-\alpha)} \right\}.$$

Proof. Applying (2.12) and (2.42),

$$\begin{aligned} TVaR_{\alpha,\lambda}(X) &= VaR_\alpha(X) + \frac{E[X] - E[X \wedge VaR_\alpha(X)]}{1-\alpha} \\ &= VaR_{\alpha,\lambda}(X) + \frac{\theta e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}} \left\{ \frac{2(1-\lambda)+\lambda e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}}}{2} \right\}}{1-\alpha} \\ &= VaR_{\alpha,\lambda}(X) + \theta e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}} \left\{ \frac{2(1-\lambda) + \lambda e^{-\frac{VaR_{\alpha,\lambda}(X)}{\theta}}}{2(1-\alpha)} \right\}. \end{aligned} \tag{2.45}$$

□

where

$$VaR_{\alpha,\lambda}(X) = \theta \Bigg(-\ln \left\{ \frac{\lambda - 1 + \sqrt{(1-\lambda)^2 + 4\lambda(1-\alpha)}}{2\lambda} \right\} \Bigg).$$

Also, $\lim_{\lambda \to 0} TVaR_{\alpha,\lambda}(X) = TVaR_\alpha(X)$.

2.2.9 Numerical Computations

We estimate the parameters of the transmuted exponential distribution using the maximum likelihood estimation. The likelihood function is given by

$$\begin{aligned} L(X_1, X_2, \cdots, X_n | \theta, \lambda) &= \prod_{i=1}^{n} f(x; \theta, \lambda) \\ &= \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \big(1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\big) \\ &= \theta^{-n} \exp\big(-\frac{1}{\theta} \sum_{i=1}^{n} x_i\big) \prod_{i=1}^{n} \big(1 - \lambda + 2\lambda e^{-\frac{x_i}{\theta}}\big) \end{aligned} \tag{2.46}$$

Taking *log* on both sides of (2.46), we have

$$\log L(X_1, X_2, \cdots, X_n|\theta, \lambda) = -n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log\left(1-\lambda+2\lambda e^{-\frac{x_i}{\theta}}\right). \quad (2.47)$$

2.2.9.1 <u>Simulation Study</u> We perform a Monte Carlo (MC) study to estimate the parameters of the exponential and transmuted exponential distribution. We generate a sample of size $n = (50, 100, 150, 250, 300, 500, 1000)$ with $M = 10,000$ simulations, using the inverse integral transformation. In this simulation, a set of uniform random numbers, u, between 0 and 1 is generated using a random number generator. The inverse integral transformation method is then applied to convert the uniform random numbers to the desired distribution by evaluating the inverse cumulative distribution function (CDF) of the transmuted distribution. This process is repeated until the desired number of samples is generated.

Setting the transmuted cdf to u_i and solving for x_i in the following equation,

$$U_i = (1 - e^{\frac{-x_i}{\theta}})(1 + \lambda e^{\frac{-x_i}{\theta}}), \quad (2.48)$$

we obtain the following expression for x_i :

$$x_i = -\theta\left(\ln\left\{\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 + 4\lambda(1-u)}}{2\lambda}\right\}\right), \quad (2.49)$$

$i = 1, 2, \ldots, n$. And using these simulated $x_i's$ in (2.49), we estimated $\theta^{(j)}$, $j = 1, 2, \ldots, M$. To evaluate the MLEs, we consider three criteria: the root mean square error (RMSE), mean absolute error (MAE), and mean relative error (MRE), which are defined as follows:

$$RMSE = \sqrt{\frac{1}{M}\sum_{i=1}^{M}[\theta^{\hat{(i)}} - \theta]^2} \quad (2.50)$$

$$MAE = \frac{1}{M}\left|\hat{\theta^{(i)}} - \theta\right| \tag{2.51}$$

$$MRE = \frac{1}{M}\sum_{i=1}^{M}\frac{\hat{\theta^{(i)}}}{\theta} \tag{2.52}$$

Smaller values of RSME and MAE are preferred, while values closer to 1 are preferred for MRE. We also compute an estimated 95% asymptotic confidence interval (CI) for the MLEs.

Table 2.5: Simulation results, $\theta = 5,000, \lambda = 0.4$

n	**Parameter**	**Estimate**	**Lower**	**Upper**	**RMSE**	**MAE**	**MRE**
100	$\hat{\theta}_\lambda$	4734	2698	6772	1073	884	0.9467
	$\hat{\lambda}_T$	0.2600	-0.3771	0.8972	0.3539	0.2872	0.6501
150	$\hat{\theta}_\lambda$	4824	2931	6717	982	809	0.9648
	$\hat{\lambda}_T$	0.2992	-0.2784	0.8768	0.3115	0.2575	0.7479
200	$\hat{\theta}_\lambda$	4866	3046	6887	938	770	0.9733
	$\hat{\lambda}_T$	0.3180	-0.2205	0.8565	0.2867	0.2389	0.7950
300	$\hat{\theta}_\lambda$	4929	3240	6618	865	707	0.9858
	$\hat{\lambda}_T$	0.3443	-0.1468	0.8355	0.2567	0.2153	0.8608
500	$\hat{\theta}_\lambda$	4988	3442	6535	789	733	0.9976
	$\hat{\lambda}_T$	0.3699	-0.0681	0.8080	0.2255	0.1888	0.9248
1000	$\hat{\theta}_\lambda$	5041	3680	6401	695	540	1.008
	$\hat{\lambda}_T$	0.3920	0.0154	0.7686	0.1923	0.1579	0.9800

Table 2.6: Simulation results, $\theta = 10,000, \lambda = 0.35$

n	**Parameter**	**Estimate**	**Lower**	**Upper**	**RMSE**	**MAE**	**MRE**
50	$\hat{\theta}_\lambda$	9,314	4, 629	13,998	2,487	2, 031	0.9314
	$\hat{\lambda}_T$	0.1624	-0.5848	0.9095	0.4249	0.3302	0.4639
100	$\hat{\theta}_\lambda$	9,665	5,635	13,695	2,083	1,675	0.9665
	$\hat{\lambda}_T$	0.244	-0.3728	0.8611	0.3321	0.2674	0.6976
150	$\hat{\theta}_\lambda$	9,795	6,058	13,532	1,918	1,531	0.9795
	$\hat{\lambda}_T$	0.2749	-0.2810	0.8309	0.2931	0.2402	0.7856
250	$\hat{\theta}_\lambda$	9,962	6,537	13,386	1,748	1,375	0.9962
	$\hat{\lambda}_T$	0.3091	-0.1823	0.80	0.2540	0.2105	0.8831
300	$\hat{\theta}_\lambda$	10,008	6,673	13,344	1,702	1,333	1.0008
	$\hat{\lambda}_T$	0.32	-0.1526	0.791	0.243	0.2014	0.9119
500	$\hat{\theta}_\lambda$	10,087	7,022	13,152	1,566	1,205	1.0081
	$\hat{\lambda}_T$	0.34	-0.086	0.762	0.2166	0.1785	0.9660
1000	$\hat{\theta}_\lambda$	10,114	7,440	12,788	1,369	1,024	1.0114
	$\hat{\lambda}_T$	0.3485	-0.0137	0.7107	0.1848	0.1487	0.9957

2.2.9.2 Real Life Application To estimate θ and λ in (2.47), we used 'optim' function in R. The aim is to find the optimal values of the parameters that maximize the likelihood of the observed data fitting the transmuted exponential distribution. The 'optim' function requires initial values for the parameters to start the optimization process. Choosing appropriate initial values is important, as it can affect the convergence of this optimization algorithm and the accuracy of the parameter estimates. Providing sensible initial values based on prior knowledge or exploratory analysis can help improve the estimation process. We used the mean of the data as an initial value of θ and 0.5 for λ.

The data utilized to estimate these parameters pertain to payments made for workers' compensation medical benefits, contained in [15]. These payments are not tied to any specific insurance policy or set of policyholders, making them independent of any particular coverage plan. Instead, they represent individual instances of compensation for medical expenses related to work-related injuries or illnesses. The dataset comprises a random sample of 20 such payments, carefully selected to ensure a representative cross-section of the overall population of workers' compensation medical benefit payments. By drawing this random sample, the intention was to capture a diverse range of payment amounts, reflecting the variability inherent in such compensatory transactions. It should be noted that each payment in the dataset represents the full amount of the loss incurred by the injured party. This ensures that the dataset accurately reflects the financial impact of these incidents, providing a comprehensive perspective on the costs associated with workers' compensation medical benefits.

Table 2.7: Dataset

2700	8200	11500	12600	15500	16100
24300	29400	34000	38400	45700	68000
85500	87700	97400	119300	134000	188400
255800	1574300				

Table 2.8: Summary Statistics of the Dataset

Min	$Q1$	Median	Mean	$Q3$	Max
$2,700$	$15,950$	$42,050$	$142,440$	$102,815$	$1,574,300$

We obtained the following estimates using the 'optim' function:

Table 2.9: Maximum Likelihood Estimates

Parameter	Exponential Distribution	Transmuted Exponential Distribution
$\hat{\theta}$	$142,440$	$193,828.8$
$\hat{\lambda}$		0.81

Note: The maximum likelihood estimate of the exponential distribution is the sample mean.

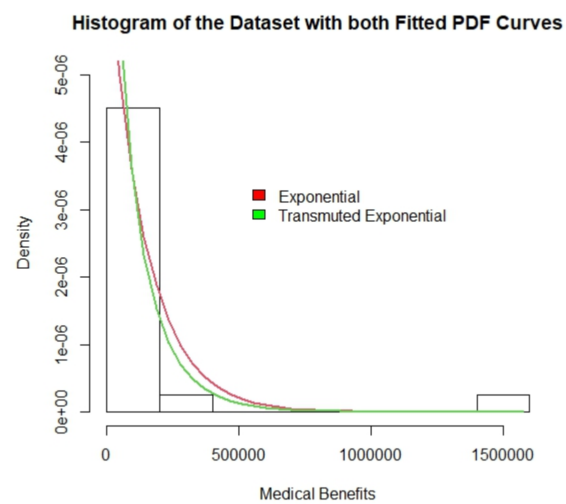

Figure 2.2: A plotted histogram with exponential and transmuted exponential distribution curves

Remarks (Figure (2.2)) :

1. Highly skewed data.

2. Transmuted exponential distribution gives a better fit.

Table 2.10: Model Comparison of Exponential and Transmuted Exponential Distributions

Model	Parameter Estimates	$-2LL$	AIC	AICC	BIC
Exponential	$\hat{\theta} = 142440$	514.66	516.66	516.88	515.96
Transmuted Exponential	$\hat{\theta} = 193828.8, \quad \hat{\lambda} = 0.81$	508.38	512.38	513.09	510.98

.

where

$$AIC = 2k - 2\log(L), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad BIC = k\log(n) - 2,$$

k is the number of parameters, n is the sample size, AIC is the Akaike Information Criterion, $AICC$ is the Akaike Information Criterion Corrected, BIC is the Bayesian Information Criterion.

From 2.10, we observe that the values of the transmuted Pareto distribution lead to a better fit than that of the Pareto distribution.

We also compare the performance of the distributions using a Likelihood Ratio Test (LRT). We test the following hypothesis:

$$H_0 : \lambda = 0 \quad versus \quad H_1 : \lambda \neq 0.$$

We obtain the following results:

$$\omega = 6.28 > \chi_1^2(\alpha = 0.05) = 3.841,$$

where $\omega = 2\left[LL_{Expo} - LL_{T.Expo}\right] = 2(257.33 - 254.19) = 6.28.$

We therefore reject the null hypothesis and conclude that incorporating the transmutation parameter provides a better fit.

Using the MLE estimates for the θ and λ, we obtain the following VaR and $TVaR$ at different risk levels:

Table 2.11: Value at Risk (VaR) at different α levels

$\alpha - level$	Exponential Distribution	Transmuted Exponential Distribution
0.1%	143	107
1%	$1,432$	$1,077$
5%	$7,306$	$5,500$
10%	$15,008$	$11,311$

Table 2.12: Tail Value at Risk ($TVaR$) at different α levels

$\alpha - level$	Exponential Distribution	Transmuted Exponential Distribution
0.1%	$142,583$	$115,444$
1%	$143,872$	$116,488$
5%	$149,746$	$121,255$
10%	$157,448$	$127,526$

From Tables 2.11 and 2.10, we observe that the transmuted exponential distribution consistently provides lower VaR estimates compared to the exponential distribution across different $\alpha - levels$. This implies that the transmuted exponential distribution is more conservative in estimating the potential losses at these confidence levels. That is, the transmuted exponential distribution is more risk-averse and assigns a higher weight to extreme events or tail risk when modeling large losses. Furthermore, the transmuted exponential distribution consistently provides lower $TVaR$ estimates compared to the exponential distribution across the chosen alpha levels. This also indicates that the transmuted exponential distribution tends to be more conservative in estimating potential losses beyond the specified confidence levels.

CHAPTER THREE

PARETO AND TRANSMUTED PARETO DISTRIBUTIONS

3.1 Coverage Modifications With Transmuted Pareto Distribution

The PDF of a Pareto distribution is given by:

$$g(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \quad x > \beta, \quad \alpha > 0. \tag{3.1}$$

The CDF of a Pareto Distribution is also given by:

$$G(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}, \quad x > \beta, \quad \alpha > 0. \tag{3.2}$$

Applying (2.3), the CDF of a Transmuted Pareto Distribution is given by:

$$\begin{aligned} F_{\lambda}(x) &= (1+\lambda)\left\{1 - \left(\frac{\beta}{x}\right)^{\alpha}\right\} - \lambda\left\{1 - \left(\frac{\beta}{x}\right)^{\alpha}\right\}^{2} \\ &= \left(1 - \left(\frac{\beta}{x}\right)^{\alpha}\right)\left\{1 + \lambda\left(\frac{\beta}{x}\right)^{\alpha}\right\} \\ &= 1 - \left(\frac{\beta}{x}\right)^{\alpha}\left\{1 - \lambda + \lambda\left(\frac{\beta}{x}\right)^{\alpha}\right\}, \quad x > \beta, \quad \alpha > 0, \quad -1 < \lambda < 1. \end{aligned} \tag{3.3}$$

Note: when $\lambda = 0$, (3.3) reduces to (3.2).

The survival function of the transmuted Pareto distribution is also given by:

$$S(x) = 1 - F(x) = \left(\left(\frac{\beta}{x}\right)^{\alpha}\right)\left\{1 - \lambda + \lambda\left(\frac{\beta}{x}\right)^{\alpha}\right\} \tag{3.4}$$

The PDF of the transmuted Pareto distribution is also given by

$$\begin{aligned} f_{\lambda}(x) &= \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\left\{1 + \lambda - 2\lambda\left(1 - \left(\frac{\beta}{x}\right)^{\alpha}\right)\right\} \\ &= \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\left\{1 - \lambda + 2\lambda\left(\frac{\beta}{x}\right)^{\alpha}\right\}, \quad x > \beta, \quad \alpha > 0, \quad -1 < \lambda < 1. \end{aligned} \tag{3.5}$$

Observe that (3.5) reduces to (3.1) when $\lambda = 0$.

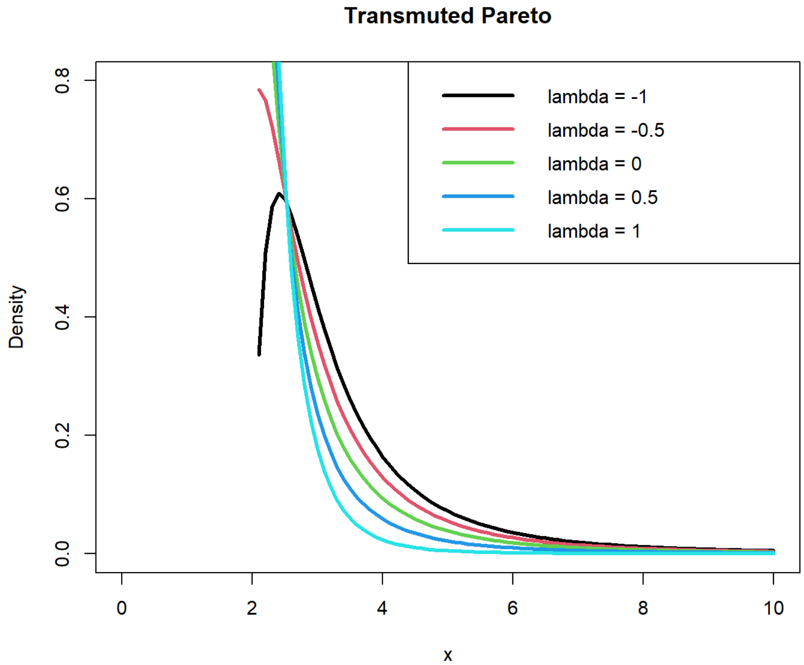

Figure 3.1: Plot of the PDF of Transmuted Pareto Distribution with $\alpha = \beta = 3$

3.1.1 Expected value of Pareto and Transmuted Pareto Distribution

Under the Pareto distribution, the expected value is given by

$$E[X] = \int_{\beta}^{\infty} f(x)dx = \int_{\beta}^{\infty} x\frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}dx = \frac{\alpha\beta}{\alpha-1}, \quad \alpha > 1 \tag{3.6}$$

The expected value of the transmuted Pareto distribution is given by

$$\begin{aligned} E_{\lambda}[X] &= \int_{\beta}^{\infty} x\frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\left\{1-\lambda+2\lambda\left(\frac{\beta}{x}\right)^{\alpha}\right\}dx \\ &= \frac{\alpha\beta(2\alpha-\lambda-1)}{(2\alpha-1)(\alpha-1)}, \quad \alpha > 1, \quad -1 < \lambda < 1. \end{aligned} \tag{3.7}$$

We observe from (3.6) and (3.7) that, with transmutation, the expected value of the Pareto distribution gets scaled up by a factor of $\frac{2\alpha-\lambda-1}{2\alpha-1}$. Also, (3.7) reduces to (3.6) when $\lambda = 0$.

