A HYBRIDIZED DISCONTINUOUS GALERKIN SCHEME FOR THE COUPLED STOKES-DARCY FLOW AND TRANSPORT

by

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To my parent, my brother and my sister

for their endless love and encouragement.

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ABSTRACT

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Adviser: Aycil Cesmelioglu, Ph.D.

The main focus of this thesis is on finding highly accurate and robust numerical methods to solve a complex flow and transport problem governed by the fully-coupled time-dependent Stokes-Darcy-transport equations. This problem has many applications one of which is groundwater contamination by pollutants transported via surface/subsurface flow. It consists of two main ingredients; the time-dependent Stokes-Darcy equations describing the flow, and the time-dependent advection-diffusion equation for the transport of chemicals via this flow.

Therefore, the first part of this thesis is dedicated to studying the time-dependent Stokes-Darcy problem that describes the free flow and porous media flow on two different parts of a domain and their interaction at the common interface. We introduce a hybridized discontinuous Galerkin (HDG) method which provides exact mass conservation and pressure robustness and handles the interface conditions via facet unknowns. We prove well-posedness and a priori error estimates in the energy norm, and provide numerical experiments that show optimal convergence and robustness of the method with respect to the problem parameters.

The second part deals with the time-dependent advection-diffusion equation where we again use an HDG method for the spatial discretization. We show the existence and uniqueness of the semi-discrete transport problem and prove a priori error estimates in the energy norm. A number of numerical experiments are presented for different boundary conditions and we observe optimal rates of convergence in each case.

Combining the two parts by a sequential algorithm, we solve the fully coupled time-dependent Stokes-Darcy-transport problem. The coupling of the flow and transport is introduced by the dependence of the fluid viscosity and source/sink terms on the concentration and by the dependence of the dispersion/diffusion tensor in the porous media domain on the advective fluid velocity. Our sequential algorithm employs a linearizing decoupling strategy based on the backward Euler time-stepping where the Stokes-Darcy and the transport equations are solved sequentially by time-lagging the concentration. The well-posedness results and a priori error estimates for the velocity and the concentration in the energy norm are presented and numerical examples demonstrating optimal convergence and mass conservation are provided.

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$(w)^t$	Tangential component of a vector w, $w - (w \cdot n)n$
Γ	Set of all boundary facets
Γ^0	Union of all facets
$[\![v]\!]$	Jump of v
$\mathcal{F}$	Set of all boundary faces and interfaces of a mesh
$\mathcal{F}^b$	Set of all boundary faces of a mesh
$\mathcal{F}^i$	Set of all interfaces of a mesh
$\mathcal{T}$	Mesh of the domain Ω
∂K	Boundary of K
h	Meshsize
$H(\mathrm{div}, \Omega)$	Space of all functions v in $L^2(\Omega)$ such that $\nabla \cdot v \in L^2(\Omega)$
$H^k(\Omega) = W^{k,2}(\Omega)$	Sobolev space of order k in Ω, $p = 2$
h_K	Diameter of K
$L_0^2(\Omega)$	Space of the functions in $L^2(\Omega)$ with zero mean value over Ω
n	Outward unit normal vector
$P_k(K)$	Space of polynomials of degree at most k on K
CG	Continuous Galerkin
DG	Disontinuous Galerkin
EDG	Embedded Discontinuous Galerkin
HDG	Hybridized Discontinuous Galerkin

CHAPTER ONE

INTRODUCTION

1.1 The Stokes-Darcy and transport problems

The coupled Stokes and Darcy problem is defined by a system of partial differential equations consisting of the Stokes equations for free flow and the Darcy equations for porous media flow. These equations are coupled at the interface separating the free flow and porous media flow regions via appropriate interface conditions. The model has various applications, for example, it can be used to describe blood flow in vessels in biomedical engineering, groundwater flow in environmental engineering [5], or flow in the industrial filtration processes [33]. Coupling the Stokes-Darcy equations to an advection-diffusion equation results in a mathematical model that describes the transport of chemicals by surface/subsurface flow in groundwater contamination problem, or by blood flow in vessels in the biochemical transport problem [22].

Different mass-conserving finite element methods have been proposed to discretize the Stokes-Darcy equations. In [37], Kanschat and Rivière proposes a finite element method in which the Stokes equations are discretized by a discontinuous Galerkin (DG) method using divergence-conforming finite elements for the velocity, and the Darcy equation is discretized by using a mixed finite element method resulting in a strongly mass conservative scheme. In [28], a strongly conservative hybridized discontinuous Galerkin (HDG)/mixed discretization for the Stokes-Darcy problem is presented where a divergence-conforming HDG method is used for the velocity and a mixed finite element method is used for the Darcy flow. In [16], a mass conserving embedded–hybridized discontinuous Galerkin

(EDG-HDG) discretization for the coupled Stokes-Darcy system was introduced by using the EDG method for the Stokes equations with continuous Stokes velocity trace unknown and the HDG method for the Darcy equation.

Similarly, different stable numerical methods have been developed to discretize the transport equation, for instance, streamline upwind Petrov-Galerkin methods [11], DG [21], [41], and EDG methods [53].

However, combining a stable and accurate discretization for the Stokes–Darcy problem and a stable and accurate discretization for the transport problem does not directly ensure that the discretization for the coupled Stokes–Darcy–transport problem will inherit stability and accuracy. It is known that [45] if the velocity approximation is not local mass conservative, the numerical solution of the coupled flow and transport problems in porous media will be unstable and may present unphysical oscillations. Furthermore, compatibility in the sense of [23] is desirable to avoid loss of accuracy or loss of global conservation. Therefore, the specifics of the combination must be determined carefully.