Table 3.1: Numerical Analysis of the Expected Values of the Transmuted Pareto Distribution with Different Lambdas for $\alpha = 5$, and $\beta = 10,000$

λ	Expected Value
-1	$13,889$
-0.5	$13,194$
0	$12,500$
0.5	$11,806$
1	$11,111$

.

Note: The expected value of the standard Pareto Distribution with $\alpha = 5$, and $\beta = 10,000$ is $12,500$. From the table above, we observe that when $\lambda = 0$, the expected value of the Transmuted Pareto distribution is the same as that of the Pareto distribution. Also, when $\lambda > 0$, Transmuted Pareto distribution has a lower expected value, compared with that of the standard Pareto distribution. On the other hand, when $\lambda < 0$, the Transmuted Pareto distribution has a relatively higher expected value.

3.1.2 Payment Per Loss Under Ordinary Deductible

Recall the definition of Payment per Loss in (2.9). The expected value of the Pareto distribution is given by:

$$\begin{aligned} E[(X-d)_+] &= \int_d^\infty (x-d)f(x)dx \\ &= \int_d^\infty (x-d)\frac{\alpha\beta^\alpha}{x^{\alpha+1}}dx \\ &= \frac{\beta^\alpha}{(\alpha-1)d^{\alpha-1}} \\ &= \frac{\beta^\alpha d^{1-\alpha}}{(\alpha-1)}, \quad \alpha > 1. \end{aligned} \tag{3.8}$$

Theorem 15. Under the transmuted Pareto distribution, the expected payment per loss for a policy with an ordinary deductible d is
$\frac{\beta^\alpha}{\alpha-1}\left[d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda\right], \quad \alpha>1, \quad -1<\lambda<1.$

Proof. Applying (2.9) and (3.4),

$$\begin{aligned}
E[Y_\lambda^L] &= \int_d^\infty S_\lambda(x)dx \\
&= \int_d^\infty \left(\left(\frac{\beta}{x}\right)^\alpha\right)\left\{1-\lambda+\lambda\left(\frac{\beta}{x}\right)^\alpha\right\}dx \\
&= \beta^\alpha\left(\int_d^\infty x^{-\alpha}dx - \lambda\int_d^\infty x^{-\alpha}dx + \lambda\beta^\alpha\int_d^\infty x^{-2\alpha}dx\right) \\
&= \beta^\alpha\left(\frac{d^{1-\alpha}}{\alpha-1} - \frac{\lambda d^{1-\alpha}}{\alpha-1} + \frac{\lambda\beta^\alpha d^{1-2\alpha}}{2\alpha-1}\right) \\
&= \frac{\beta^\alpha}{\alpha-1}\left[d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda\right], \quad \alpha>1, \quad -1<\lambda<1.
\end{aligned} \tag{3.9}$$

□

Observe that, (3.8) gets scaled by a factor in (3.9), and when $\lambda=0$, (3.9) reduces to (3.8).

Table 3.2: Numerical Analysis of the Expected Payment Per Loss for the Transmuted Pareto Distribution with Different Lambdas at $d=500$, $\alpha=3$ and $\beta=5,000$

λ	Pareto	Transmuted Pareto
0.5		$125,000$
0.1		$225,000$
0	$250,000$	$250,000$
-0.1		$275,000$
-0.5		$375,000$

.

In Table 3.2, we compute the expected payment per loss for a Pareto and Transmuted Pareto distribution. Observe that, when $\lambda=0$, the expected payments are the same. Also, as λ decreases, the expected payment per loss for the

Transmuted Pareto distribution also increases. Thus, a negative correlation exists between the $\lambda' s$ and the expected payments.

3.1.3 Payment Per Loss Under Franchise Deductible

We defined a policy with a franchise deductible in (2.13). Under the Pareto distribution, (2.14) is

$$E[Y^L] = \frac{\beta^\alpha d^{1-\alpha}}{\alpha - 1} + d\left(\frac{\beta}{d}\right)^\alpha = \frac{\beta^\alpha d^{1-\alpha}}{\alpha - 1} + d^{1-\alpha}\beta^\alpha, \quad \alpha > 1. \tag{3.10}$$

Theorem 16. The expected payment per loss for the transmuted Pareto distribution for a franchise deductible policy is

$$\frac{\beta^\alpha}{\alpha - 1}\left[d^{1-\alpha}(1-\lambda) + d^{1-2\alpha}\lambda\right] + d^{1-\alpha}\beta^\alpha\left\{1 - \lambda + \lambda\left(\frac{\beta}{d}\right)^\alpha\right\}, \quad \alpha > 1.$$

Proof. Applying (2.13), (3.3) and (3.9),

$$\begin{aligned} E[Y^L_\lambda] &= E[(X-d)_+] + d[1 - F_X(d)] \\ &= \frac{\beta^\alpha}{\alpha - 1}\left[d^{1-\alpha}(1-\lambda) + d^{1-2\alpha}\lambda\right] + d\left(\frac{\beta}{d}\right)^\alpha\left\{1 - \lambda + \lambda\left(\frac{\beta}{d}\right)^\alpha\right\} \\ &= \frac{\beta^\alpha}{\alpha - 1}\left[d^{1-\alpha}(1-\lambda) + d^{1-2\alpha}\lambda\right] + d^{1-\alpha}\beta^\alpha\left\{1 - \lambda + \lambda\left(\frac{\beta}{d}\right)^\alpha\right\}, \quad \alpha > 1. \end{aligned} \tag{3.11}$$

□

Comparing $E[Y^L]$ under a franchise deductible policy for the Pareto distribution in (3.10) to that of the transmuted Pareto in (3.11), with transmutation, $E[Y^L]$ for the Pareto distribution gets scaled by a factor. Also, when $\lambda = 0$, (3.11) reduces to (3.10).

3.1.4 Payment Per Loss Under a Policy Limit

For a policy limit u, (2.20) and (2.21) give the definition and the expected payment per loss, respectively. For a Pareto distribution, the expected payment per loss with a policy limit of u is given by

$$E[Y^L] = \int_\beta^u \left(\frac{\beta}{x}\right)^\alpha dx = \frac{\beta^\alpha}{\alpha - 1}(\beta^{1-\alpha} - u^{1-\alpha}), \quad \alpha > 1. \tag{3.12}$$

Theorem 17. For an insurance policy with a policy limit u, the expected payment per loss under a transmuted Pareto distribution is

$$\beta^\alpha \left[\frac{\beta^{1-\alpha} - u^{1-\alpha}}{\alpha - 1}(1 - \lambda) + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}}{2\alpha - 1}\beta^\alpha \lambda\right], \quad \alpha > 1.$$

Proof. Applying (2.21) and (3.4),

$$\begin{aligned}
E[Y_\lambda^L] &= \int_\beta^u S_\lambda(x) dx \\
&= \int_\beta^u \left(\frac{\beta}{x}\right)^\alpha \left\{1 - \lambda + \lambda\left(\frac{\beta}{x}\right)^\alpha\right\} dx \\
&= \beta^\alpha \left[\int_\beta^u x^{-\alpha} dx - \lambda \int_\beta^u x^{-\alpha} dx + \lambda\beta^\alpha \int_\beta^u x^{-2\alpha} dx\right] \\
&= \beta^\alpha \left[\frac{\beta^{1-\alpha} - u^{1-\alpha}}{\alpha - 1} - \frac{\beta^{1-\alpha} - u^{1-\alpha}}{\alpha - 1}\lambda + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}}{2\alpha - 1}\beta^\alpha\lambda\right] \\
&= \beta^\alpha \left[\frac{\beta^{1-\alpha} - u^{1-\alpha}}{\alpha - 1}(1 - \lambda) + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}}{2\alpha - 1}\beta^\alpha\lambda\right], \quad \alpha > 1.
\end{aligned} \tag{3.13}$$

□

Remark: When $\lambda = 0$, (3.13) reduces to (3.12).

3.1.5 Payment Per Payment Under Ordinary Deductible

Under the Pareto distribution, the expected payment per payment (2.17) for an ordinary deductible policy is given by

$$\begin{aligned} E[Y^P] &= \frac{\beta^\alpha d^{1-\alpha}}{\alpha-1} \Big/ \left(\frac{\beta}{d}\right)^\alpha \\ &= \frac{d}{\alpha-1}, \quad \alpha > 1. \end{aligned} \tag{3.14}$$

Theorem 18. The expected payment per payment by an insurer under a transmuted Pareto distribution is $\frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right), \quad \alpha > 1.$

Proof. From (2.17) and (3.13),

$$\begin{aligned} E[Y_\lambda^P] &= \frac{E_\lambda[(X-d)_+]}{1-F_{\lambda,X}(d)} \\ &= \frac{\beta^\alpha}{\alpha-1}\left[d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda\right] \Big/ \left(\frac{\beta}{d}\right)^\alpha \left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\} \\ &= \frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right), \quad \alpha > 1. \end{aligned} \tag{3.15}$$

□

We observe that, given that a loss has exceeded the deductible, the expected payment per payment for a Pareto distribution in (3.14) is $\frac{d}{\alpha-1}$. The result in (3.15) shows that the transmuted Pareto distribution scales (3.14) by a factor. Also,(3.15) reduces to (3.14) when $\lambda = 0$.

3.1.6 Payment Per Payment Under Franchise Deductible

Under the Pareto distribution, the expected payment per payment for a franchise deductible insurance policy is given by

$$E[Y^P] = \frac{E[(X-d)_+] + d[1-F_X(d)]}{1-F_X(d)} = \frac{d}{\alpha-1} + d = \frac{\alpha d}{\alpha-1}, \quad \alpha > 1. \tag{3.16}$$

Theorem 19. For a franchise deductible of d, the expected payment per payment of a transmuted Pareto distribution is

$$E[Y_\lambda^P] = \frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right)+d.$$

Proof. Applying (3.4) and (3.15),

$$\begin{aligned} E[Y_\lambda^P] &= \frac{E_\lambda[(X-d)_+]+d[1-F_{\lambda,X}(d)]}{1-F_{\lambda,X}(d)} \\ &= \frac{E_\lambda[(X-d)_+]}{[1-F_{\lambda,X}(d)]}+d \\ &= \frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right)+d, \quad \alpha>1. \end{aligned} \tag{3.17}$$

□

Note that (3.17) reduces to (3.16) when $\lambda=0$.

3.1.7 Summary of Results

Table 3.3: Summary of Results for Y^L and Y^P Under Different Deductibles and Policy Limit for the Transmuted Pareto Distribution

Policy	$E[Y_\lambda^L]$	$E[Y_\lambda^P]$
Ordinary Deductible	$\frac{\beta^\alpha}{\alpha-1}\left[d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda\right]$	$\frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right)$
Franchise Deductible	$\frac{\beta^\alpha}{\alpha-1}\left[d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda\right]+ d^{1-\alpha}\beta^\alpha\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}$	$\frac{d^\alpha}{\alpha-1}\left(\frac{d^{1-\alpha}(1-\lambda)+d^{1-2\alpha}\lambda}{\left\{1-\lambda+\lambda\left(\frac{\beta}{d}\right)^\alpha\right\}}\right)+d$
Policy Limit	$\beta^\alpha\left[\frac{\beta^{1-\alpha}-u^{1-\alpha}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-u^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right]$	

3.1.8 Loss Elimination Ratio (LER) and Increase Limit Factor (ILF)

From definition (2.23), under the Pareto distribution, LER is given by

$$\begin{aligned} LER &= \frac{E[X \wedge d]}{E[X]} \\ &= \frac{\beta^{\alpha}}{\alpha-1}(\beta^{1-\alpha} - d^{1-\alpha}) \Big/ \frac{\alpha\beta}{\alpha-1} \\ &= \frac{\beta^{\alpha-1}}{\alpha}(\beta^{1-\alpha} - d^{1-\alpha}). \end{aligned} \tag{3.18}$$

Theorem 20. The LER under a transmuted Pareto distribution is
$LER_\lambda = \frac{\beta^{\alpha-1}}{\alpha}\left[\frac{\beta^{1-\alpha}-d^{1-\alpha}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha}-d^{1-2\alpha}}{2\alpha-1}\beta^{\alpha}\lambda\right]\left[\frac{(\alpha-1)(2\alpha-1)}{(2\alpha-\lambda-1)}\right], \quad \alpha > 1.$

Proof. Applying (2.23), (3.7) and (3.13),

$$\begin{aligned} LER_\lambda &= \frac{E[X \wedge d]}{E[X]} \\ &= \beta^{\alpha}\left[\frac{\beta^{1-\alpha} - d^{1-\alpha}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - d^{1-2\alpha}}{2\alpha-1}\beta^{\alpha}\lambda\right] \Big/ \left[\frac{\alpha\beta}{\alpha-1}\left(\frac{2\alpha-\lambda-1}{2\alpha-1}\right)\right] \\ &= \frac{\beta^{\alpha-1}}{\alpha}\left[\frac{\beta^{1-\alpha} - d^{1-\alpha}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - d^{1-2\alpha}}{2\alpha-1}\beta^{\alpha}\lambda\right]\left[\frac{(\alpha-1)(2\alpha-1)}{(2\alpha-\lambda-1)}\right], \quad \alpha > 1. \end{aligned} \tag{3.19}$$

□

Observe that when $\lambda = 0$, (3.19) reduces to (3.18).

Under the Pareto distribution,

$$\begin{aligned} ILF &= \frac{E[X \wedge u]}{E[X \wedge d]} \\ &= \frac{\beta^{\alpha}}{\alpha-1}(\beta^{1-\alpha} - u^{1-\alpha}) \Big/ \frac{\beta^{\alpha}}{\alpha-1}(\beta^{1-\alpha} - d^{1-\alpha}) \\ &= \frac{\beta^{1-\alpha} - u^{1-\alpha}}{\beta^{1-\alpha} - d^{1-\alpha}}, \quad u > d, \quad \alpha > 1. \end{aligned} \tag{3.20}$$

Theorem 21. The ILF under a transmuted Pareto distribution is

$$\left[\frac{\beta^{1-\alpha} - u^{1-\alpha}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}}{2\alpha-1}\beta^{\alpha}\lambda\right] \Big/ \left[\frac{\beta^{1-\alpha} - d^{1-\alpha}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - d^{1-2\alpha}}{2\alpha-1}\beta^{\alpha}\lambda\right].$$

Proof. Applying (2.24) and (3.13),

$$
\begin{aligned}
ILF_\lambda &= \frac{E[X\wedge u]}{E[X\wedge d]} \\
&= \beta^\alpha\left[\frac{\beta^{1-\alpha}-u^{1-\alpha}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-u^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right]\Big/ \\
&\quad \beta^\alpha\left[\frac{\beta^{1-\alpha}-d^{1-\alpha}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-d^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right] \\
&= \left[\frac{\beta^{1-\alpha}-u^{1-\alpha}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-u^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right]\Big/ \\
&\quad \left[\frac{\beta^{1-\alpha}-d^{1-\alpha}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-d^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right], \quad \alpha>1, \quad u>d.
\end{aligned}
\tag{3.21}
$$

□

Remark: When $\lambda = 0$, (3.21) reduces to (3.20).