There are only a few papers in the literature on a special case of this coupled problem, where the coupling is only one-way. In this loosely coupled problem, the Stokes-Darcy flow is assumed to be unaffected by the changes in the concentration. The first of these papers, [52], uses a combination of mixed finite element and local discontinuous Galerkin methods. Then [46] uses discontinuous Galerkin methods, [36] uses the mixed finite element method and Streamline Upwind Petrov–Galerkin method, and [12] uses the embedded-hybridized discontinuous Galerkin method. The published literature about the fully coupled Stokes–Darcy-transport problem is very sparse. Within our knowledge, there are only two papers that studied the fully coupled Stokes-Darcy-transport problem. The first one [17] is a theoretical paper analyzing a weak solution for the

(Navier-)Stokes/Darcy-transport problem. The second one [47] uses the mixed finite element method for the Stokes-Darcy problem and classical piecewise linear finite element for the concentration problem.

In this thesis, we propose a sequential algorithm that decouples the problem into two subproblems (Stokes-Darcy and transport) which are both solved by the HDG method. This method maintains the nice properties of the DG method such as local mass conservation while having fewer degrees of freedom for high order polynomials. Furthermore, the HDG method for the Stokes-Darcy subproblem produces strong mass conservation and pressure robustness, and by selecting the polynomial degrees carefully we achieve compatibility and avoid loss of accuracy.

1.2 A brief introduction to HDG methods

Among many techniques developed for the approximate solution of partial differential equations (PDEs), the continuous Galerkin (CG) Method is often used. The CG method is based on a weak formulation of the PDEs that is defined on an infinite-dimensional space. The solution is approximated by using a finite dimensional subspace of the infinite-dimensional space such as the space of piecewise polynomials that are continuous across element interfaces. One drawback of the standard formulations of the CG method is that it does not perform well with convection-dominated problems and it lacks local mass conservation property which is crucial for transport problems.

A method that overcomes these drawbacks of the CG method is the discontinuous Galerkin Method (DG) which was first introduced in [43] in 1973 for the linear hyperbolic problems and was later extended to other types of PDEs [2, 3]. It is also built on a weak formulation, but the continuity requirement on the finite-dimensional subspace is relaxed and inter element continuity is imposed weakly. There are many advantages of the DG method. For example, it is locally mass conservative, can handle discontinuous parameters, is highly parallelizable and is ideal for adaptivity and parallel computing because of the relaxed continuity requirement. It allows for flexibility in the choice of approximation spaces compared to mixed finite element methods and can achieve high order accuracy by using high degree approximating polynomials easily.

A disadvantage of the DG methods is that they produce more degrees of freedom than the continuous Galerkin (CG) methods. In order to reduce the computational cost associated with the DG methods, the hybridized discontinuous Galerkin (HDG) methods were introduced and applied to a variety of problems, for instance, second order elliptic problems [20], Stokes problems [19, 44], Navier-Stokes problems [13], etc. The HDG method introduces trace unknowns that approximate the solution on the element boundaries and numerical fluxes that allow inter element communication only via these trace unknowns. This makes it possible to apply static condensation thus reducing the system to a system for only the single-valued trace unknowns. After solving this reduced system for the trace unknowns, the main unknowns can be obtained independently inside each element. The trace system has less globally coupled degrees of freedom compared to a standard DG system especially for high orders [35], thus it requires less computational time and memory. For higher order approximations, after static condenstation, the HDG system is comparable in size to the standard CG system [38, 54]. Figure 1.1 illustrates the degrees of freedom for the CG method, the DG method, and the HDG method in a four-element mesh. The blue nodes represent the degrees of freedom of the approximate solution, while the red nodes represent the degrees of freedom of the trace of solution.

1.3 Thesis Approach

This thesis presents a high order numerical scheme based on the hybridizable discontinuous Galerkin methods to solve the fully coupled time-dependent Stokes-Darcy-transport problem. There are two components of this problem; the time-dependent Stokes-Darcy problem which provides the velocity by which a contaminant is transported and that affects how the contaminant disperses and diffuses, and the transport problem which provides the concentration of the contaminant affecting the viscosity and the source and sink terms in the Stokes-Darcy problem.

We first solve the Stokes-Darcy problem without the coupling to the transport equation by extending the work in [16] to the time dependent case. We derive an HDG formulation for time dependent Stokes-Darcy problem. The main reason to consider this discretization is its strong mass conservation property and robustness. We prove the well-posedness of the method by using an inf-sup condition and present its a priori error analysis.

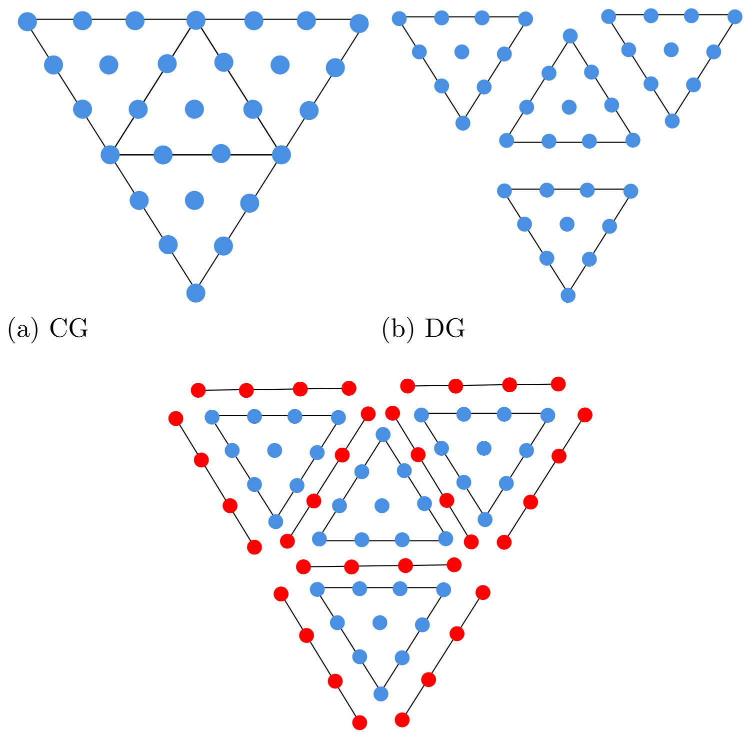
Next, we present the time-dependent transport problem assuming a given divergence-free velocity field neglecting the coupling to the Stokes-Darcy problem. We discretize the transport equation by an HDG method based on the one defined in [53] for the scalar advection-diffusion-reaction equation. The main motivation is that HDG methods are as effective as DG methods for convection dominated problems while being more efficient. We show the existence and uniqueness of the semi-discrete transport solution and prove a priori error estimates in the energy norm.