3.1.9 Effects of Inflation and Coinsurance

3.1.9.1 Inflation Factor Recall (2.27) provides the expected payment per loss for a deductible d and uniform inflation factor $(1+r)$. For a Pareto distribution, (2.27) gives

$$
\begin{aligned}
E[Y^L] &= (1+r)\left[\frac{\alpha\beta}{\alpha-1}-\frac{\beta^\alpha}{\alpha-1}\left(\beta^{1-\alpha}-(\frac{d}{1+r})^{1-\alpha}\right)\right] \\
&= (1+r)\left[\frac{\alpha\beta-\beta+\beta^\alpha d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}\right], \quad \alpha>1.
\end{aligned}
\tag{3.22}
$$

Theorem 22. Given an ordinary deductible d and a uniform inflation factor $(1+r)$, the expected payment per loss under a transmuted Pareto distribution is

$$
(1+r)\left[\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)}-\beta^\alpha\left(\frac{\beta^{1-\alpha}-d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\right)\right].
$$

Proof. Applying (2.21), (3.7) and (3.13),

$$
\begin{aligned}
E[Y_\lambda^L] &= (1+r)\left\{E[X] - E\left[X \wedge \frac{d}{1+r}\right]\right\} \\
&= (1+r)\left[\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)} - \int_\beta^{\frac{d}{1+r}} S_\lambda(x)dx\right] \\
&= (1+r)\left[\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)} - \int_\beta^{\frac{d}{1+r}} \left(\frac{\beta}{x}\right)^\alpha \left\{1-\lambda+\lambda\left(\frac{\beta}{x}\right)^\alpha\right\}dx\right] \\
&= (1+r)\left[\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)} - \beta^\alpha\left(\frac{\beta^{1-\alpha}-(\frac{d}{1+r})^{1-\alpha}}{\alpha-1}(1-\lambda)\right.\right. \qquad (3.23) \\
&\quad \left.\left. + \frac{\beta^{1-2\alpha}-(\frac{d}{1+r})^{1-2\alpha}}{2\alpha-1}\beta^\alpha\lambda\right)\right] \\
&= (1+r)\left[\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)} - \beta^\alpha\left(\frac{\beta^{1-\alpha}-d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)\right.\right. \\
&\quad \left.\left. + \frac{\beta^{1-2\alpha}-d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\right)\right], \quad \alpha > 1.
\end{aligned}
$$

□

Observe that when $\lambda = 0$, (3.23) reduces to (3.22).

3.1.10 Coinsurance, Inflation Factor, Deductible, and Policy Limit

We define the expected payment per loss when a policy has an ordinary deductible d, a policy limit u, an inflation factor r, and a coinsurance factor c in (2.31). Also, given the above-mentioned coverage modifications, the expected payment per payment is defined in (2.32). Applying (2.31), the expected payment per loss under the Pareto distribution is given by

$$
\begin{aligned}
E[Y^L] &= c(1+r)\left[E\big(X \wedge \frac{u}{1+r}\big) - E\big(X \wedge \frac{d}{1+r}\big)\right] \\
&= c(1+r)\left[\frac{\beta^\alpha}{\alpha-1}\left(\beta^{1-\alpha} - (\frac{u}{1+r})^{1-\alpha}\right) - \frac{\beta^\alpha}{\alpha-1}\left(\beta^{1-\alpha} - (\frac{d}{1+r})^{1-\alpha}\right)\right] \\
&= c(1+r)\frac{\beta^\alpha}{\alpha-1}\left[(\frac{d}{1+r})^{1-\alpha} - (\frac{u}{1+r})^{1-\alpha}\right], \quad \alpha > 1.
\end{aligned}
\qquad (3.24)
$$

Also, the expected payment per payment for the Pareto distribution defined in (2.32) is

$$\begin{aligned}
E[Y^P] &= \frac{E[Y^L]}{1 - F_X(\frac{d}{1+r})} \\
&= c(1+r)\frac{\beta^\alpha}{\alpha-1}\Big[(\frac{d}{1+r})^{1-\alpha} - (\frac{u}{1+r})^{1-\alpha}\Big]\Big/\Big(\frac{\beta}{\frac{d}{1+r}}\Big)^\alpha \qquad (3.25)\\
&= \frac{cd^\alpha(1+r)^{1-\alpha}}{\alpha-1}\Big[(\frac{d}{1+r})^{1-\alpha} - (\frac{u}{1+r})^{1-\alpha}\Big], \quad \alpha > 1.
\end{aligned}$$

Theorem 23. For a transmuted Pareto distribution, the expected payment per loss with a deductible, coinsurance, policy limit, and inflation factor is

$$\begin{aligned}
&c(1+r)\Bigg[\Bigg\{\beta^\alpha\Bigg(\frac{\beta^{1-\alpha} - u^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\Bigg)\Bigg\} \\
&-\Bigg\{\beta^\alpha\Bigg(\frac{\beta^{1-\alpha} - d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\Bigg)\Bigg\}\Bigg], \quad \alpha > 1.
\end{aligned}$$

Proof. Applying (2.31) to the transmuted Pareto distribution,

$$\begin{aligned}
E[Y^L] &= c(1+r)\Big[E\big(X \wedge \frac{u}{1+r}\big) - E\big(X \wedge \frac{d}{1+r}\big)\Big] \\
&= c(1+r)\Bigg[\Bigg\{\beta^\alpha\Bigg(\frac{\beta^{1-\alpha} - u^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - u^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\Bigg)\Bigg\} \\
&\quad -\Bigg\{\beta^\alpha\Bigg(\frac{\beta^{1-\alpha} - d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda) + \frac{\beta^{1-2\alpha} - d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^\alpha\lambda\Bigg)\Bigg\}\Bigg], \quad \alpha > 1.
\end{aligned} \qquad (3.26)$$

□

Remark: With $\lambda = 0$, (3.26) reduces to (3.25).

Theorem 24. For a transmuted Pareto distribution, the expected payment per payment with a deductible, coinsurance, policy limit, and inflation factor is

$$c(1+r)\beta^{\alpha}\Bigg[\Bigg\{\Bigg(\frac{\beta^{1-\alpha}-u^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-u^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^{\alpha}\lambda\Bigg)\Bigg\}$$
$$-\Bigg\{\beta^{\alpha}\Bigg(\frac{\beta^{1-\alpha}-d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^{\alpha}\lambda\Bigg)\Bigg\}\Bigg]$$
$$\Bigg/\Bigg(\frac{\beta}{\frac{d}{1+r}}\Bigg)^{\alpha}\Bigg\{1-\lambda+\lambda\Bigg(\frac{\beta}{\frac{d}{1+r}}\Bigg)^{\alpha}\Bigg\},\quad \alpha>1.$$

Proof. Applying (2.32)

$$\begin{aligned}E[Y^P] &= \frac{E[Y^L]}{1-F_X(\frac{d}{1+r})}\\
&= c(1+r)\beta^{\alpha}\Bigg[\Bigg\{\Bigg(\frac{\beta^{1-\alpha}-u^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-u^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^{\alpha}\lambda\Bigg)\Bigg\}\\
&-\Bigg\{\beta^{\alpha}\Bigg(\frac{\beta^{1-\alpha}-d^{1-\alpha}(1+r)^{\alpha-1}}{\alpha-1}(1-\lambda)+\frac{\beta^{1-2\alpha}-d^{1-2\alpha}(1+r)^{2\alpha-1}}{2\alpha-1}\beta^{\alpha}\lambda\Bigg)\Bigg\}\Bigg]\\
&\Bigg/\Bigg(\frac{\beta}{\frac{d}{1+r}}\Bigg)^{\alpha}\Bigg\{1-\lambda+\lambda\Bigg(\frac{\beta}{\frac{d}{1+r}}\Bigg)^{\alpha}\Bigg\},\quad \alpha>1.\end{aligned} \tag{3.27}$$

□

3.1.11 Value at Risk (VaR) and Tail Value at Risk (TVaR)

3.1.11.1 Value at Risk (VaR) In (2.1.8.1), we explained Value at Risk (VaR). (2.37), (2.38) and (2.40) also defines how VaR is computed mathematically. The $VaR_p(X)$ of a Pareto distribution at p level is given by

$$\pi_p = VaR_p(X) = \beta(1-p)^{-\frac{1}{\alpha}}$$

.

Theorem 25. The Value-at-Risk of a transmuted Pareto distribution is $\pi_{p,\lambda} = \beta\left[\frac{(\lambda-1)+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\right]^{-\frac{1}{\alpha}}$, where $0<p<1$ is the confidence level,

$\lambda \neq 0$ is the transmutation parameter, β is the scale parameter and α is the shape parameter.

Proof. Applying (3.3) and the definition of $VaR_p(X)$ in (2.33),

$$\left(\frac{\beta}{\pi_{p,\lambda}}\right)^{\alpha}\left\{1-\lambda+\lambda\left(\frac{\beta}{\pi_{p,\lambda}}\right)^{\alpha}\right\}=1-p.$$

Solving the quadratic equation above for $\pi_{p,\lambda}$, We have

$$\pi_{p,\lambda}=\beta\left[\frac{(\lambda-1)+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\right]^{-\frac{1}{\alpha}}, \quad \lambda \neq 0. \tag{3.28}$$

□

3.1.11.2 Tail Value at Risk (TVaR) For a Pareto distribution, the $TVaR_p(X)$ at p security level is

$$TVaR_p(X)=\frac{\alpha\beta(1-p)^{-\frac{1}{\alpha}}}{\alpha-1}, \quad \alpha>1.$$

Theorem 26. At a p level, the $TVaR_p(X)$ of a transmuted Pareto distribution is

$$TVaR_{p,\lambda}(X)=VaR_{p,\lambda}(X)+\frac{E_\lambda[X]}{1-p}-\beta^{\alpha}\left[\frac{\beta^{1-\alpha}-VaR_{p,\lambda}(X)^{1-\alpha}}{(\alpha-1)(1-p)}(1-\lambda)\right.$$
$$\left.+\frac{\beta^{1-2\alpha}-VaR_{p,\lambda}(X)^{1-2\alpha}}{(2\alpha-1)(1-p)}\beta^{\alpha}\lambda\right].$$

where

$$E_\lambda[X]=\frac{\alpha\beta(2\alpha-\lambda-1)}{(\alpha-1)(2\alpha-1)}, \quad VaR_{p,\lambda}=\beta\left[\frac{(\lambda-1)+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\right]^{-\frac{1}{\alpha}}, \quad \lambda\neq 0.$$

Proof. Applying (2.44),

$$\begin{aligned} TVaR_{p,\lambda}(X) &= VaR_{p,\lambda}(X) + \frac{E_\lambda[X] - E_\lambda[X \wedge VaR_{p,\lambda}(X)]}{1-p} \\ &= VaR_{p,\lambda}(X) + \Bigg(E_\lambda[X] - \Bigg\{ \beta^\alpha \Bigg[\frac{\beta^{1-\alpha} - VaR_{p,\lambda}(X)^{1-\alpha}}{\alpha - 1}(1-\lambda) \\ &+ \frac{\beta^{1-2\alpha} - VaR_{p,\lambda}(X)^{1-2\alpha}}{2\alpha - 1}\beta^\alpha \lambda \Bigg] \Bigg\} \Bigg) \Bigg/ (1-p) \\ &= VaR_{p,\lambda}(X) + \frac{E_\lambda[X]}{1-p} - \beta^\alpha \Bigg[\frac{\beta^{1-\alpha} - VaR_{p,\lambda}(X)^{1-\alpha}}{(\alpha-1)(1-p)}(1-\lambda) \\ &+ \frac{\beta^{1-2\alpha} - VaR_{p,\lambda}(X)^{1-2\alpha}}{(2\alpha-1)(1-p)}\beta^\alpha \lambda \Bigg], \quad \alpha > 1, \quad 0 < p < 1, \end{aligned} \tag{3.29}$$

where

$$E_\lambda[X] = \frac{\alpha\beta(2\alpha - \lambda - 1)}{(\alpha-1)(2\alpha-1)}, \quad VaR_{p,\lambda}(X) = \beta \left[\frac{(\lambda - 1) + \sqrt{(1-\lambda)^2 + 4\lambda(1-p)}}{2\lambda} \right]^{-\frac{1}{\alpha}}.$$

□

3.2 Numerical Analysis of Value-at-Risk and Tail-Value-at-Risk

3.2.1 Likelihood Function and Parameter Estimates

The likelihood function for the Pareto distribution is given by

$$\begin{aligned} L(\alpha, \beta; X_i) &= \prod_{i=1}^{n} f(x_i; \alpha, \beta) \\ &= \prod_{i=1}^{n} \left(\frac{\alpha\beta^\alpha}{x_i^{\alpha+1}} \right) \\ &= \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} \frac{1}{x_i^{\alpha+1}} \end{aligned} \tag{3.30}$$

Taking log of the likelihood in (3.30), we obtain

$$\ln[L(\alpha, \beta; X_i)] = n \ln\alpha + n\alpha \ln\beta - (\alpha + 1) \sum_{i=1}^{n} \ln(x_i) \tag{3.31}$$

Also, the likelihood function for the Transmuted Pareto Distribution is given by:

$$\begin{aligned} L(\alpha,\beta,\lambda;x_i) &= \prod_{i=1}^{n}\left(\frac{\alpha\beta^{\alpha}}{x_i^{\alpha+1}}\right)\left[1-\lambda+2\lambda\left(\frac{\beta}{x_i}\right)^{\alpha}\right] \\ &= \alpha^{n}\beta^{n\alpha}\prod_{i=1}^{n}\frac{1}{x_i^{\alpha+1}}\prod_{i=1}^{n}\left[1-\lambda+2\lambda\left(\frac{\beta}{x_i}\right)^{\alpha}\right] \end{aligned} \tag{3.32}$$

Taking log of the likelihood function in (3.32), we have:

$$\begin{aligned} \ln\Big[L(\alpha,\beta,\lambda;x_i)\Big] &= n\ln\alpha + n\alpha\ln\beta - (\alpha+1)\sum_{i=1}^{n}\ln(x_i) \\ &\quad + \sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda\left(\frac{\beta}{x_i}\right)^{\alpha}\right] \end{aligned} \tag{3.33}$$

Observe that $x > \beta$, so the maximum likelihood estimate of the β for both Pareto and Transmuted Pareto distribution is the minimum of x_i. Thus, the first-order statistic $(x_{(1)})$.

The maximum likelihood estimate of α and λ is obtained by solving the nonlinear system above.

We use a dataset contained in [12]. This dataset provides a comprehensive record of 35 hurricanes that occurred from 1949 to 1980, capturing the total damage incurred during these storm events. To accurately account for changes in the cost of living over time, the reported losses have been adjusted for inflation using the Residential Construction Index. As a result, all loss values are presented in 1981 dollars, providing a consistent and comparable basis for analysis. It is worth noting that the dataset specifically includes hurricanes where the trended losses surpassed the threshold of $5,000,000$. By focusing on significant events with substantial economic impact, the dataset offers valuable insights into the most severe and financially consequential hurricanes within the designated timeframe. The table below shows are summary of the data.

Table 3.4: Losses(10^3)

6766	7123	10562	14474	15351	16983	18383
19030	25304	29112	30146	33727	40596	41409
47905	49397	52600	59917	63123	77809	102942
103217	123680	140136	192013	198446	227338	329511
361200	421680	513586	545778	750389	863881	1638000

Table 3.5: Summary Statistics of the Dataset (Losses(10^3))

Min	$Q1$	Median	Mean	$Q3$	Max
$6,766$	$27,208$	$59,917$	$204,900$	$212,892$	$1,638,000$

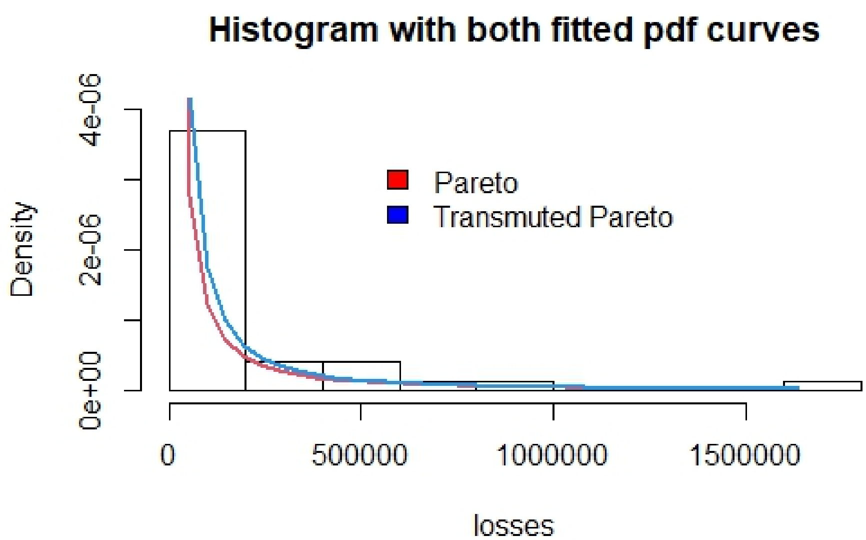

Figure 3.2: Plot of the Histogram of Data

Table 3.6: Model Comparison: Losses(10^3)

Model	Parameter Estimates	$-2LL$	AIC	AICC	BIC
Pareto	$\hat{\alpha} = 0.42, \quad \hat{\beta} = 6766$	888.38	892.38	892.7	895.43
Transmuted Pareto	$\hat{\alpha} = 0.57, \quad \hat{\beta} = 6766, \quad \hat{\lambda} = -0.63$	881.48	887.48	888.28	892.06

where

$$AIC = 2k - 2\log(L), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad BIC = k\log(n) - 2\log L,$$

k is the number of parameters, n is the sample size, AIC is the Akaike Information Criterion, $AICC$ is the Akaike Information Criterion Corrected, BIC is the Bayesian Information Criterion.

From 4.3, we observe that the values of the transmuted Pareto distribution lead to a better fit than that of the Pareto distribution.

We also compare the performance of the distributions using a Likelihood Ration Test (LRT). We test the following hypothesis:

$$H_0 : \lambda = 0 \quad versus \quad H_1 : \lambda \neq 0.$$

We obtain the following results:

$$\omega = 6.9 > \chi_1^2(\alpha = 0.05) = 3.841.$$

where $\omega = 2\left[LL_{Par} - LL_{T.Par}\right] = 2(444.19 - 440.74) = 6.9.$

We therefore reject the null hypothesis and conclude that incorporating the transmutation parameter provides a better fit.

Table 3.7: Value at Risk (VaR) at different security levels for $\beta = 6,766$, $\alpha = 3$, $\lambda = -0.63$: Losses(10^3)

p-level	Pareto	Transmuted Pareto
0.1%	$6,768$	$6,672$
1%	$6,789$	$6,780$
5%	$6,883$	$6,838$
10%	$7,008$	$6,914$

From Tables 3.7, We observe that the transmuted Pareto distribution provides lower VaR estimates compared to that of the Pareto distribution.

Table 3.8: Tails Value at Risk ($TVaR$) at different p - levels for $\beta = 6,766$, $\alpha = 3$, $\lambda = -0.63$: Losses(10^3)

$\alpha - level$	Pareto	Transmuted Pareto
0.1%	$10,152$	$19,485$
1%	$10,183$	$19,601$
5%	$10,324$	$20,134$
10%	$10,512$	$20,854$

From Tables 3.8, the transmuted Pareto distribution consistently provides higher $TVaR$ estimates compared to that of Pareto distribution across the chosen alpha levels.

CHAPTER FOUR

WEIBULL AND TRANSMUTED WEIBULL DISTRIBUTIONS

4.1 Coverage Modifications with Transmuted Weibull Distribution

The probability density function of a Weibull distribution is given by:

$$g(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(- (\frac{x}{\beta})^{\alpha}\right), \quad x > 0 \tag{4.1}$$

The cumulative distribution function is also given by

$$G(x) = 1 - \exp\left(- (\frac{x}{\beta})^{\alpha}\right), \quad x > 0 \tag{4.2}$$

Applying (2.3), (4.1), and (4.2), the CDF and the of a transmuted Weibull distribution is given by

$$\begin{aligned} F_{\lambda}(x) &= (1+\lambda)\left\{1 - \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right\} - \lambda\left\{1 - \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right\}^2 \\ &= \left(1 - \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right)\left(1 + \lambda \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right) \\ &= 1 - \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\left\{1 - \lambda + \lambda \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right\}, \quad x > 0, \quad -1 < \lambda < 1. \end{aligned} \tag{4.3}$$

The corresponding survival function or the transmuted Weibull is given by:

$$S_{\lambda}(x) = \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\left\{1 - \lambda + \lambda \exp\left(- (\frac{x}{\beta})^{\alpha}\right)\right\}, \quad x > 0, \quad -1 < \lambda < 1. \tag{4.4}$$

Also, applying (2.4), (4.1), and (4.2), the PDF of the transmuted Weibull is given by

$$f_{\lambda}(x) = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} \exp\left(-(\frac{x}{\beta})^{\alpha}\right)\left\{1-\lambda+2\lambda \exp\left(-(\frac{x}{\beta})^{\alpha}\right)\right\}, \quad x > 0, \quad -1 < \lambda < 1. \tag{4.5}$$

Observe that, when $\lambda = 0$, (4.3) reduces to (4.2) and (4.5) reduces to (4.1), which are the CDF and PDF of the Weibull distribution, respectively.

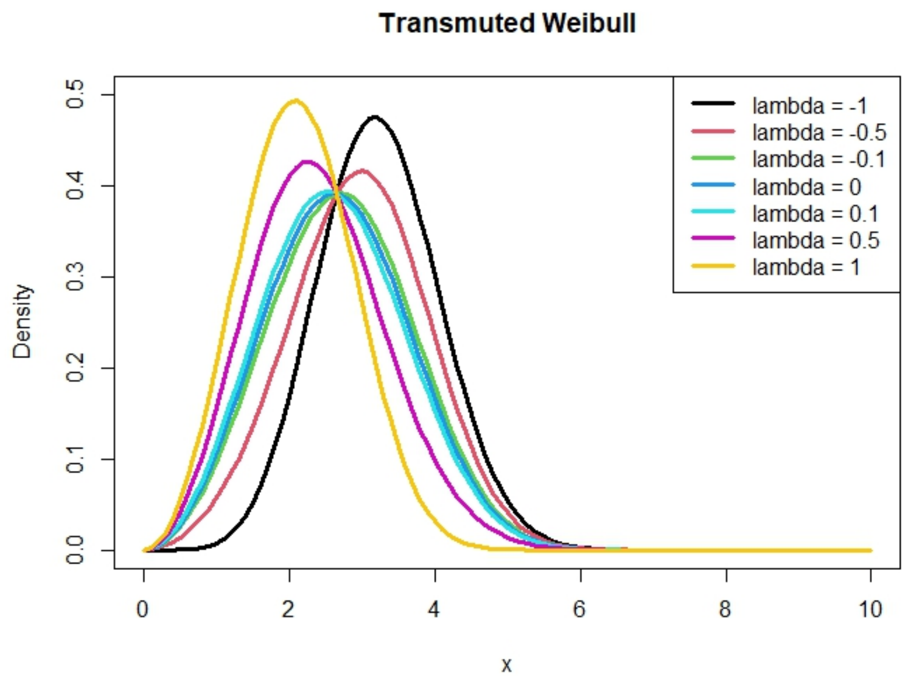

Figure 4.1: Plot of the PDF of transmuted Weibull distribution ($\alpha = \beta = 3$)

4.1.1 Expected Value of the Weibull and Transmuted Weibull Distribution

Using the PDF of Weibull distribution,

$$\begin{aligned} E[X] &= \int_0^\infty x f(x;\beta,\alpha)dx \\ &= \int_0^\infty x\frac{\alpha}{\beta}\Big(\frac{x}{\beta}\Big)^{\alpha-1}\exp\Big(-(\frac{x}{\beta})^\alpha\Big)dx \\ &= \frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha}). \end{aligned} \tag{4.6}$$

Using the survival function of the transmuted Weibull distribution defined in (4.4), the mean of the transmuted Weibull distribution is also given by:

$$
\begin{aligned}
E[X_\lambda] &= \int_0^\infty S(x;\beta,\alpha,\lambda)dx \\
&= \int_0^\infty \exp\Big(-(\frac{x}{\beta})^\alpha\Big)\Big\{1-\lambda+\lambda\exp\Big(-(\frac{x}{\beta})^\alpha\Big)\Big\}dx \\
&= \int_0^\infty \exp\Big(-(\frac{x}{\beta})^\alpha\Big)dx - \lambda\int_0^\infty \exp\Big(-(\frac{x}{\beta})^\alpha\Big) + \lambda\int_0^\infty \exp\Big(-(\frac{x}{\beta})^{2\alpha}\Big)dx.
\end{aligned} \tag{4.7}
$$

Using integration by parts, let

$$
U = \left(\frac{x}{\beta}\right)^\alpha, \quad x = u^{\frac{1}{\alpha}}\beta, \quad dx = \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}du.
$$

Substituting, (4.7) becomes

$$
\begin{aligned}
E[X_\lambda] &= \int_0^\infty \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}\mathrm{e}^{-u}du - \lambda\int_0^\infty \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}\mathrm{e}^{-u}du + \lambda\int_0^\infty \frac{\beta}{2\alpha}u^{\frac{1}{2\alpha}-1}\mathrm{e}^{-u}du \\
&= \frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha}) - \lambda\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha}) + \lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha}) \\
&= \frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})(1-\lambda) + \lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha}).
\end{aligned} \tag{4.8}
$$

We observe from (4.8) and (4.6) that the expected value of the Weibull distribution is $\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})$, which gets scaled up with transmutation for $-1 \leq \lambda < 0$. We also observed that when $\lambda = 0$, (4.8) reduces to (4.6).

4.1.2 Payment Per Loss Under Ordinary Deductible

Recall the definition of Payment per Loss in (2.9). The expected value of the Weibull distribution is given by:

$$
\begin{aligned}
E[Y^L] &= \int_d^\infty S(x)dx \\
&= \int_d^\infty \exp\Big\{-(\frac{x}{\beta})^\alpha\Big\}dx.
\end{aligned} \tag{4.9}
$$

Let

$$u = \left(\frac{x}{\beta}\right)^{\alpha}, \quad x = u^{\frac{1}{\alpha}}\beta, \quad dx = \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}du.$$

Equation (4.9) therefore, becomes:

$$\begin{aligned} E[Y^L] &= \int_{(\frac{d}{\beta})^{\alpha}}^{\infty} \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}\mathrm{e}^{-u}du \\ &= \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}. \end{aligned} \tag{4.10}$$

where

$$\Gamma(a, x) = \int_{x}^{\infty} t^{a-1}\mathrm{e}^{-t}dt.$$

Theorem 27. Under the transmuted Weibull distribution, the expected payment per loss for a policy with an ordinary deductible d is

$$\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda, \quad \alpha > 0, \quad -1 < \lambda < 1.$$

Proof.

$$\begin{aligned} E[Y_{\lambda}^{L}] &= \int_{d}^{\infty} S_{\lambda}(x)dx \\ &= \int_{d}^{\infty} \exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\left(1-\lambda+\lambda\exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right)dx \\ &= (1-\lambda)\int_{d}^{\infty} \exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}dx + \lambda\int_{d}^{\infty} \exp\left\{-\left(\frac{x}{\beta}\right)^{2\alpha}\right\}dx \\ &= (1-\lambda)\int_{(\frac{d}{\beta})^{\alpha}}^{\infty} \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}\mathrm{e}^{-u}du + \lambda\int_{(\frac{d}{\beta})^{2\alpha}}^{\infty} \frac{\beta}{2\alpha}u^{\frac{1}{2\alpha}-1}\mathrm{e}^{-u}du \\ &= \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda, \quad \alpha, \beta > 0, \quad -1 < \lambda < 1. \end{aligned} \tag{4.11}$$

□

Observe that, (4.10) gets scaled by a factor in (4.11), and when $\lambda = 0$, (4.11) reduces to (4.10).

4.1.3 Payment Per Loss Under Franchise Deductible

We defined a policy with a franchise deductible in (2.13). Under the Weibull distribution, (2.14) is

$$E[Y^L] = \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\} + d\left[\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}\right], \quad \alpha, \beta > 0. \tag{4.12}$$

Theorem 28. The expected payment per loss for the transmuted Weibull distribution for a franchise deductible policy is

$$\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda+d\left[\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}\left(1-\lambda+\lambda\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}\right)\right].$$

Proof.

$$\begin{aligned}E[Y_{\lambda}^{L}] &= E_{\lambda}[(X-d)_{+}] + d[1 - F_{\lambda,X}(d)] \\ &= \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda \\ &+ d\left[\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}\left(1-\lambda+\lambda\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}\right)\right], \quad \alpha, \beta > 0, \quad -1 < \lambda < 1.\end{aligned} \tag{4.13}$$

□

Comparing $E[Y^L]$ under a franchise deductible policy for the Weibull distribution in (4.12) to that of the transmuted Weibull in (4.13), $E[Y^L]$ for the Weibull distribution gets scaled by a factor.Also, when $\lambda = 0$, (4.13) reduces to (4.12).

4.1.4 Payment Per Loss Under a Policy Limit

For a policy limit u, (4.13) and (2.21) give the definition and the expected payment per loss, respectively. For a Weibull distribution, the expected payment per loss with a policy limit of u is given by

$$\begin{aligned}
E[Y^L] &= \int_0^u S_X(x)dx \\
&= \int_0^u \exp\Big\{-(\frac{x}{\beta})^\alpha\Big\}dx \\
&= \int_0^{(\frac{u}{\beta})^\alpha} \mathrm{e}^{-t}\frac{\beta}{\alpha}t^{\frac{1}{\alpha}-1}dt \\
&= \frac{\beta}{\alpha}\Gamma\Big(\frac{1}{\alpha}, (\frac{u}{\beta})^\alpha\Big).
\end{aligned} \tag{4.14}$$

Theorem 29. For an insurance policy with a policy limit u, the expected payment per loss under a transmuted Weibull distribution is

$$\frac{\beta}{\alpha}\Gamma\Big(\frac{1}{\alpha}, (\frac{u}{\beta})^\alpha\Big)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Big(\frac{1}{2\alpha}, (\frac{u}{\beta})^{2\alpha}\Big)\lambda.$$

Proof. Applying (2.21) and (4.4),

$$\begin{aligned}
E[Y_\lambda^L] &= \int_0^u S_\lambda(x)dx \\
&= \int_0^u \exp\Big(-(\frac{x}{\beta})^\alpha\Big)\Big\{1-\lambda+\lambda\exp\Big(-(\frac{x}{\beta})^\alpha\Big)\Big\}dx \\
&= (1-\lambda)\int_0^u \mathrm{e}^{-(\frac{x}{\beta})^\alpha}dx + \lambda\int_0^u \mathrm{e}^{-(\frac{x}{\beta})^{2\alpha}}dx \\
&= (1-\lambda)\int_0^{(\frac{u}{\beta})^\alpha} \frac{\beta}{\alpha}t^{\frac{1}{\alpha}-1}\mathrm{e}^{-t}dt + \lambda\int_0^{(\frac{u}{\beta})^{2\alpha}} \frac{\beta}{2\alpha}t^{\frac{1}{2\alpha}-1}\mathrm{e}^{-t}dt \\
&= \frac{\beta}{\alpha}\Gamma\Big(\frac{1}{\alpha}, (\frac{u}{\beta})^\alpha\Big)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Big(\frac{1}{2\alpha}, (\frac{u}{\beta})^{2\alpha}\Big)\lambda.
\end{aligned} \tag{4.15}$$

□

Remark: When $\lambda = 0$, (4.15) reduces to (4.14).

4.1.5 Payment Per Payment Under Ordinary Deductible

Under the Weibull distribution, the expected payment per payment (2.17) for an ordinary deductible policy is given by

$$
\begin{aligned}
E[Y^P] &= \frac{E[(X-d)_+]}{1-F_X(d)} \\
&= \frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}}{\exp\left(-\left(\frac{d}{\beta}\right)^{\alpha}\right)} \\
&= \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}\exp\left(\left(\frac{d}{\beta}\right)^{\alpha}\right)
\end{aligned}
\tag{4.16}
$$

Theorem 30. The expected payment per payment by an insurer under a transmuted Weibull distribution is

$$
\left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}}\right]\left(\exp\left\{\left(\frac{d}{\beta}\right)^{\alpha}\right\}\right), \quad \alpha, \beta > 0, \quad -1 < \lambda < 1.
$$

Proof. From (2.17) and (4.11),

$$
\begin{aligned}
E[Y_\lambda^P] &= \frac{E_\lambda[(X-d)_+]}{1-F_{\lambda,X}(d)} \\
&= \frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{\exp\left(-\left(\frac{d}{\beta}\right)^{\alpha}\right)\left\{1-\lambda+\lambda\exp\left(-\left(\frac{d}{\beta}\right)^{\alpha}\right)\right\}} \\
&= \left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}}\right]\left(\exp\left\{\left(\frac{d}{\beta}\right)^{\alpha}\right\}\right).
\end{aligned}
\tag{4.17}
$$

□

Remark: We Observe that (4.17) is greater than (4.16), and they are equal when $\lambda = 0$.

4.1.6 Payment Per Payment Under Franchise Deductible

Under the Weibull distribution, the expected payment per payment for a franchise deductible insurance policy is given by

$$\begin{aligned} E[Y^P] &= \frac{E[(X-d)+]}{1-F_X(d)} + d \\ &= \frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\} \exp\left\{(\frac{d}{\beta})^{\alpha}\right\} + d \end{aligned} \tag{4.18}$$

Theorem 31. *For a franchise deductible of d, the expected payment per payment of a transmuted Weibull distribution is*

$$E[Y^P_\lambda] = \left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}}\right]\left(\exp\left\{(\frac{d}{\beta})^{\alpha}\right\}\right) + d.$$

Proof. Applying (4.4) and (4.17),

$$\begin{aligned} E[Y^P_\lambda] &= \frac{E_\lambda[(X-d)_+]}{1-F_{\lambda,X}(d)} + d \\ &= \left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-(\frac{d}{\beta})^{\alpha}\right\}}\right]\left(\exp\left\{(\frac{d}{\beta})^{\alpha}\right\}\right) + d. \end{aligned} \tag{4.19}$$

□

Note that (4.19) reduces to (4.18) when $\lambda = 0$.