In the final step, we combine the Stokes-Darcy discretization and the transport discretization to obtain the solution to the fully coupled time-dependent Stokes-Darcy problem. The proposed algorithm solves the Stokes-Darcy and transport problems sequentially by time-lagging the concentration solution and therefore linearizes the nonlinear problem. In each Stokes-Darcy solve, our HDG method provides a divergence-free and divergence-conforming velocity, the error estimates for the velocity do not depend on the approximation error of the pressure.

In each part, we provide numerical examples to substantiate the attractive properties of the numerical methods.

1.4 Thesis Outline

The remainder of this thesis is organized as follows. In Chapter 2, we investigate a numerical method for the time-dependent coupled Stokes-Darcy problem. Chapter 3 deals with the time-dependent advection-diffusion type transport problem driven by a given divergence-free velocity field. Chapter 4 discusses the full coupling of the transport problem with the Stokes-Darcy problem. The final chapter states some conclusions and possible extensions of this research.



(a) CG

(b) DG

(c) HDG

Figure 1.1: An illustration of the degrees of freedom for the CG method, the DG method, and the HDG method on a four-element mesh of triangles in 2D using cubic polynomial approximation.

CHAPTER TWO

TIME-DEPENDENT STOKES-DARCY PROBLEM

In this chapter, we focus on the time-dependent Stokes-Darcy problem that describes the coupling of free flow and porous media flow on different parts of a domain and their interaction at the common interface. Let $\Omega \subset \mathbb{R}^{\text{dim}}$, $\text{dim} = 2, 3$ be a domain consisting of two disjoint subdomains, Ω^s and Ω^d, such that $\Omega = \Omega^s \cup \Omega^d$. Each subdomain is assumed to be a polygonal/polyhedral region. On the boundaries $\partial\Omega$ of Ω and $\partial\Omega^i$ of Ω^i, $i = s, d$, we denote the outward unit normal by n and n^i, $i = s, d$, respectively. We will use n to denote also the unit normal vector on Γ^I pointing from Ω^s to Ω^d, that is, $n = n^s = -n^d$ on Γ^I the polygonal interface between Ω^s and Ω^d. We denote the external boundary of Ω^s by $\Gamma^s := \partial\Omega \cap \partial\Omega^s$, and the external boundary of Ω^d by $\Gamma^d := \partial\Omega \cap \partial\Omega^d$.

2.1 Time-dependent Stokes-Darcy system

Let $J = [0, T]$ be a time interval of interest. The time-dependent Stokes-Darcy problem is given by the following set of PDEs:

$$\partial_t u - \nabla \cdot (2\mu\epsilon(u)) + \nabla p = f^s \qquad \text{in } \Omega^s \times J, \tag{2.1a}$$

$$\kappa^{-1} u + \nabla p = 0 \qquad \text{in } \Omega^d \times J, \tag{2.1b}$$

$$-\nabla \cdot u = \chi^d g^d \qquad \text{in } \Omega \times J, \tag{2.1c}$$

$$u = 0 \qquad \text{on } \Gamma^s \times J, \tag{2.1d}$$

$$u \cdot n = 0 \qquad \text{on } \Gamma^d \times J, \tag{2.1e}$$

where $\epsilon(u) := (\nabla u + (\nabla u)^T)/2$ is the strain tensor and χ^d is the characteristic function of Ω^d. The coefficient $\mu > 0$ is the kinematic viscosity, $\kappa > 0$ is the

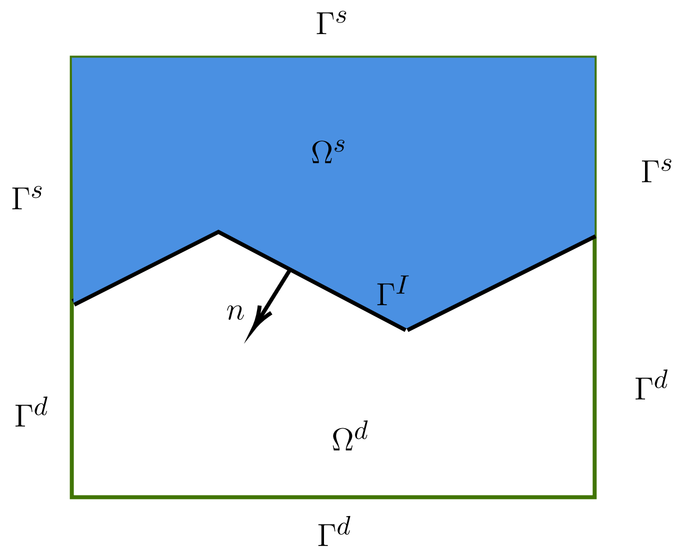


Figure 2.1: A sample domain for the Stokes–Darcy problem

permeability constant, $f^s : \Omega^s \times J \to \mathbb{R}^{\dim}$ is the body force, and $g^d : \Omega^d \times J \to \mathbb{R}$ is the source or sink term in the porous medium. Whenever there is a need to differentiate the values of a function on Stokes and Darcy parts of the domain, we will use superscripts. That is, for a function v defined on Ω, $v^s := v|_{\Omega^s}, v^d := v|_{\Omega^d}$.