4.1.7 Summary of Results

Table 4.1: Summary of Results for Y^L and Y^P Under Different Deductibles and Policy Limit for the Transmuted Weibull Distribution

Policy	$E[Y_\lambda^L]$	$E[Y_\lambda^P]$
Ordinary Deductible	$\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha},\left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda$	$\left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha},\left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}}\right]\times\left(\exp\left\{\left(\frac{d}{\beta}\right)^{\alpha}\right\}\right)$
Franchise Deductible	$\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha},\left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda+d\left[\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}\left(1-\lambda+\lambda\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}\right)\right]$	$\left[\frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha},\left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda}{1-\lambda+\lambda\exp\left\{-\left(\frac{d}{\beta}\right)^{\alpha}\right\}}\right]\times\left(\exp\left\{\left(\frac{d}{\beta}\right)^{\alpha}\right\}\right)+d$
Policy Limit	$\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha},\left(\frac{u}{\beta}\right)^{\alpha}\right)(1-\lambda)+\frac{\beta}{2\alpha}\Gamma\left(\frac{1}{2\alpha},\left(\frac{u}{\beta}\right)^{2\alpha}\right)\lambda$	

4.1.8 Loss Elimination Ratio (LER) and Increase Limit Factor (ILF)

From definition (2.23) under the Weibull distribution, LER is given by

$$\begin{aligned}LER &= \frac{E[X\wedge d]}{E[X]}\\ &= \frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}}{\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)}\\ &= \frac{\Gamma\left\{\frac{1}{\alpha},\left(\frac{d}{\beta}\right)^{\alpha}\right\}}{\Gamma\left(\frac{1}{\alpha}\right)}\end{aligned} \tag{4.20}$$

Theorem 32. The LER under a transmuted Weibull distribution is

$$LER_\lambda = \frac{\Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta})^\alpha\right)(1-\lambda) + \frac{\lambda}{2}\Gamma\left(\frac{1}{2\alpha}, (\frac{d}{\beta})^{2\alpha}\right)\lambda}{\Gamma\left(\frac{1}{\alpha}\right)(1-\lambda) + \frac{\lambda}{2}\Gamma\left(\frac{1}{2\alpha}\right)}.$$

Proof. Applying (2.23), (4.8) and (4.15)

$$\begin{aligned} LER_\lambda &= \frac{E[X \wedge d]}{E[X]} \\ &= \frac{\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta})^\alpha\right)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left(\frac{1}{2\alpha}, (\frac{d}{\beta})^{2\alpha}\right)\lambda}{\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})(1-\lambda) + \lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha})} \\ &= \frac{\Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta})^\alpha\right)(1-\lambda) + \frac{\lambda}{2}\Gamma\left(\frac{1}{2\alpha}, (\frac{d}{\beta})^{2\alpha}\right)\lambda}{\Gamma\left(\frac{1}{\alpha}\right)(1-\lambda) + \frac{\lambda}{2}\Gamma\left(\frac{1}{2\alpha}\right)}. \end{aligned} \tag{4.21}$$

□

Observe that when $\lambda = 0$, (4.21) reduces to (4.20).

Under the Weibull distribution,

$$\begin{aligned} ILF &= \frac{E[X \wedge u]}{E[X \wedge d]} \\ &= \frac{\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}, (\frac{u}{\beta})^\alpha\right)}{\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta})^\alpha\right)} \\ &= \frac{\Gamma\left(\frac{1}{\alpha}, (\frac{u}{\beta})^\alpha\right)}{\Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta})^\alpha\right)} \end{aligned} \tag{4.22}$$

Theorem 33. The ILF under a transmuted Weibull distribution is

$$\frac{\Gamma\left\{\frac{1}{\alpha}, \left(\frac{u}{\beta}\right)^\alpha\right\}(1-\lambda) + \frac{\lambda}{2}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{u}{\beta}\right)^{2\alpha}\right\}}{\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^\alpha\right\}(1-\lambda) + \frac{\lambda}{2}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}}.$$

Proof.

$$
\begin{aligned}
ILF_\lambda &= \frac{E[X \wedge u]}{E[X \wedge d]} \\
&= \frac{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{u}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{u}{\beta}\right)^{2\alpha}\right\}\lambda}{\frac{\beta}{\alpha}\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}\lambda} \qquad (4.23) \\
&= \frac{\Gamma\left\{\frac{1}{\alpha}, \left(\frac{u}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\lambda}{2}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{u}{\beta}\right)^{2\alpha}\right\}}{\Gamma\left\{\frac{1}{\alpha}, \left(\frac{d}{\beta}\right)^{\alpha}\right\}(1-\lambda) + \frac{\lambda}{2}\Gamma\left\{\frac{1}{2\alpha}, \left(\frac{d}{\beta}\right)^{2\alpha}\right\}}
\end{aligned}
$$

□

Remark: When $\lambda = 0$, (4.23) reduces to (4.22).

4.1.9 Effects of Inflation and Coinsurance

4.1.9.1 Inflation Factor Recall (2.27) provides the expected payment per loss for a deductible d and uniform inflation factor $(1+r)$. For a Weibull distribution, (2.27) gives

$$
\begin{aligned}
E[Y^L] &= (1+r)\left\{E[X] - E\left[X \wedge \frac{d}{1+r}\right]\right\} \\
&= (1+r)\left[\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha}) - \frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}, (\frac{\frac{d}{1+r}}{\beta})^{\alpha}\right)\right] \qquad (4.24) \\
&= (1+r)\frac{\beta}{\alpha}\left[\Gamma(\frac{1}{\alpha}) - \Gamma\left(\frac{1}{\alpha}, \left\{\frac{d}{(1+r)\beta}\right\}^{\alpha}\right)\right].
\end{aligned}
$$

Theorem 34. Given an ordinary deductible d and a uniform inflation factor $(1+r)$, the expected payment per loss under a transmuted Weibull distribution is

$$
(1+r)\frac{\beta}{\alpha}\left[\left\{\Gamma(\frac{1}{\alpha}) - \Gamma\left(\frac{1}{\alpha}, (\frac{d}{(1+r)\beta})^{\alpha}\right)\right\}(1-\lambda) + \frac{\lambda}{2}\left\{\Gamma(\frac{1}{2\alpha}) - \Gamma\left(\frac{1}{2\alpha}, (\frac{d}{(1+r)\beta})^{2\alpha}\right)\right\}\right].
$$

Proof. Applying (2.21), (4.8) and (4.15),

$$
\begin{aligned}
E[Y_\lambda^L] &= (1+r)\bigg\{E_\lambda[X] - E\bigg[X \wedge \frac{d}{1+r}\bigg]\bigg\}\\
&= (1+r)\bigg[\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})(1-\lambda) + \lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha}) - \bigg\{\frac{\beta}{\alpha}\Gamma\bigg(\frac{1}{\alpha},(\frac{\frac{d}{1+r}}{\beta})^{\alpha}\bigg)(1-\lambda)\\
&+ \frac{\beta}{2\alpha}\Gamma\bigg(\frac{1}{2\alpha},(\frac{\frac{d}{1+r}}{\beta})^{2\alpha}\bigg)\lambda\bigg\}\bigg]\\
&= (1+r)\frac{\beta}{\alpha}\bigg[\Gamma(\frac{1}{\alpha})(1-\lambda) + \frac{\lambda}{2}\Gamma(\frac{1}{2\alpha}) - \Gamma\bigg(\frac{1}{\alpha},(\frac{\frac{d}{1+r}}{\beta})^{\alpha}\bigg)(1-\lambda) - \frac{\lambda}{2}\Gamma\bigg(\frac{1}{2\alpha},(\frac{\frac{d}{1+r}}{\beta})^{2\alpha}\bigg)\bigg]\\
&= (1+r)\frac{\beta}{\alpha}\bigg[\bigg\{\Gamma(\frac{1}{\alpha}) - \Gamma\bigg(\frac{1}{\alpha},(\frac{d}{(1+r)\beta})^{\alpha}\bigg)\bigg\}(1-\lambda) + \frac{\lambda}{2}\bigg\{\Gamma(\frac{1}{2\alpha}) - \Gamma\bigg(\frac{1}{2\alpha},(\frac{d}{(1+r)\beta})^{2\alpha}\bigg)\bigg\}\bigg].
\end{aligned}
\tag{4.25}
$$

□

Observe that when $\lambda = 0$, (4.25) reduces to (4.24).

4.1.10 Coinsurance, Inflation Factor, Deductible, and Policy Limit

We define the expected payment per loss when a policy has an ordinary deductible d, a policy limit u, an inflation factor r, and a coinsurance factor c in (2.31). Also, given the above-mentioned coverage modifications, the expected payment per payment is defined in (2.32). Applying (2.31), the expected payment per loss under the Weibull distribution is given by

$$
\begin{aligned}
E[Y^L] &= c(1+r)\bigg[E\big(X \wedge \frac{u}{1+r}\big) - E\big(X \wedge \frac{d}{1+r}\big)\bigg]\\
&= c(1+r)\bigg[\frac{\beta}{\alpha}\Gamma\bigg(\frac{1}{\alpha},(\frac{\frac{u}{1+r}}{\beta})^{\alpha}\bigg) - \frac{\beta}{\alpha}\Gamma\bigg(\frac{1}{\alpha},(\frac{\frac{d}{1+r}}{\beta})^{\alpha}\bigg)\bigg]\\
&= c(1+r)\frac{\beta}{\alpha}\bigg[\Gamma\bigg(\frac{1}{\alpha},\bigg\{\frac{u}{(1+r)\beta}\bigg\}^{\alpha}\bigg) - \Gamma\bigg(\frac{1}{\alpha},\bigg\{\frac{d}{(1+r)\beta}\bigg\}^{\alpha}\bigg)\bigg].
\end{aligned}
\tag{4.26}
$$

Also, the expected payment per payment for the Weibull distribution defined in (2.32) is

$$
\begin{aligned}
E[Y^P] &= \frac{E[Y^L]}{1 - F_X(\frac{d}{1+r})} \\
&= \left(1+r)\frac{\beta}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}, \left\{\frac{u}{(1+r)\beta}\right\}^{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \left\{\frac{d}{(1+r)\beta}\right\}^{\alpha}\right)\right]\right/ \exp\left(-(\frac{\frac{d}{1+r}}{\beta})^{\alpha}\right) \\
&= (1+r)\frac{\beta}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}, \left\{\frac{u}{(1+r)\beta}\right\}^{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \left\{\frac{d}{(1+r)\beta}\right\}^{\alpha}\right)\right]\exp\left((\frac{d}{(1+r)\beta})^{\alpha}\right).
\end{aligned}
\tag{4.27}
$$

Theorem 35. For a transmuted Weibull distribution, the expected payment per loss with a deductible, coinsurance, policy limit, and inflation factor is

$$
\begin{aligned}
&c(1+r)\frac{\beta}{\alpha}\Bigg[\left\{\Gamma\left(\frac{1}{\alpha}, (\frac{u}{\beta(1+r)})^{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, (\frac{d}{\beta(1+r)})^{\alpha}\right)\right\}(1-\lambda) \\
&+ \left\{\frac{1}{2}\Gamma\left(\frac{1}{2\alpha}, (\frac{u}{\beta(1+r)})^{2\alpha}\right) - \frac{1}{2}\Gamma\left(\frac{1}{2\alpha}, (\frac{d}{\beta(1+r)})^{2\alpha}\right)\right\}\lambda\Bigg]
\end{aligned}
$$

Proof. Applying (2.31) to the transmuted Weibull distribution,

$$
\begin{aligned}
E[Y^L_\lambda] &= c(1+r)\Bigg[E\big(X \wedge \frac{u}{1+r}\big) - E\big(X \wedge \frac{d}{1+r}\big)\Bigg] \\
&= c(1+r)\Bigg[\Bigg\{\frac{\beta}{\alpha}\Gamma\Bigg(\frac{1}{\alpha},(\frac{\frac{u}{1+r}}{\beta})^{\alpha}\Bigg)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{\frac{u}{1+r}}{\beta})^{2\alpha}\Bigg)\lambda\Bigg\} \\
&- \Bigg\{\frac{\beta}{\alpha}\Gamma\Bigg(\frac{1}{\alpha},(\frac{\frac{d}{1+r}}{\beta})^{\alpha}\Bigg)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{\frac{d}{1+r}}{\beta})^{2\alpha}\Bigg)\lambda\Bigg\}\Bigg] \\
&= c(1+r)\Bigg[\Bigg\{\frac{\beta}{\alpha}\Gamma\Bigg(\frac{1}{\alpha},(\frac{u}{\beta(1+r)})^{\alpha}\Bigg)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{u}{\beta(1+r)})^{2\alpha}\Bigg)\lambda\Bigg\} \\
&- \Bigg\{\frac{\beta}{\alpha}\Gamma\Bigg(\frac{1}{\alpha},(\frac{d}{\beta(1+r)})^{\alpha}\Bigg)(1-\lambda) + \frac{\beta}{2\alpha}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{d}{\beta(1+r)})^{2\alpha}\Bigg)\lambda\Bigg\}\Bigg] \\
&= c(1+r)\frac{\beta}{\alpha}\Bigg[\Bigg\{\Gamma\Bigg(\frac{1}{\alpha},(\frac{u}{\beta(1+r)})^{\alpha}\Bigg) - \Gamma\Bigg(\frac{1}{\alpha},(\frac{d}{\beta(1+r)})^{\alpha}\Bigg)\Bigg\}(1-\lambda) \\
&+ \Bigg\{\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{u}{\beta(1+r)})^{2\alpha}\Bigg) - \frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{d}{\beta(1+r)})^{2\alpha}\Bigg)\Bigg\}\lambda\Bigg]
\end{aligned}
\tag{4.28}
$$

□

Remark: With $\lambda = 0$, (4.28) reduces to (4.26).

Theorem 36. For a transmuted Weibull distribution, the expected payment per payment with a deductible, coinsurance, policy limit, and inflation factor is

$$
\begin{aligned}
& c(1+r)\frac{\beta}{\alpha}\Bigg[\Bigg\{\Gamma\Bigg(\frac{1}{\alpha},(\frac{u}{\beta(1+r)})^{\alpha}\Bigg) - \Gamma\Bigg(\frac{1}{\alpha},(\frac{d}{\beta(1+r)})^{\alpha}\Bigg)\Bigg\}(1-\lambda) \\
& + \Bigg\{\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{u}{\beta(1+r)})^{2\alpha}\Bigg) - \frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{d}{\beta(1+r)})^{2\alpha}\Bigg)\Bigg\}\lambda\Bigg] \exp\Bigg((\frac{d}{(1+r)\beta})^{\alpha}\Bigg) \\
& \Bigg/ \Bigg\{1 - \lambda + \lambda\exp\Bigg(-(\frac{d}{(1+r)\beta})^{\alpha}\Bigg)\Bigg\}.
\end{aligned}
$$

Proof. Applying (2.32)

$$
\begin{aligned}
E[Y_\lambda^P] &= \frac{E[Y^L]}{1-F_X(\frac{d}{1+r})} \\
&= c(1+r)\frac{\beta}{\alpha}\Bigg[\Bigg\{\Gamma\Bigg(\frac{1}{\alpha},(\frac{u}{\beta(1+r)})^\alpha\Bigg)-\Gamma\Bigg(\frac{1}{\alpha},(\frac{d}{\beta(1+r)})^\alpha\Bigg)\Bigg\}(1-\lambda) \\
&+\Bigg\{\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{u}{\beta(1+r)})^{2\alpha}\Bigg)-\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{d}{\beta(1+r)})^{2\alpha}\Bigg)\Bigg\}\lambda\Bigg] \\
&\Bigg/\exp\Bigg(-(\frac{\frac{d}{1+r}}{\beta})^\alpha\Bigg)\Bigg\{1-\lambda+\lambda\exp\Bigg(-(\frac{\frac{d}{1+r}}{\beta})^\alpha\Bigg)\Bigg\} \\
&= c(1+r)\frac{\beta}{\alpha}\Bigg[\Bigg\{\Gamma\Bigg(\frac{1}{\alpha},(\frac{u}{\beta(1+r)})^\alpha\Bigg)-\Gamma\Bigg(\frac{1}{\alpha},(\frac{d}{\beta(1+r)})^\alpha\Bigg)\Bigg\}(1-\lambda) \\
&+\Bigg\{\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{u}{\beta(1+r)})^{2\alpha}\Bigg)-\frac{1}{2}\Gamma\Bigg(\frac{1}{2\alpha},(\frac{d}{\beta(1+r)})^{2\alpha}\Bigg)\Bigg\}\lambda\Bigg]\exp\Bigg((\frac{d}{(1+r)\beta})^\alpha\Bigg) \\
&\Bigg/\Bigg\{1-\lambda+\lambda\exp\Bigg(-(\frac{d}{(1+r)\beta})^\alpha\Bigg)\Bigg\}.
\end{aligned} \tag{4.29}
$$

□

Remark: With $\lambda = 0$, (4.29) reduces to (4.27).