On the interface $\Gamma^I \times J$, we assume the following interface conditions:

$$u^s \cdot n = u^d \cdot n, \tag{2.2a}$$

$$p^s - 2\mu\epsilon(u^s)n \cdot n = p^d, \tag{2.2b}$$

$$-2\mu(\epsilon(u^s)n)^t = \alpha\kappa^{-1/2}(u^s)^t, \tag{2.2c}$$

where the tangential component of a vector w is denoted by $(w)^t := w - (w \cdot n)n$ and $\alpha > 0$ is an experimentally determined parameter. Equation (2.2a) and (2.2b) represent the flux continuity and the equilibrium of normal forces on the interface $\Gamma^I \times J$, respectively. Note that (2.2b) is a consequence of the balance of the normal forces that is given by

$$\sigma^s n^s \cdot n^s + \sigma^d n^d \cdot n^d = 0 \text{ on } \Gamma^I \times J,$$

where $\sigma^s = -p^s\mathbb{I} + 2\mu\epsilon(u^s)$ in $\Omega^s \times J$ and $\sigma^d = p^s\mathbb{I}$ in $\Omega^d \times J$. The condition on the tangential fluid velocity on $\Gamma^I \times J$ described by (2.2c) is called the Beavers-Joseph-Saffman condition. It is a modification by Saffman [48] of the proportionality assumption made by Beavers and Joseph [6] on the slip velocity and the shear stress.

Finally, we prescribe the following initial condition:

$$u^s(x, 0) = u_0^s(x) \quad \text{in } \Omega^s. \tag{2.3}$$

2.2 The weak formulation

In this thesis, we use the notation from [27] and [1] for the Sobolev spaces. To describe the weak formulation, we introduce the following spaces

$$H^{\mathrm{div}}(\Omega) = \left\{ v \in [L^2(\Omega)]^{\mathrm{dim}} : \ \nabla \cdot v \in L^2(\Omega) \right\}.$$

Then the function spaces for the weak formulation is

$$\begin{aligned} V &= \left\{ v \in H^{\mathrm{div}}(\Omega) : \ v|_{\Omega^s} \in H^1(\Omega^s), \, v = 0 \ \text{ on } \ \Gamma^s, \, v \cdot n = 0 \text{ on } \ \Gamma^d \right\}, \\ Q &= \left\{ q \in L^2(\Omega) : \ \int_\Omega q \, \mathrm{d}x = 0 \right\}. \end{aligned}$$

With these definitions, we now derive the weak formulation. Taking the scalar product of (2.1a) and (2.1b) with any test function $v \in V$ and integrating on Ω^s and Ω^d, respectively, we have

$$\begin{aligned} \int_{\Omega^s} \partial_t u \cdot v \, \mathrm{d}x - \int_{\Omega^s} (\nabla \cdot 2\mu\epsilon(u)) \cdot v \, \mathrm{d}x + \int_{\Omega^s} \nabla p \cdot v \, \mathrm{d}x &= \int_{\Omega^s} f^s \cdot v \, \mathrm{d}x \\ \int_{\Omega^d} \kappa^{-1} u \cdot v \, \mathrm{d}x + \int_{\Omega^d} \nabla p \cdot v \, \mathrm{d}x &= 0. \end{aligned}$$

Integrating by parts and noting that $v = 0$ on Γ^s and, $v \cdot n = 0$ on Γ^d, we obtain

$$\int_{\Omega^s} \partial_t u \cdot v \,\mathrm{d}x + \int_{\Omega^s} 2\mu\epsilon(u) : \epsilon(v)\,\mathrm{d}x - \int_{\Gamma^I} 2\mu\epsilon(u^s)n^s \cdot v \,\mathrm{d}s \\ - \int_{\Omega^s} p\nabla \cdot v \,\mathrm{d}x + \int_{\Gamma^I} p^s n^s \cdot v^s \,\mathrm{d}s = \int_{\Omega^s} f^s \cdot v \,\mathrm{d}x, \tag{2.5a}$$

$$\int_{\Omega^d} \kappa^{-1} u \cdot v \,\mathrm{d}x - \int_{\Omega^d} p\nabla \cdot v \,\mathrm{d}x + \int_{\Gamma^I} p^d n^d \cdot v^d \,\mathrm{d}s = 0, \tag{2.5b}$$

where for $A \in \mathbb{R}^{m,n}$ and $B \in \mathbb{R}^{m,n}$, $A : B$ is defined as

$$A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}.$$

Observe that multiplying (2.2b) with $n = n^s$, then adding to (2.2c), we have

$$p^s n^s - 2\mu\epsilon(u^s)n^s = p^d n^s + \frac{\alpha}{\sqrt{\kappa}}(u^s)^t \text{ on } \Gamma^I \times J. \tag{2.6}$$

Also note that since $v \in V \subset H^{\mathrm{div}}(\Omega)$, $v \cdot n$ is continuous. Then recalling that $v^s \cdot n^s = v^d \cdot n^s$ and $n^d = -n^s$ on Γ^I, the sum of all interface integrals in (2.5) can be simplified as follows:

$$\begin{aligned} -\int_{\Gamma^I} 2\mu\epsilon(u^s)n^s \cdot v^s \,\mathrm{d}s &+ \int_{\Gamma^I} p^s n^s \cdot v^s \,\mathrm{d}s + \int_{\Gamma^I} p^d n^d \cdot v^d \,\mathrm{d}s \\ &= \int_{\Gamma^I} (p^d n^s + \frac{\alpha}{\sqrt{\kappa}}(u^s)^t) \cdot v^s \,\mathrm{d}s - \int_{\Gamma^I} p^d v^d \cdot n^s \,\mathrm{d}s \\ &= \int_{\Gamma^I} p^d [v^s \cdot n^s - v^d \cdot n^s] \,\mathrm{d}s + \int_{\Gamma^I} \frac{\alpha}{\sqrt{\kappa}}(u^s)^t \cdot v^s \,\mathrm{d}s \\ &= \int_{\Gamma^I} \frac{\alpha}{\sqrt{\kappa}}(u^s)^t (v^s)^t \,\mathrm{d}s. \end{aligned} \tag{2.7}$$

By summing (2.5a) and (2.5b), and by using (2.7), we have

$$\int_{\Omega^s} \partial_t u \cdot v \,\mathrm{d}x + a(u,v) + b(v,p) = \int_{\Omega^s} f^s \cdot v^s \,\mathrm{d}x,$$

where the bilinear forms are defined as follows:

$$a_D(u,v) = \int_{\Omega^d} \kappa^{-1} u \cdot v \,\mathrm{d}x \qquad \forall u, v \in V,$$

$$
\begin{aligned}
a_S(u,v) &= \int_{\Omega^s} 2\mu\epsilon(u) : \epsilon(v)\,\mathrm{d}x && \forall u, v \in V,\\
a_I(u,v) &= \int_{\Gamma^I} \alpha\kappa^{-1/2}(u^s)^t(v^s)^t\,\mathrm{d}s && \forall u, v \in V,\\
a(u,v) &= a_D(u,v) + a_S(u,v) + a_I(u,v) && \forall u, v \in V,\\
b(v,q) &= -\int_{\Omega} q\nabla\cdot v\,\mathrm{d}x && \forall v \in V, \quad \forall q \in Q.
\end{aligned}
$$

Similarly, by multiplying (2.1c) with a test function $q \in Q$, then integrating over Ω, we obtain

$$
b(u,q) = \int_{\Omega^d} g^d q\,\mathrm{d}x.
$$

Then the weak formulation of the time-dependent Stokes-Darcy problem reads: find $(u,p) \in L^2(0,T; V \times Q)$ such that for a.e. $t \in J$

$$
\int_{\Omega^s} \partial_t u \cdot v\,\mathrm{d}x + a(u,v) + b(v,p) = \int_{\Omega^s} f^s \cdot v^s\,\mathrm{d}x, \quad \forall v \in V, \tag{2.8a}
$$

$$
b(u,q) = \int_{\Omega^d} g^d q\,\mathrm{d}x, \quad \forall q \in Q \tag{2.8b}
$$

$$
u^s(x) = u_0^s \quad \text{in } \Omega^s. \tag{2.8c}
$$

To complete this section, we state the following theorem [14, Remark 3.6] about the existence and uniqueness of the solution to (2.8). For the steady-state case, the well-posedness of (2.8) can be found in [29, Proposition 2.1].

Theorem 1. *Given $f^s \in L^2(0,T,L^2(\Omega^s))$ and $g^d \in L^2(0,T,L^2(\Omega^d))$, problem* (2.8) *is well-posed.*

2.3 The numerical method

In this section, we first present a semi-discrete scheme to solve eq. (2.8). Our semi-discrete scheme is based on the hybridized discontinuous Galerkin (HDG) method as introduced in [16] for the stationary Stokes–Darcy problem.

We denote by $\mathcal{T}^j := \{K\}$ a mesh of Ω^j, $j = s, d$, with non-overlapping elements K and assume that $\mathcal{T}^s$ and $\mathcal{T}^d$ match at the interface Γ^I. Let $\mathcal{T} := \mathcal{T}^s \cup \mathcal{T}^d$. The diameter of an element $K \in \mathcal{T}$ is denoted by h_K and n denotes the outward unit normal vector on the boundary ∂K of K. We set $h := \max_{K \in \mathcal{T}} h_K$. We denote by $\mathcal{F}^i$ and $\mathcal{F}^b$ the set of all interfaces and the set of all boundary faces, respectively, and we set $\mathcal{F} := \mathcal{F}^i \cup \mathcal{F}^b$. The union of all facets is denoted by Γ_0 and the union of all facets in $\overline{\Omega}^j$ is denoted by Γ_0^j, $j = s, d$. Finally, we denote the set of all facets that lie on Γ^I and all facets that lie in $\overline{\Omega}^j$ by $\mathcal{F}^I$ and $\mathcal{F}^j$, $j = s, d$, respectively.

We introduce the following finite element function spaces on Ω,

$$
\begin{aligned}
V_h &:= \left\{ v_h \in [L^2(\Omega)]^{\dim} : \, v_h \in [P_k(K)]^{\dim} \quad \forall\, K \in \mathcal{T} \right\}, \\
Q_h &:= \left\{ q_h \in L^2(\Omega) : \, q_h \in P_{k-1}(K) \quad \forall\, K \in \mathcal{T} \right\} \cap L_0^2(\Omega), \\
Q_h^j &:= \left\{ q_h \in L^2(\Omega^j) : \, q_h \in P_{k-1}(K) \quad \forall\, K \in \mathcal{T}^j \right\}, \; j = s, d,
\end{aligned}
$$

where $P_k(D)$ denotes the space of polynomials of degree at most k on D. On Γ_0^s and Γ_0^d, we consider the following spaces.

$$
\begin{aligned}
\bar{V}_h &:= \left\{ \bar{v}_h \in [L^2(\Gamma_0^s)]^{\dim} : \, \bar{v}_h \in [P_k(F)]^{\dim} \quad \forall\, F \in \mathcal{F}^s, \, \bar{v}_h = 0 \, \text{on} \, \Gamma^s \right\}, \\
\bar{Q}_h^j &:= \left\{ \bar{q}_h^j \in L^2(\Gamma_0^j) : \, \bar{q}_h^j \in P_k(F) \quad \forall\, F \in \mathcal{F}^j \right\}, \; j = s, d.
\end{aligned}
$$