4.1.11 Value at Risk (VaR) and Tail Value at Risk(TVaR)

4.1.11.1 Value at Risk (VaR) In (2.1.8.1), we explained Value at Risk (VaR). (2.37), (2.38) and (2.39) also defines how VaR is computed mathematically. The $VaR_p(X)$ of a Weibull distribution at p level is given by

$$\pi_p = VaR_p(X) = \beta[-\ln(1-p)]^{\frac{1}{\alpha}}.$$

Theorem 37. The Value-at-Risk of a transmuted Weibull distribution is $\pi_{p,\lambda} = \beta\Bigg\{-\ln\Bigg[\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\Bigg]\Bigg\}^{\frac{1}{\alpha}}$, where $0 < p < 1$ is the confidence level, $\lambda \neq 0$ is the transmutation parameter, β is the scale parameter and α is the shape parameter.

Proof. Applying (4.3) and the definition of $VaR_p(X)$ in (2.38),

$$\exp\Big(-(\frac{x}{\beta})^{\alpha}\Big)\Big\{1-\lambda+\lambda\exp\Big(-(\frac{x}{\beta})^{\alpha}\Big)\Big\}=1-p.$$

Solving the quadratic equation above for $\pi_{p,\lambda}$, We have

$$\pi_{p,\lambda}=\pi_{p,\lambda}=\beta\Big\{-\ln\Big[\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\Big]\Big\}^{\frac{1}{\alpha}},\quad \lambda\neq 0. \tag{4.30}$$

□

4.1.11.2 Tail Value at Risk (TVaR) For a Weibull distribution, the $TVaR_p(X)$ at p security level is

$$TVaR_p(X)=\frac{\alpha\beta(1-p)^{-\frac{1}{\alpha}}}{\alpha-1},\quad \alpha>1.$$

Theorem 38. At a p level, the $TVaR_p(X)$ of a transmuted Weibull distribution is

$$\begin{aligned}TVaR_{p,\lambda}(X)=VaR_{p,\lambda}(X)+\frac{E_\lambda[X]}{1-p}-\frac{\beta}{\alpha(1-p)}\Gamma\Big(\frac{1}{\alpha},(\frac{VaR_{p,\lambda}(X)}{\beta})^{\alpha}\Big)(1-\lambda)\\ -\frac{\beta}{2\alpha(1-p)}\Gamma\Big(\frac{1}{2\alpha},(\frac{VaR_{p,\lambda}(X)}{\beta})^{2\alpha}\Big)\lambda,\end{aligned}$$

where

$$E_\lambda[X]=\frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})(1-\lambda)+\lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha}),\quad VaR_{p,\lambda}(X)=\beta\Big\{-\ln\Big[\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\Big]\Big\}^{\frac{1}{\alpha}}.$$

Proof. Applying (2.44),

$$\begin{aligned}TVaR_{p,\lambda}(X) &= VaR_{p,\lambda}(X) + \frac{E_\lambda[X] - E_\lambda[X \wedge VaR_{p,\lambda}(X)]}{1-p}\\ &= VaR_{p,\lambda}(X) + \left[E_\lambda[X] - \left\{\frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}, (\frac{VaR_{p,\lambda}(X)}{\beta})^\alpha\right)(1-\lambda)\right.\right.\\ &\left.\left. + \frac{\beta}{2\alpha}\Gamma\left(\frac{1}{2\alpha}, (\frac{VaR_{p,\lambda}(X)}{\beta})^{2\alpha}\right)\lambda\right\}\right]\Big/(1-p)\\ &= VaR_{p,\lambda}(X) + \frac{E_\lambda[X]}{1-p} - \frac{\beta}{\alpha(1-p)}\Gamma\left(\frac{1}{\alpha}, (\frac{VaR_{p,\lambda}(X)}{\beta})^\alpha\right)(1-\lambda)\\ &- \frac{\beta}{2\alpha(1-p)}\Gamma\left(\frac{1}{2\alpha}, (\frac{VaR_{p,\lambda}(X)}{\beta})^{2\alpha}\right)\lambda, \quad 0<p<1,\end{aligned} \tag{4.31}$$

where

$$E_\lambda[X] = \frac{\beta}{\alpha}\Gamma(\frac{1}{\alpha})(1-\lambda)+\lambda\frac{\beta}{2\alpha}\Gamma(\frac{1}{2\alpha}), \quad VaR_{p,\lambda}(X) = \beta\left\{-\ln\left[\frac{\lambda-1+\sqrt{(1-\lambda)^2+4\lambda(1-p)}}{2\lambda}\right]\right\}^{\frac{1}{\alpha}}.$$

□

4.2 Numerical Analysis of Value-at-Risk and Tail-Value-at-Risk

4.2.1 Likelihood Function and Parameter Estimates

The likelihood function for the Weibull distribution is given by

$$\begin{aligned}L(\beta, \alpha; X_i) &= \prod_{i=1}^n f(x_i; \alpha, \beta)\\ &= \prod_{i=1}^n \frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\exp\left(-(\frac{x}{\beta})^\alpha\right)\end{aligned} \tag{4.32}$$

Taking log of the likelihood function of (4.32), we obtain

$$\ln[L(\beta, \alpha; X_i)] = n\ln\alpha - n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n(\frac{x_i}{\beta})^\alpha. \tag{4.33}$$

Also, the likelihood function for the Transmuted Weibull Distribution is given by:

$$L(\beta,\alpha,\lambda;X_i)=\prod_{i=1}^{n}\frac{\alpha}{\beta}(\frac{x_i}{\beta})^{\alpha-1}\exp\Big(-(\frac{x_i}{\beta})^{\alpha}\Big)\Big\{1-\lambda+2\lambda\exp\Big(-(\frac{x_i}{\beta})^{\alpha}\Big)\Big\} \tag{4.34}$$

Taking log of the likelihood function, we have:

$$\begin{aligned}\ln\Big[L(\beta,\alpha,\lambda;X_i)\Big] &= n\ln\alpha-n\alpha\ln\beta+(\alpha-1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}(\frac{x_i}{\beta})^{\alpha}\\ &+\sum_{i=1}^{n}\ln\Big[1-\lambda+2\lambda\exp\Big(-(\frac{x_i}{\beta})^{\alpha}\Big)\Big].\end{aligned} \tag{4.35}$$

4.2.2 Simulation Study

We perform a Monte Carlo (MC) study to estimate the parameters of the transmuted Weibull distribution. We generate a sample of size $n=(20,50,100,150,200,400)$ with $M=10,000$ simulations, using the inverse integral transformation of the cdf of the transmuted Weibull distribution.

Table 4.2: Simulation results, $\beta = 10,000, \alpha = 5, \lambda = 0.1$

n	**Parameter**	**Estimate**	**Lower**	**Upper**	**RMSE**	**MAE**	**MRE**
20	$\hat{\beta}_\lambda$	9945	8745	11145	615	497	0.9945
	$\hat{\alpha}_\lambda$	5.301	3.2429	7.3592	1.092	0.8350	1.0602
	$\hat{\lambda}$	0.0817	-0.6005	0.7640	0.3486	0.2882	0.8175
50	$\hat{\beta}_\lambda$	9962	8984	10940	501	400	0.9962
	$\hat{\alpha}_\lambda$	5.0466	3.779	6.3139	0.6482	0.5191	1.0093
	$\hat{\lambda}$	0.0818	-0.5932	0.7430	0.3418	0.2676	0.818
100	$\hat{\beta}_\lambda$	9973	9127	10819	433	337	0.9973
	$\hat{\alpha}_\lambda$	4.9849	4.047	5.923	0.4789	0.3916	0.9970
	$\hat{\lambda}$	0.0825	-0.5329	0.6952	0.3138	0.2311	0.825
150	$\hat{\beta}_\lambda$	9984	9204	10764	398	301	0.9984
	$\hat{\alpha}_\lambda$	4.977	4.19	5.764	0.4024	0.3299	0.9954
	$\hat{\lambda}$	0.0875	-0.4876	0.6626	0.2937	0.2075	0.8749
200	$\hat{\beta}_\lambda$	9993	9248	10739	380	281	0.99932
	$\hat{\alpha}_\lambda$	4.976	4.2676	5.6844	0.3622	0.2974	0.9952
	$\hat{\lambda}$	0.0930	-0.4581	0.6442	0.2813	0.1919	0.9302
200	$\hat{\beta}_\lambda$	9993	9248	10739	380	281	0.99932
	$\hat{\alpha}_\lambda$	4.976	4.2676	5.6844	0.3622	0.2974	0.9952
	$\hat{\lambda}$	0.0930	-0.4581	0.6442	0.2813	0.1919	0.9302
400	$\hat{\beta}_\lambda$	10003	9362	10645	327	225	1.003
	$\hat{\alpha}_\lambda$	4.974	4.421	5.526	0.2832	0.2288	0.9947
	$\hat{\lambda}$	0.0998	-0.3837	0.5833	0.2467	0.1549	0.9981

4.2.3 Real Life Application

The data utilized to estimate these parameters is the same data used in Chapter Two, which pertains to payments made for workers' compensation medical benefits, contained in [15].

Table 4.3: Model Comparison

Model	Parameter Estimates	$-2LL$	AIC	AICC	BIC
Weibull	$\hat{\alpha} = 0.68406$, $\hat{\beta} = 142,440$	506.4	510.4	511.11	512.39
Transmuted Weibull	$\hat{\alpha} = 0.6965$, $\hat{\beta} = 156,683$, $\hat{\lambda} = 0.73638$	503.6	509.6	511.10	512

$$AIC = 2k - 2\log(L), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad BIC = k\log(n) - 2\log L,$$

k is the number of parameters, n is the sample size, AIC is the Akaike Information Criterion, $AICC$ is the Akaike Information Criterion Corrected, BIC is the Bayesian Information Criterion.

From 4.3, we observe that the values of the transmuted Weibull distribution lead to a better fit than that of the Weibull distribution.

We also compare the performance of the distributions using a Likelihood Ration Test (LRT). We test the following hypothesis:

$$H_0 : \lambda = 0 \quad versus \quad H_1 : \lambda \neq 0.$$

We obtain the following results:

$$\omega = 6.28 > \chi_1^2(\alpha = 0.05) = 3.841.$$

where $\omega = 2\Big[LL_{Wei} - LL_{T.Wei}\Big] = 2(257.33 - 254.19) = 6.28.$

We therefore reject the null hypothesis and conclude that incorporating the transmutation parameter provides a better fit.

Table 4.4: Value at Risk (VaR) at different p levels

$p-level$	Weibull	Transmuted Weibull
0.1%	$1,424,163$	$1,884,204$
1%	$660,042$	$873,163$
5%	$383,364$	$506,929$
10%	$301,582$	$398,534$

From Tables 4.4, the transmuted Weibull distribution consistently provides higher VaR estimates compared to that of Weibull distribution across the chosen alpha levels.

CHAPTER FIVE

COMBINED APPLICATION OF COPULA AND TRANSMUTATION

5.1 Copula and Bivariate Transmuted Generalized Exponential Distribution

Copula functions possess the capability to connect individual marginal distributions to a joint distribution. For specified univariate marginal distribution functions $[F_1(t_1), F_2(t_2), \ldots, F_m(t_m)$, the function

$$F(t_1, t_2, \ldots, t_m) = C[F_1(t_1), F_2(t_2), \ldots, F_m(t_m)]. \tag{5.1}$$

which is defined using copula function C, results in a joint distribution function with univariate marginal distributions. We exemplify the combination of these two concepts with the generalized exponential distribution.

The generalized exponential distribution with two parameters has a density given by

$$g(x; \alpha, v) = \alpha v \big[1 - \mathrm{e}^{-vx}\big]^{\alpha - 1} \mathrm{e}^{-vx} \tag{5.2}$$

where $x > 0$, $\alpha > 0$, $v > 0$ are shape and scale parameters, respectively.

The generalized exponential CDF is given by

$$G(x) = P(T \leq x) = \alpha v \int_0^t \big[1 - \mathrm{e}^{-vt}\big]^{\alpha - 1} \mathrm{e}^{-vt} dt$$

Let $w = \mathrm{e}^{-vt} \quad \implies dw = -v\mathrm{e}^{-vt}$

Therefore:

$$\begin{aligned} G(x) &= \alpha v \int_1^{\mathrm{e}^{-vx}} [1 - w]^{\alpha - 1} \frac{-dw}{v} \\ &= \alpha \int_{\mathrm{e}^{-vx}}^1 (1 - w)^{\alpha - 1} dw \\ &= (1 - \mathrm{e}^{-vx})^{\alpha}, \quad x > 0. \end{aligned} \tag{5.3}$$

The transmuted PDF with the transmutation parameter λ is

$$f_\lambda(x) = g(x)[1 + \lambda - 2\lambda G(x)] \tag{5.4}$$

and

$$F_\lambda(x) = (1 + \lambda)G(x) - \lambda G^2(x). \tag{5.5}$$

Therefore, from (5.2) and (5.3),

$$f_\lambda(x) = \alpha v \left[1 - \mathrm{e}^{-vx}\right]^{\alpha-1} \mathrm{e}^{-vx} \left\{1 + \lambda - 2\lambda(1 - \mathrm{e}^{-vx})^{\alpha}\right\} \tag{5.6}$$

and

$$F_\lambda(x) = (1 + \lambda)(1 - \mathrm{e}^{-vx})^{\alpha} - \lambda(1 - \mathrm{e}^{-vx})^{2\alpha}, \tag{5.7}$$

where $x, v, \alpha > 0$, $-1 < \lambda < 1$.

We have

$$F(x_1, x_2, \ldots, x_m) = C\Big(F_1(x_1), F_2(x_2), \ldots, F_m(x_m)\Big).$$

Taking $m = 2$ in (5.1) and applying the Farlie-Gumbel-Morgenstern copula, we have

$$F(x, y) = F_1(x)F_2(y)\Big[1 + \theta(1 - F_1(x))(1 - F_2(y))\Big]. \tag{5.8}$$

Thus,

$$\begin{aligned} F(x, y) &= C\Big(F_1(x), F_2(y)\Big) \\ &= \left[1 - \mathrm{e}^{\ln(1-F_1(x))}\right]\left[1 - \mathrm{e}^{\ln(1-F_2(y))}\right]\left\{1 + \theta \mathrm{e}^{\ln(1-F_1(x))+\ln(1-F_2(y))}\right\}. \end{aligned} \tag{5.9}$$

Applying (5.3), we obtain

$$F(x, y) = (1-\mathrm{e}^{-v_1x})^{\alpha_1}(1-\mathrm{e}^{-v_2y})^{\alpha_2}\left[1+\theta\left[\left\{1-(1-\mathrm{e}^{-v_1x})^{\alpha_1}\right\}\left\{1-(1-\mathrm{e}^{-v_2y})^{\alpha_2}\right\}\right]\right], \tag{5.10}$$

where $-1 < \theta < 1$ is a correlation parameter.

This is a bivariate generalized exponential CDF of X and Y based on the Farlie - Gumbel- Morgenstern Copula.