Note that the facet velocity field is defined only in Γ_0^s while the facet pressure is defined separately in both Γ_0^s and Γ_0^d so that $\bar{p}^s \neq \bar{p}^d$ on Γ^I, and thus the interface condition eq. (2.2b) is not violated. For ease of notation, we use boldface notation and define $\boldsymbol{V}_h := V_h \times \bar{V}_h$, $\boldsymbol{Q}_h := Q_h \times \bar{Q}_h^s \times \bar{Q}_h^d$ and $\boldsymbol{Q}_h^j := Q_h^j \times \bar{Q}_h^j$ for $j = s, d$. Consequently, the elements in $\mathbf{V}_h$, $\mathbf{Q}_h$ and $\mathbf{Q}_h^j$, for $j = s, d$, are represented by

$\mathbf{v}_h := (v_h, \bar{v}_h) \in \mathbf{V}_h$, $\mathbf{q}_h := (q_h, \bar{q}_h^s, \bar{q}_h^d) \in \mathbf{Q}_h$ and $\mathbf{q}_h^j := (q_h, \bar{q}_h^j) \in \mathbf{Q}_h^j$, $j = s, d$. We also set $\mathbf{X}_h := \mathbf{V}_h \times \mathbf{Q}_h$.

Next, let

$$V := \Big\{ v \in L^2(\Omega) : \, v^s \in [H^2(\Omega^s)]^{\dim}, \, v^d \in [H^1(\Omega^d)]^{\dim}, \\ v = 0 \text{ on } \Gamma^s, \; v \cdot n = 0 \text{ on } \Gamma^d, \, v^s \cdot n = v^d \cdot n \text{ on } \Gamma^I \Big\},$$

$$Q := \Big\{ q \in L_0^2(\Omega) : \, q^s \in H^1(\Omega^s), \, q^d \in H^2(\Omega^d) \Big\},$$

where $L_0^2(\Omega)$ is the space of L^2-functions with zero mean value. We set $X := V \times Q$ and denote by $\bar{V}$ and $\bar{Q}$ the trace space of V, restricted to Γ_0^s and the trace space of Q restricted to Γ_0, respectively. We introduce the trace operator $\gamma_V : V \to \bar{V}$ to restrict functions in V to Γ_0^s, and the trace operators $\gamma_{Q^j} : Q^j \to \bar{Q}^j$ to restrict functions in Q^j to Γ_0^j, $j = s$, d. Consistent with the previous notation, we set $\mathbf{V} := V \times \bar{V}$, $\mathbf{Q} := Q \times \bar{Q}$ and define the extended spaces

$$\mathbf{V}(h) := \mathbf{V}_h + \mathbf{V}, \quad \mathbf{Q}(h) := \mathbf{Q}_h + \mathbf{Q}, \quad \mathbf{X}(h) := \mathbf{V}(h) \times \mathbf{Q}(h).$$

The restrictions of $\mathbf{V}(h)$ and $\mathbf{Q}(h)$ to Ω^j will be specified by superscripts, that is, by $\mathbf{V}^j(h)$ and $\mathbf{Q}^j(h)$, $j = s$, d. For ease of notation, we write $\|\cdot\|_{m,p,D}$ instead of $\|\cdot\|_{W^{m,p}(D)}$ with the following simplifications. When $m = 0$, $W^{0,p}(D)$ coincides with $L^p(D)$ and when $p = 2$, $H^m(D) = W^{m,p}(D)$. For $p = 2$, we use $\|\cdot\|_{m,D}$ to denote $\|\cdot\|_{W^{m,2}(D)}$ and for $m = 0$, $p = 2$, we use $\|\cdot\|_D$ instead of $\|\cdot\|_{0,D}$ even for vector valued functions. We refer the reader to [1] for the definitions of the standard Sobolev spaces $W^{m,p}(D)$ and the norms corresponding to them $\|\cdot\|_{W^{m,p}(D)}$.

On $\mathbf{V}^s(h)$, we define the norms

$$|||\mathbf{v}|||_{v,s}^2 := \sum_{K \in \mathcal{T}^s} (\|\nabla v\|_K^2 + h_K^{-1} \|v - \bar{v}\|_{\partial K}^2),$$

and

$$|||\mathbf{v}|||_{v',s}^2 := |||\mathbf{v}|||_{v,s}^2 + \sum_{K\in\mathcal{T}^s} h_K^2 |v|_{H^2(K)}^2.$$

On $\mathbf{V}(h)$, we define

$$|||\mathbf{v}|||_v^2 := |||\mathbf{v}|||_{v,s}^2 + \|v\|_{\Omega^d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2, \tag{2.9}$$

and

$$|||\mathbf{v}|||_{v'}^2 := |||\mathbf{v}|||_v^2 + \sum_{K\in\mathcal{T}^s} h_K^2 |v|_{H^2(K)}^2 = |||\mathbf{v}|||_{v',s}^2 + \|v\|_{\Omega^d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2. \tag{2.10}$$

Finally, for $\mathbf{q}^j \in \mathbf{Q}^j(h)$, $j = s$, d, and for $\mathbf{q} \in \mathbf{Q}(h)$, we define

$$|||\mathbf{q}^j|||_{p,j}^2 := \|q\|_{\Omega^j}^2 + \sum_{K\in\mathcal{T}^j} h_K \|\bar{q}^j\|_{\partial K}^2, \quad |||\mathbf{q}|||_p^2 := \sum_{j=s,d} |||\mathbf{q}^j|||_{p,j}^2.$$

Next, we derive the semi-discrete scheme. Let $\sigma^s = 2\mu\epsilon(u^s) - p^s\mathbb{I}$ denote the Stokes stress tensor. Then multiplying (2.1a) by a test function $v \in V_h$, integrating over an element K, using integration by parts and summing over all elements, we get

$$\sum_{K\in\mathcal{T}^s}\int_K \partial_t u^s\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_K \sigma^s : \nabla v\,\mathrm{d}x - \sum_{K\in\mathcal{T}^s}\int_{\partial K}\hat{\sigma}^s n\cdot v\,\mathrm{d}s = \sum_{K\in\mathcal{T}^s}\int_K f^s\cdot v\,\mathrm{d}x, \tag{2.11}$$

where the flux $\hat{\sigma}^s$ in the third term is replaced with the numerical flux

$$\hat{\sigma}^s = 2\mu\epsilon(u^s) - \bar{p}^s\mathbb{I} - \frac{2\mu\beta}{h}(u^s - \bar{u})\otimes n. \tag{2.12}$$

In other words,

$$\begin{aligned}\sum_{K\in\mathcal{T}^s}\int_K \partial_t u\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_K 2\mu\epsilon(u):\epsilon(v)\,\mathrm{d}x - \sum_{K\in\mathcal{T}^s}\int_K p\nabla\cdot v\,\mathrm{d}x \\ - \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu\epsilon(u)n\cdot v\,\mathrm{d}s + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta\mu}{h}(u-\bar{u})\cdot v\,\mathrm{d}s \\ + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{p}v\cdot n\,\mathrm{d}s = \sum_{K\in\mathcal{T}^s}\int_K f^s\cdot v\,\mathrm{d}x.\end{aligned} \tag{2.13}$$

To guarantee the continuity of the normal flux, we impose

$$\sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{v}\cdot\hat{\sigma}^s n\,\mathrm{d}s=\int_{\Gamma^I}\bar{v}\cdot\sigma^s n\,\mathrm{d}s.$$

Using eq. (2.6),

$$\sigma^s n^s=-p^d n^s-\alpha\kappa^{-1/2}(u^s)^t\ \text{ on }\Gamma^I\times J.$$

Therefore, due to the fact that $\bar{v}\cdot\bar{u}^t=(\bar{v})^t\cdot\bar{u}^t$, and switching from p^d to $\bar{p}^d$ and u^s to $\bar{u}$,

$$-\sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{v}\cdot\hat{\sigma}^s n\,\mathrm{d}s=\int_{\Gamma^I}\bar{v}\cdot n^s\bar{p}^d\,\mathrm{d}s+\int_{\Gamma^I}\alpha\kappa^{-1/2}\bar{v}^t\cdot\bar{u}^t\,\mathrm{d}s. \tag{2.14}$$

Plugging $\hat{\sigma}^s$ into (2.14), we get

$$\begin{aligned}\sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{v}\cdot n\bar{p}^s\,\mathrm{d}s-\sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu\epsilon(u)n\cdot\bar{v}\,\mathrm{d}s&\\ +\sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta\mu}{h}(u-\bar{u})\cdot\bar{v}\,\mathrm{d}s&\\ =\int_{\Gamma^I}\bar{v}\cdot n^s\bar{p}^d\,\mathrm{d}s+\int_{\Gamma^I}\alpha\kappa^{-1/2}\bar{v}^t\cdot\bar{u}^t\,\mathrm{d}s.&\end{aligned} \tag{2.15}$$

Since $\bar{v}\cdot n$ and $\bar{p}^s$ are single-valued on interior facets and $\bar{v}=0$ on Γ^s,

$$\sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{v}\cdot n^s\bar{p}^s\,\mathrm{d}s=\int_{\Gamma^s\cup\Gamma^I}\bar{v}\cdot n^s\bar{p}^s\,\mathrm{d}s=\int_{\Gamma^I}\bar{v}\cdot n^s\bar{p}^s\,\mathrm{d}s.$$

Thus, we can rewrite (2.15) as

$$\begin{aligned}\int_{\Gamma^I}\bar{v}\cdot n^s\bar{p}^s\,\mathrm{d}s-\sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu\epsilon(u^s)n^s\cdot\bar{v}\,\mathrm{d}s&\\ +\sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta\mu}{h}(u-\bar{u})\cdot\bar{v}\,\mathrm{d}s&\\ =\int_{\Gamma^I}\bar{v}\cdot n^s\bar{p}^d\,\mathrm{d}s+\int_{\Gamma^I}\alpha\kappa^{-1/2}\bar{v}^t\cdot\bar{u}^t\,\mathrm{d}s.&\end{aligned} \tag{2.16}$$

After subtracting (2.16) from (2.13) and adding a consistent symmetrizing term, we obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^s}\int_K \partial_t u\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_K 2\mu\epsilon(u):\epsilon(v)\,\mathrm{d}x - \sum_{K\in\mathcal{T}^s}\int_K p\nabla\cdot v\,\mathrm{d}x \\
&\quad - \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu\epsilon(u)n\cdot(v-\bar{v})\,\mathrm{d}s - \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu\epsilon(v)n\cdot(u-\bar{u})\,\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}^s}\int_{\partial K} \frac{2\beta\mu}{h}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s + \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{p}^s v\cdot n\,\mathrm{d}s \\
&\quad - \int_{\Gamma^I} \bar{p}^s\bar{v}\cdot n^s\,\mathrm{d}s + \int_{\Gamma^I} \bar{p}^d\bar{v}\cdot n^s\,\mathrm{d}s + \int_{\Gamma^I} \alpha\kappa^{-1/2}\bar{v}^t\cdot\bar{u}^t\,\mathrm{d}s \\
&\qquad = \sum_{K\in\mathcal{T}^s}\int_K f^s\cdot v\,\mathrm{d}x.
\end{aligned}
$$

Multiplying (2.1c) by a test function $q \in Q_h$, then integrating on each element $K \in \mathcal{T}^s$, we get

$$\sum_{K\in\mathcal{T}^s}\int_K q\nabla\cdot u\,\mathrm{d}x = 0. \tag{2.17}$$

To impose normal continuity of the Stokes velocity, we set

$$
\begin{aligned}
0 &= \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s(u-\bar{u})\cdot n\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s u\cdot n\,\mathrm{d}s - \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s\bar{u}\cdot n\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s u\cdot n\,\mathrm{d}s - \int_{\Gamma^I} \bar{q}^s\bar{u}\cdot n^s\,\mathrm{d}s,
\end{aligned}
$$

where we used single-valuedness of $\bar{u}, \bar{q}^s$ and $\bar{u} = 0$ on Γ^s. Therefore, combining the previous two equations, we have

$$-\sum_{K\in\mathcal{T}^s}\int_K q\nabla\cdot u\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s u\cdot n\,\mathrm{d}s - \int_{\Gamma^I} \bar{q}^s\bar{u}\cdot n^s\,\mathrm{d}s = 0. \tag{2.18}$$