Next, obtaining the transmuted joint CDF $F_{\lambda_1,\lambda_2}(x,y)$, we apply (5.7) and (5.8) to have the bivariate transmuted generalized CDF of X and Y:

$$\begin{aligned} F_{\lambda_1,\lambda_2}(x,y) = &\Bigg[(1+\lambda_1)(1-\mathrm{e}^{-v_1 x})^{\alpha_1} - \lambda_1(1-\mathrm{e}^{-v_1 x})^{2\alpha_1}\Bigg]\Bigg[(1+\lambda_2)(1-\mathrm{e}^{-v_2 y})^{\alpha_2} \\ &- \lambda(1-\mathrm{e}^{-v_2 y})^{2\alpha_2}\Bigg]\Bigg[1+\theta\Bigg\{\Bigg(1-\big[(1+\lambda_1)(1-\mathrm{e}^{-v_1 x})^{\alpha_1} - \lambda_1(1-\mathrm{e}^{-v_1 x})^{2\alpha_1}\big]\Bigg)\times \\ &\Bigg(1-\big[(1+\lambda_2)(1-\mathrm{e}^{-v_2 y})^{\alpha_2} - \lambda_2(1-\mathrm{e}^{-v_2 y})^{2\alpha_2}\big]\Bigg)\Bigg\}\Bigg]. \end{aligned} \tag{5.11}$$

Then the transmuted JPDF for X and Y follows from (5.11) via

$$f(x,y;\alpha_1,\alpha_2,v_1,v_2,\lambda_1,\lambda_2) = \frac{\partial^2 F_{\lambda_1,\lambda_2}(x,y)}{\partial x \partial y}.$$

But instead of obtaining f_{λ_1,λ_2} from (5.11) directly, we can get it from (5.8) indirectly and less painfully. From 5.10, we have

$$\begin{aligned}
f(x,y) &= \frac{\partial^2}{\partial y \partial x} F(x,y) \\
&= \frac{\partial^2}{\partial y \partial x}\left[F(x)F(y)\left\{1+\theta(1-F(x))(1-F(y))\right\}\right] \\
&= \frac{\partial}{\partial y}\left[F(y)\frac{\partial}{\partial x}F(x)\left\{1+\theta(1-F(x))(1-F(y))\right\}\right] \\
&= \frac{\partial}{\partial y}\left[F(y)\left(f(x)\big[F(x)F(y)\left\{1+\theta(1-F(x))(1-F(y))\right\}\big] + F(x)\left[-\theta f(x)(1-F(y))\right]\right)\right] \\
&= f(x)\frac{\partial}{\partial y}\left[F(y)\left\{1+\theta(1-F(x))(1-F(y))\right\}\right] - \theta F(x)f(x)\frac{\partial}{\partial y}\big[F(y)(1-F(y))\big] \\
&= f(x)\left[f(y)\left\{1+\theta(1-F(x))(1-F(y))\right\} + F(y)\theta(1-F(x))(-f(y))\right] \\
&- \theta F(x)f(x)\left[f(y)(1-F(y)) + F(y)(-f(y))\right] \\
&= f(x)f(y)\left[1+\theta(1-F(x))(1-F(y))\right] - \theta f(x)(1-F(x))f(y)(F(y)) \\
&- \theta f(x)F(x)f(y)(1-F(y)) + \theta f(x)F(x)f(y)F(y).
\end{aligned} \tag{5.12}$$

That is,

$$\begin{aligned}
f(x,y) = f(x)f(y)\Big\{&\big[1+\theta(1-F(x))(1-F(y))\big] - \theta(1-F(x))F(y) \\
&- \theta F(x)(1-F(y)) + \theta F(x)F(y)\Big\} \\
= f(x)f(y)\Big\{&1+\theta-\theta F(x)-\theta F(y)+\theta F(x)F(y)-\theta F(y) \\
&+ \theta F(x)F(y) - \theta F(x) + \theta F(x)F(y) + \theta F(x)F(y)\Big\}.
\end{aligned}$$

That is,

$$f(x,y) = f(x)f(y)\left\{(1+\theta) - 2\theta\left[F(x)+F(y)-2F(x)F(y)\right]\right\}. \tag{5.13}$$

Therefore, with the transmutation parameters λ_1 and λ_2, it follows from (5.13) that

$$f_{\lambda_1,\lambda_2}(x,y) = f_{\lambda_1}(x)f_{\lambda_2}(y)\Big\{(1+\theta)-2\theta\Big[F_{\lambda_1}(x)+F_{\lambda_2}(Y)-2F_{\lambda_1}(x)F_{\lambda_2}(y)\Big]\Big\}. \quad (5.14)$$

Note that $\theta = 0$ implies noncorrelation.

Now from (5.6) and (5.7) we can write $f_{\lambda_1,\lambda_2}(x,y)$ as

$$\begin{aligned} f_{\lambda_1,\lambda_2}(x,y) &= \alpha_1 v_1 \Big[1-\mathrm{e}^{-v_1 x}\Big]^{\alpha_1-1}\mathrm{e}^{-v_1 x}\Big\{1+\lambda_1-2\lambda_1(1-\mathrm{e}^{-v_1 x})^{\alpha_1}\Big\} \\ &\times \alpha_2 v_2 \Big[1-\mathrm{e}^{-v_2 y}\Big]^{\alpha_2-1}\mathrm{e}^{-v_2 y}\Big\{1+\lambda_2-2\lambda_2(1-\mathrm{e}^{-v_2 y})^{\alpha_2}\Big\} \\ &\times \Bigg[(1+\theta)-2\theta\Big\{(1+\lambda_1)(1-\mathrm{e}^{-v_1 x})^{\alpha_1}-\lambda_1(1-\mathrm{e}^{-v_1 x})^{2\alpha_1} \\ &+(1+\lambda_2)(1-\mathrm{e}^{-v_2 y})^{\alpha_2}-\lambda_2(1-\mathrm{e}^{-v_2 y})^{2\alpha_2} \\ &-2\Big((1+\lambda_1)(1-\mathrm{e}^{-v_1 x})^{\alpha_1}-\lambda_1(1-\mathrm{e}^{-v_1 x})^{2\alpha_1}\Big) \\ &\Big((1+\lambda_2)(1-\mathrm{e}^{-v_2 y})^{\alpha_2}-\lambda_2(1-\mathrm{e}^{-v_2 y})^{2\alpha_2}\Big)\Big\}\Bigg]. \end{aligned} \quad (5.15)$$

where α_1, α_2 are shape parameters, v_1, v_2 are scale parameters; λ_1, λ_2 are transmutation parameters; and θ is the correlation parameter contained in the copula.

Next is to write the likelihood function for insurance with deductibles. A general insurance setting may involve a deductible on either loss X or Y or on both variables. Let d_x and d_y be deductible for X and Y, respectively, with $d_x = 0$ and/or $d_y = 0$ if there are no deductibles. For $X < d_x$, the value is not observed and similarly when $Y < d_y$, so that observations are conditional on $X > d_x$ and $Y > d_y$. In addition, observations may be exact loss (x and/or y) or the loss may only be known to be in an interval, say, $v < x < w$ and/or $v < y < w$. When there is a policy limit, say, amount u on X, then the observation is $X = u$ but, in fact, the actual loss lies in the interval $[u, \infty)$.

There are four contributions to the likelihood function [16]. The four contributions arise from the following possibilities:

(a) x observed, y observed

(b) x observed, y in (v, w)

(c) x in (v, w), y observed

(d) x in (v, w), y in (v, w)

The four possibilities form the basis for contributions to the likelihood function. For the sake of transparency, we systematically build up the likelihood function to include the transmutation and copula.

The first form is for the basic bivariate distribution in (x, y) without transmutation or copula. The four parts listed above translate into the following contributions (or formulas):

(a) Individual observations for both X and Y:

$$\frac{f_{X,Y}(x,y)}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \tag{5.16}$$

(b) Individual observation for X, grouped observation for Y on interval (v, w):

$$\frac{\partial F(x,w)/\partial x - \partial F(x,v)/\partial x}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \tag{5.17}$$

(c) Individual observations for Y, grouped observation for X on interval (v, w):

$$\begin{aligned} &\frac{\partial F(w,y)/\partial y - \partial F(v,y)/\partial y}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \\ &= \frac{f_Y(y)[F_X(w) - F_X(v)]}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \end{aligned} \tag{5.18}$$

(d) Grouped observations for both X and Y on intervals (v_X, w_X) and (v_Y, w_Y), respectively.

$$\frac{F_{X,Y}(w_X, w_Y) - F_{X,Y}(v_X, w_Y) - F_{X,Y}(w_X, v_Y) + F_{X,Y}(v_X, v_Y)}{1 - F_X(d_x) - F_Y(d_y) + F_{X,Y}(d_x, d_y)} \tag{5.19}$$

5.1.1 Likelihood Function

We subdivide n observations into the four classes. For $i = 1, 2, 3, \ldots, n$, we state

C_1 : both x_i and y_i are individual observations.

C_2 : x_i is an individual observation and y_i is a grouped observation.

C_3 : x_i is a grouped observation and y_i is an individual observation.

C_4 : both x_i and y_i are grouped observations.

The likelihood function for a continuous model is given by:

From (a), (b), (c), (d) above, the likelihood function is given by

$$\begin{aligned} L = &\prod_{i \in C_1} \frac{f(x_i, y_i)}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \\ &\times \prod_{i \in C_2} \frac{\partial F(x_i, w)/\partial x_i - \partial F(x_i, v)/\partial x_i}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \\ &\times \prod_{i \in C_3} \frac{\partial F(w, y_i)/\partial y_i - \partial F(v, y_i)/\partial y_i}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \\ &\times \prod_{i \in C_4} \frac{F_{X,Y}(w_X, w_Y) - F_{X,Y}(v_X, w_Y) - F_{X,Y}(w_X, v_Y) + F_{X,Y}(v_X, v_Y)}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)}. \end{aligned} \tag{5.20}$$

Let us define the indicator variables δ_{1i} and δ_{2i} by

$$\delta_{1i} = \begin{cases} 1 & \text{if } x_i \quad is \quad an \quad observed \quad loss \\ 0 & \text{if } x_i \quad is \quad a \quad grouped \quad loss, \end{cases} \tag{5.21}$$

and

$$\delta_{2i} = \begin{cases} 1 & \text{if } y_i \quad is \quad an \quad observed \quad loss \\ \\ 0 & \text{if } y_i \quad is \quad a \quad grouped \quad loss, \end{cases} \tag{5.22}$$

where $i = 1, 2, \ldots, n$ and n is the number of observations.

With δ_{1i} and $\delta_2 i$ in (5.21) and (5.22), we can rewrite the likelihood function in (5.20) as

$$\begin{aligned} L = & \prod_{i \in C_1} \left[\frac{f(x_i, y_i)}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \right]^{\delta_{1i}\delta_{2i}} \\ & \times \prod_{i \in C_2} \left[\frac{\partial F(x_i, w)/\partial x_i - \partial F(x_i, v)/\partial x_i}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \right]^{\delta_{1i}(1-\delta_{2i})} \\ & \times \prod_{i \in C_3} \left[\frac{\partial F(w, y_i)/\partial y_i - \partial F(v, y_i)/\partial y_i}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \right]^{(1-\delta_{1i})\delta_{2i}} \\ & \times \prod_{i \in C_4} \left[\frac{F_{X,Y}(w_X, w_Y) - F_{X,Y}(v_X, w_Y) - F_{X,Y}(w_X, v_Y) + F_{X,Y}(v_X, v_Y)}{1 - F_X(d_X) - F_Y(d_Y) + F_{X,Y}(d_X, d_Y)} \right]^{(1-\delta_{1i})(1-\delta_{2i})}. \end{aligned} \tag{5.23}$$

Note that, if there is no grouped observation, this likelihood function reduces to the standard likelihood function under left truncation; exactly the first term. Similarly, the last term represents the likelihood function when all the observations are grouped that is, observed within an interval. Under transmutation, the likelihood function can be written as

$$
\begin{aligned}
L = &\prod_{i \in C_1} \left[\frac{f_{\lambda_1,\lambda_2}(x_i, y_i)}{1 - F_{\lambda_1}(d_X i) - F_{\lambda}(d_Y i) + F_{\lambda_1,\lambda_2}(d_X i, d_Y i)} \right]^{\delta_{1i}\delta_{2i}} \\
&\times \prod_{i \in C_2} \left[\frac{f_{\lambda_1}(x_i)[F_{\lambda_2}(w_{Y_i}) - F_{\lambda_2}(v_{Y_i})]}{1 - F_{\lambda_1}(d_X i) - F_{\lambda}(d_Y i) + F_{\lambda_1,\lambda_2}(d_X i, d_Y i)} \right]^{\delta_{1i}(1-\delta_{2i})} \\
&\times \prod_{i \in C_3} \left[\frac{f_{\lambda_2}(y_i)[F_{\lambda_1}(w_{X_i}) - F_{\lambda_1}(v_{X_i})]}{1 - F_{\lambda_1}(d_X i) - F_{\lambda}(d_Y i) + F_{\lambda_1,\lambda_2}(d_X i, d_Y i)} \right]^{(1-\delta_{1i})\delta_{2i}} \\
&\times \prod_{i \in C_4} \left[\frac{F_{\lambda_1,\lambda_2}(w_{X_i}, w_{Y_i}) - F_{\lambda_1,\lambda_2}(v_{X_i}, w_{Y_i}) - F_{\lambda_1,\lambda_2}(w_{X_i}, v_{Y_i}) + F_{\lambda_1,\lambda_2}(v_{X_i}, v_{Y_i})}{1 - F_{\lambda_1}(d_X i) - F_{\lambda}(d_Y i) + F_{\lambda_1,\lambda_2}(d_X i, d_Y i)} \right]^{(1-\delta_{1i})(1-\delta_{2i})}.
\end{aligned}
\tag{5.24}
$$

where λ_1 and λ_2 are transmutation parameters included in the distributions of X and Y, respectively.

Putting the actual transmuted PDF's and CDF's in (5.24) and simplifying, the likelihood function becomes

$$
\begin{aligned}
&\prod_{i=1}^{n}\Bigg[\Bigg(f(x_i)f(y_i)\big[1-\lambda_1\big(2F(x_i)-1\big)\big]\big[1-\lambda_2\big(2F(y_i)-1\big)\big]\Big[(1+\theta)-2\theta\Big\{F(x_i)\big(1+\lambda_1 S(x_i)\big)\\
&+F(y_i)\big(1+\lambda_2 S(y_i)\big)-2\big[1-\lambda_1\big(2F(x_i)-1\big)\big]\big[1-\lambda_2\big(2F(y_i)-1\big)\big]\Big\}\Big]\Bigg)\Big/\Delta\Bigg]^{\delta_{1i}\delta_{2i}}\\
&\times\Bigg[\Bigg(\Bigg(f(x_i)\Big[1-\lambda_1\big(2F(x_i)-1\big)\Big]\Big[F(w_{y_i})\Big\{1-\lambda_2[F(w_{y_i})-1]\Big\}-F(v_{y_i})\Big\{1-\lambda_2[F(v_{y_i})-1]\Big\}\Big]\Bigg)\\
&\Big/\Delta\Bigg]^{\delta_{1i}(1-\delta_{2i})}\\
&\times\Bigg[\Bigg(f(y_i)\big[1-\lambda_2\big(2F(y_i)-1\big)\big]\Big[F(w_{x_i})\{1-\lambda_1\big(F(w_{x_i})-1\big)\}\Big]-\Big[F(v_{x_i})\{1-\lambda_1\big(F(v_{x_i})-1\big)\}\Big]\Bigg)\\
&\Big/\Delta\Bigg]^{(1-\delta_{1i})\delta_{2i}}\\
&\times\Bigg[F(w_{x_i})\big(1+\lambda_1 S(w_{x_i})\big)F(w_{y_i})\big(1+\lambda_2 S(w_{y_i})\big)\cdot\Bigg\{1+\theta\Big[\big(1-F(w_{x_i})(1+\lambda_1 S(w_{x_i}))\big)\\
&\big(1-F(w_{y_i})(1+\lambda_2 S(w_{y_i}))\big)\Big]\Bigg\}-F(v_{x_i})\big(1+\lambda_1 S(w_{x_i})\big)F(w_{y_i})\big(1+\lambda_2 S(w_{y_i})\big)\\
&\cdot\Bigg\{1+\theta\Big[\big(1-F(v_{x_i})(1+\lambda_1 S(v_{x_i}))\big)\big(1-F(w_{y_i})(1+\lambda_2 S(w_{y_i}))\big)\Big]\Bigg\}\\
&-F(w_{x_i})\big(1+\lambda_1 S(w_{x_i})\big)F(v_{y_i})\big(1+\lambda_2 S(v_{y_i})\big)\cdot\Bigg\{1+\theta\Big[\big(1-F(w_{x_i})(1+\lambda_1 S(w_{x_i}))\big)\\
&\big(1-F(v_{y_i})(1+\lambda_2 S(v_{y_i}))\big)\Big]\Bigg\}\\
&+F(v_{x_i})\big(1+\lambda_1 S(v_{x_i})\big)F(v_{y_i})\big(1+\lambda_2 S(v_{y_i})\big)\cdot\Bigg\{1+\theta\Big[\big(1-F(v_{x_i})(1+\lambda_1 S(v_{x_i}))\big)\\
&\big(1-F(v_{y_i})(1+\lambda_2 S(v_{y_i}))\big)\Big]\Bigg\}\Bigg)\Big/\Delta\Bigg]^{(1-\delta_{1i})(1-\delta_{2i})},
\end{aligned}
\tag{5.25}
$$

where

$$\Delta = \Big(1 - F(d_{x_i})\big(1 + \lambda_1 S(d_{x_i})\big) - F(d_{y_i})\big(1 + \lambda_2 S(d_{y_i})\big) + \Big\{\big[1 - \lambda_1\big(2F(d_{x_i}) - 1\big)\big]$$

$$\big[1 - \lambda_2\big(2F(d_{y_i}) - 1\big)\big] \cdot \Big[1 + \theta\big[\big(-\lambda_1[2F(d_{x_i}) - 1\big)\big(-\lambda_2[2F(d_{y_i}) - 1\big)]\big]\Big]\Big\}\Big).$$

Reducing to only individual observations in X and Y, *i.e*, $\delta_{1i} = 1$ and $\delta_{2i} = 1$, the likelihood function reduces to only the first contribution. Thus, under that scenario, the likelihood function in (5.25) becomes

$$\begin{aligned} L[v_1, v_2, \lambda_1, \lambda_2, \theta | x, y] = \prod_{i=1}^{n} \Bigg(& (v_1 \mathrm{e}^{-v_1 x_i})(v_2 \mathrm{e}^{-v_2 y_i}) \Big[1 - \lambda_1 \Big(2(v_1 \mathrm{e}^{-v_1 x_i}) - 1\Big)\Big] \times \\ & \Big[1 - \lambda_2 \Big(2(v_2 \mathrm{e}^{-v_2 y_i}) - 1\Big)\Big] \Big\{(1+\theta) - 2\theta \Big[(1 - \mathrm{e}^{-v_1 x_i})(1 + \lambda_1 \mathrm{e}^{-v_1 x_i}) \\ & + (1 - \mathrm{e}^{-v_2 y_i})(1 + \lambda_2 \mathrm{e}^{-v_2 y_i}) - 2\Big(1 - \lambda_1(2(1 - \mathrm{e}^{-v_1 x_i}) - 1)\Big) \\ & \Big(1 - \lambda_2(2(1 - \mathrm{e}^{-v_2 y_i}) - 1)\Big)\Big]\Big\}\Bigg) \Bigg/ \\ & \Bigg(1 - (1 - \mathrm{e}^{-v_1 x_i})(1 + \lambda_1 \mathrm{e}^{-v_1 x_i}) - (1 - \mathrm{e}^{-v_2 y_i})(1 + \lambda_2 \mathrm{e}^{-v_2 y_i}) \\ & + \Bigg[\Big[1 - \lambda_1(2(1 - \mathrm{e}^{-v_1 x_i}) - 1)\Big]\Big[1 - \lambda_2(2(1 - \mathrm{e}^{-v_2 y_i}) - 1)\Big] \\ & \Big\{1 + \theta\Big[\big[-\lambda_1(2(1 - \mathrm{e}^{-v_1 x_i}) - 1)\big]\big[-\lambda_2(2(1 - \mathrm{e}^{-v_2 y_i}) - 1)\big]\Big]\Big\}\Bigg]\Bigg). \end{aligned} \tag{5.26}$$

Simplifying, we get

$$L[v_1, v_2, \lambda_1, \lambda_2, \theta; x, y] = v_1^n v_2^n \mathrm{e}^{-v_1 \sum_{i=1}^n x_i} \mathrm{e}^{-v_2 \sum_{i=1}^n y_i} \mathrm{e}^{\sum_{i=1}^n \ln[1-\lambda_1(2(1-\mathrm{e}^{-v_1 x_i})-1)]}$$

$$\mathrm{e}^{\sum_{i=1}^n \ln[1-\lambda_2(2(1-\mathrm{e}^{-v_2 y_i})-1)]} \times \mathrm{e}^{\sum_{i=1}^n \ln\left[(1+\theta)-2\theta\left((1-\mathrm{e}^{-v_1 x_i})(1+\lambda_1 \mathrm{e}^{-v_1 x_i})\right.\right.}$$

$$+ (1 - \mathrm{e}^{-v_2 y_i})(1 + \lambda_2 \mathrm{e}^{-v_2 y_i}) - 2\left[1 - \lambda_1\left(2(v_1 \mathrm{e}^{-v_1 x_i}) - 1\right)\right]$$

$$\left.\left.\left[1 - \lambda_2\left(2(v_2 \mathrm{e}^{-v_2 y_i}) - 1\right)\right]\right)\right]$$

$$\bigg/ (\Delta_{(1)})^n. \tag{5.27}$$

where $\Delta_{(1)}$ is the denominator in (5.26) evaluated at the deductible of 1 million, that is, at $dx_i = dy_i = 1$ million for $i = 1, 2, \ldots, n$.

5.2 Bayesian Methodology

5.2.1 Noniformative Prior

Inclusion of Bayes methodology via a joint uniform prior for the parameters $v_1, v_2, \lambda_1, \lambda_2$, and θ is next via

$$\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta) \propto 1. \tag{5.28}$$

From (5.27) and (5.28), the kernel of the joint posterior density of $v_1, v_2, \lambda_1, \lambda_2$, and θ, given x and y, is

$$\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta | x, y) \propto v_1^n v_2^n \mathrm{e}^{-v_1 \sum_{i=1}^n x_i} \mathrm{e}^{-v_2 \sum_{i=1}^n y_i} \mathrm{e}^{\sum_{i=1}^n \ln[1-\lambda_1(2(1-\mathrm{e}^{-v_1 x_i})-1)]}$$

$$\mathrm{e}^{\sum_{i=1}^n \ln[1-\lambda_2(2(1-\mathrm{e}^{-v_2 y_i})-1)]} \times \mathrm{e}^{\sum_{i=1}^n \ln\Big[(1+\theta)-2\theta\Big((1-\mathrm{e}^{-v_1 x_i})(1+\lambda_1 \mathrm{e}^{-v_1 x_i})}$$

$$+ (1 - \mathrm{e}^{-v_2 y_i})(1 + \lambda_2 \mathrm{e}^{-v_2 y_i}) - 2\Big[1 - \lambda_1\Big(2(v_1 \mathrm{e}^{-v_1 x_i}) - 1\Big)\Big]$$

$$\Big[1 - \lambda_2\Big(2(v_2 \mathrm{e}^{-v_2 y_i}) - 1\Big)\Big]\Big)\Big] \Big/ (\Delta_{(1)})^n. \tag{5.29}$$

We can re-write the joint posterior density of $\boldsymbol{\psi}(v_1, v_2, \lambda_1, \lambda_2, \theta)$ in (5.29) under uniform prior as

$$\begin{aligned}
\Pi\big(\boldsymbol{\psi} | x, y\big) \propto\ & f_{v_1}\Big(n, \sum_{i=1}^n x_i\Big) f_{v_2}\Big(n, \sum_{i=1}^n y_i\Big) \\
& \cdot f_{\lambda_1}\Big(\exp \sum_{i=1}^n \ln\big[1 - \lambda_1\big(2(1 - \mathrm{e}^{-v_1 x_i}) - 1\big)\big]\Big) \Big/ (\Delta_{(1)})^n \\
& \cdot f_{\lambda_2}\Big(\exp \sum_{i=1}^n \ln\big[1 - \lambda_2\big(2(1 - \mathrm{e}^{-v_2 y_i}) - 1\big)\big]\Big) \Big/ (\Delta_{(1)})^n \\
& \cdot f_{\theta}\Big(\exp \sum_{i=1}^n \ln\Big[(1 + \theta) - 2\theta\Big((1 - \mathrm{e}^{-v_1 x_i})(1 + \lambda_1 \mathrm{e}^{-v_1 x_i}) \\
& + (1 - \mathrm{e}^{-v_2 y_i})(1 + \lambda_2 \mathrm{e}^{-v_2 y_i}) - 2\Big[1 - \lambda_1\Big(2(v_1 \mathrm{e}^{-v_1 x_i}) - 1\Big)\Big] \\
& \Big[1 - \lambda_2\Big(2(v_2 \mathrm{e}^{-v_2 y_i}) - 1\Big)\Big]\Big)\Big]\Big) \Big/ (\Delta_{(1)})^n,
\end{aligned} \tag{5.30}$$

where f_{v_1} and f_{v_2} represent gamma density functions of V_1 and V_2, respectively, while f_{λ_1} and f_{λ_2} represent the probability density functions of λ_1 and λ_2, respectively, which are not well-known probability density functions. Similarly, f_θ represents a probability density function of θ, and it is also not well known.

To obtain the complete form of the joint posterior density, it follows from (5.28) and (5.29) that the joint posterior density under a uniform prior in (5.28) for parameters $v_1, v_2, \lambda_1, \lambda_2$ and θ, given data x and y, can be obtained from

$$\Pi(\boldsymbol{\psi}|x,y) = \frac{L(v_1, v_2, \lambda_1, \lambda_2, \theta; x, y)\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta)}{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty}\int_{0}^{\infty} L(v_1, v_2, \lambda_1, \lambda_2, \theta; x, y)\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta)dv_1 dv_2 d\lambda_1 d\lambda_2 d\theta}, \tag{5.31}$$

where the parameters' vector is $\boldsymbol{\psi} = (v_1, v_2, \lambda_1, \lambda_2, \theta)'$.

5.2.2 Informative Prior

Let the informative priors be given as

$$\begin{aligned} &v_1 \sim Gamma(a_1, b_1), \quad v_2 \sim Gamma(a_2, b_2), \\ &\lambda_1 \sim UNIF(c_1, c_2), \quad \lambda_2 \sim UNIF(d_1, d_2), \\ &\theta \sim UNIF(e_1, e_2), \end{aligned} \tag{5.32}$$

where $a_1, b_1, a_2, b_2, c_1, c_2, d_1, d_2, e_1$ and e_2 are hyperparameters.

This the informative joint prior density for $v_1, v_2, \lambda_1, \lambda_2$, and θ is given by

$$\begin{aligned} \Pi(\boldsymbol{\psi}) &= \Pi(v_1, v_2, \lambda_1, \lambda_2, \theta) \\ &= v_1^{a_1} \mathrm{e}^{-b_1 v_1} \cdot v_2^{a_2} \mathrm{e}^{-b_2 v_2} \cdot \frac{1}{c_2 - c_1} \cdot \frac{1}{d_2 - d_1} \cdot \frac{1}{e_2 - e_1} \end{aligned} \tag{5.33}$$

where $\boldsymbol{\psi}$ $(v_1, v_2, \lambda_1, \lambda_2, \theta)'$ is a vector of parameters and $a_1, b_1, a_2, b_2, c_1, c_2, d_1, d_2, e_1$ and e_2 are hyperparameters, $-1 < c_1, c_2, d_1, d_2, e_1, e_2 < 1$.

It follows from (5.27) and (5.33) that the joint posterior density under informative prior for parameters $v_1, v_2, \lambda_1, \lambda_2, \theta$, given data x and y, is

$$\Pi(\boldsymbol{\psi}|x,y) = \frac{L(v_1, v_2, \lambda_1, \lambda_2, \theta|x, y)\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta)}{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty}\int_{0}^{\infty} L(v_1, v_2, \lambda_1, \lambda_2, \theta|x, y)\Pi(v_1, v_2, \lambda_1, \lambda_2, \theta)dv_1 dv_2 d\lambda_1 d\lambda_2 d\theta}. \tag{5.34}$$

Similar to (5.30), the kernel of the joint posterior density function under the informative prior is given by

$$\begin{aligned}
\Pi\big(\boldsymbol{\psi}|x,y\big) \propto\ & f_{v_1}\Big(n+a_1, b_1+\sum_{i=1}^{n} x_i\Big) f_{v_2}\Big(n+a_2, b_2+\sum_{i=1}^{n} y_i\Big)\\
& \cdot f_{\lambda_1}\Big(l_1 \exp\sum_{i=1}^{n} \ln\big[1-\lambda_1\big(2(1-\mathrm{e}^{-v_1x_i})-1\big)\big]\Big)\Big/(\Delta_{(1)})^n\\
& \cdot f_{\lambda_2}\Big(l_2 \exp\sum_{i=1}^{n} \ln\big[1-\lambda_2\big(2(1-\mathrm{e}^{-v_2y_i})-1\big)\big]\Big)\Big/(\Delta_{(1)})^n\\
& \cdot f_{\theta}\Big(l_3+\exp\sum_{i=1}^{n} \ln\Big[(1+\theta)-2\theta\Big((1-\mathrm{e}^{-v_1x_i})(1+\lambda_1\mathrm{e}^{-v_1x_i})\\
& +(1-\mathrm{e}^{-v_2y_i})(1+\lambda_2\mathrm{e}^{-v_2y_i})-2\Big[1-\lambda_1\Big(2(v_1\mathrm{e}^{-v_1x_i})-1\Big)\Big]\\
& \Big[1-\lambda_2\Big(2(v_2\mathrm{e}^{-v_2y_i})-1\Big)\Big]\Big)\Big]\Big)\Big/(\Delta_{(1)})^n,
\end{aligned} \tag{5.35}$$

where f_{v_1} and f_{v_2} represent gamma density functions of v_1 and v_2, respectively, while f_{λ_1} and f_{λ_2} represent the probability density functions of λ_1 and λ_2, respectively, which are not well-known probability density functions. Similarly, f_θ represents a probability density function of θ, and it is also well known. The hyperparameters (a_1, b_1), (a_2, b_2), l_1, l_2 and l_3 are for $v_1, v_2, \lambda_1, \lambda_2, \theta$, respectively.

CHAPTER SIX

CONCLUSION AND FUTURE WORK

In this research, we have embarked on a journey through the intricate world of actuarial science and statistics, exploring the fundamental importance of transmuted distributions in modeling extreme insurance losses. Recognizing the imperative to address extreme events such as natural disasters, terrorist attacks, and pandemics, often accompanied by large losses, we applied the concept of transmuted distributions. These distributions, with the incorporation of transmutation parameters, offer a powerful tool for adapting standard probability distributions to better capture the characteristics of data with large values and accompanying significant skewness. Through this transformation, we sought to improve the accuracy of predictions and enhance risk management strategies within the insurance industry.

Chapters 2, 3, and 4 guided us through the exploration of specific traditional distributions (Exponential [2], Pareto [3], and Weibull) commonly employed in actuarial science, alongside their transmuted counterparts. These chapters illuminated the potential of transmuted distributions in modeling insurance loss data, providing a flexible and robust framework for actuaries and risk assessors. Comparative analyses between traditional and transmuted models showcase the advantages of the latter in capturing extreme values and improving the overall fit to the data.

In Chapter 5, we introduce an innovative approach that combines copula functions and transmuted distributions. This novel approach allows us to capture the intricate dependency structures within insurance portfolios while effectively addressing skewness and extreme events. Moreover, we utilize Bayesian methodologies to find the posterior distribution with both informative and non-informative priors, enhancing the precision and adaptability of our models.

Looking ahead, the path for research in this field extends to exciting prospects. Future work will focus on the estimation of the parameters of these posterior distributions from Chapter 5, which comes from copula and transmutation, using the Markov Chain Monte Carlo (MCMC) method. This advanced technique will enable us to efficiently explore high-dimensional parameter spaces and provide more accurate estimates, further enhancing our ability to model and manage insurance risks.

REFERENCES

[1] Hamza Abubakar and Shamsul Rijal Muhammad Sabri, *A bayesian approach to weibull distribution with application to insurance claims data*, Journal of Reliability and Statistical Studies (2023), 1–24.

[2] S Addai and T Ogunyemi, *Modeling extreme insurance losses using transmuted exponential distribution*, Journal of Computational and Applied Mathematics (2023).

[3] ______, *Modeling extreme insurance losses using transmuted pareto distribution*, Journal of Insurance: Mathematics and Economics (2023).

[4] Gokarna R Aryal, *Transmuted log-logistic distribution*, Journal of Statistics Applications & Probability **2** (2013), no. 1, 11.

[5] Gokarna R Aryal and Chris P Tsokos, *On the transmuted extreme value distribution with application*, Nonlinear Analysis: Theory, Methods & Applications **71** (2009), no. 12, e1401–e1407.

[6] ______, *Transmuted weibull distribution: A generalization of theweibull probability distribution*, European Journal of pure and applied mathematics **4** (2011), no. 2, 89–102.

[7] Muhammad Aslam, Sajid Ali, Rahila Yousaf, and Ismail Shah, *Mixture of transmuted pareto distribution: Properties, applications and estimation under bayesian framework*, Journal of the Franklin Institute **357** (2020), no. 5, 2934–2957.

[8] Sher B Chhetri, Alfred A Akinsete, Gokarna Aryal, and Hongwei Long, *The kumaraswamy transmuted pareto distribution*, Journal of Statistical Distributions and Applications **4** (2017), no. 1, 1–24.

[9] Paul Embrechts and Ruodu Wang, *Seven proofs for the subadditivity of expected shortfall*, Dependence Modeling **3** (2015), no. 1.

[10] Omer L Gebizlioglu, Birdal Şenoğlu, and Yeliz Mert Kantar, *Comparison of certain value-at-risk estimation methods for the two-parameter weibull loss distribution*, Journal of Computational and Applied Mathematics **235** (2011), no. 11, 3304–3314.

[11] Anwar Hassan, Sameer Ahmad Wani, and Sumeera Shafi, *Pak. j. statist. 2020 vol. 36 (1), 73-89 lindley-quasi xgamma distribution: Properties and applications*, Pak. J. Statist **36** (2020), no. 1, 73–89.

[12] Robert V Hogg and Stuart A Klugman, *Loss distributions*, John Wiley & Sons, 2009.

[13] Terna G Ieren, Sauta S Abdulkadir, Abraham Okolo, Danjuma Jibasen, and Ikeme J Dike, *Statistical properties and applications of a transmuted exponential-inverse exponential distribution*, Equity Journal of Science and Technology **7** (2021), no. 2, 105–105.

[14] Muhammad Shuaib Khan, Robert King, and Irene Lena Hudson, *Transmuted generalized exponential distribution: A generalization of the exponential distribution with applications to survival data*, Communications in Statistics-Simulation and Computation **46** (2017), no. 6, 4377–4398.

[15] Stuart A Klugman, Harry H Panjer, and Gordon E Willmot, *Loss models: from data to decisions*, vol. 715, John Wiley & Sons, 2012.

[16] Stuart A Klugman and Rahul Parsa, *Fitting bivariate loss distributions with copulas*, Insurance: mathematics and economics **24** (1999), no. 1-2, 139–148.

[17] William T Shaw and Ian RC Buckley, *The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map*, arXiv preprint arXiv:0901.0434 (2009).