Multiplying (2.1b) by a test function $v \in V_h$ and integrating over an element $K \in \mathcal{T}^d$, using integration by parts, replacing p^d with $\bar{p}^d$, and summing over all elements in $\mathcal{T}^d$ yield

$$\sum_{K\in\mathcal{T}^d}\int_K \kappa^{-1}u\cdot v\,\mathrm{d}x - \sum_{K\in\mathcal{T}^d}\int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^d}\int_{\partial K}\bar{p}^d v\cdot n\,\mathrm{d}x = 0.$$

Next, multiplying (2.1c) by a test function $q \in Q_h$ and integrating over an element K, summing over all elements on Ω^d, we obtain

$$-\sum_{K\in\mathcal{T}^d}\int_K q^d\nabla\cdot u^d\,\mathrm{d}x = \sum_{K\in\mathcal{T}^d}\int_K g^d q^d\,\mathrm{d}x.$$

To impose normal continuity of the Darcy velocity, we set

$$\sum_{K\in\mathcal{T}^d}\int_{\partial K}\bar{q}u\cdot n\,\mathrm{d}s = \int_{\Gamma^I}\bar{q}^d u^d\cdot n^d\,\mathrm{d}s.$$

Therefore, combining the last three equations, we have

$$\sum_{K\in\mathcal{T}^d}\int_K \kappa^{-1}u\cdot v\,\mathrm{d}x - \sum_{K\in\mathcal{T}^d}\int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^d}\int_{\partial K}\bar{p}^d v\cdot n\,\mathrm{d}x = 0, \tag{2.19}$$

and

$$-\sum_{K\in\mathcal{T}^d}\int_K q^d\nabla\cdot u^d\,\mathrm{d}x + \sum_{K\in\mathcal{T}^d}\int_{\partial K}\bar{q}^d u^d\cdot n^d\,\mathrm{d}s - \int_{\Gamma^I}\bar{q}^d u^d\cdot n^d\,\mathrm{d}s = \sum_{K\in\mathcal{T}^d}\int_K g^d q^d\,\mathrm{d}x. \tag{2.20}$$

Adding (2.18) and (2.20), we obtain

$$\begin{aligned}
-\sum_{K\in\mathcal{T}^d}\int_K q^d\nabla\cdot u^d\,\mathrm{d}x + \sum_{K\in\mathcal{T}^d}\int_{\partial K}\bar{q}^d u^d\cdot n^d\,\mathrm{d}s - \int_{\Gamma^I}\bar{q}^d u^d\cdot n^d\,\mathrm{d}s&\\
-\sum_{K\in\mathcal{T}^s}\int_K q^s\nabla\cdot u^s\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{q}^s u^s\cdot n^s\,\mathrm{d}s - \int_{\Gamma^I}\bar{q}^s\bar{u}^s\cdot n^s\,\mathrm{d}s&\\
= \sum_{K\in\mathcal{T}^d}\int_K g^d q^d\,\mathrm{d}x.&
\end{aligned} \tag{2.21}$$

Gathering what we have so far, we propose the following semi-discrete scheme for the time-dependent Stokes–Darcy system defined by eqs. (2.1) to (2.3): for $t > 0$, find $\big(\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)\big) \in \boldsymbol{X}_h$ such that

$$\sum_{K\in\mathcal{T}^s} \int_K \partial_t u_h(t) \cdot v_h \,\mathrm{d}x + B_h((\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)), (\boldsymbol{v}_h, \boldsymbol{q}_h))$$
$$= \sum_{K\in\mathcal{T}^s} \int_K f^s(t) \cdot v_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}^d} \int_K g^d(t) q_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h \tag{2.22}$$

with initial condition $u_h(0) = \pi_h u_0$ where π_h denotes an appropriate projection into V_h. Here,

$$B_h((\mathbf{u}_h, \mathbf{p}_h), (\mathbf{v}_h, \mathbf{q}_h)) := a_h(\mathbf{u}_h, \mathbf{v}_h) + \sum_{j=s,d} b_h^j(\mathbf{p}_h^j, v_h) + b_h^{I,j}(\bar{p}_h^j, \bar{v}_h)$$
$$+ \sum_{j=s,d} b_h^j(\mathbf{q}_h^j, u_h) + b_h^{I,j}(\bar{q}_h^j, \bar{u}_h), \tag{2.23}$$

where the bilinear form $a_h(\cdot,\cdot)$ is defined as

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := a_h^s(\mathbf{u}_h, \mathbf{v}_h) + a_h^d(u_h, v_h) + a_h^I(\bar{u}_h, \bar{v}_h), \tag{2.24}$$

with

$$\begin{aligned}
a_h^s(\mathbf{u}, \mathbf{v}) :=& \sum_{K\in\mathcal{T}^s} \int_K 2\mu\epsilon(u) : \epsilon(v) \,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{2\beta\mu}{h_K} (u - \bar{u}) \cdot (v - \bar{v}) \,\mathrm{d}s \\
&- \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu\epsilon(u) n \cdot (v - \bar{v}) \,\mathrm{d}s - \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu\epsilon(v) n \cdot (u - \bar{u}) \,\mathrm{d}s, \\
a_h^d(u, v) :=& \int_{\Omega^d} \kappa^{-1} u \cdot v \,\mathrm{d}x, \\
a_h^I(\bar{u}, \bar{v}) :=& \int_{\Gamma^I} \alpha\kappa^{-1/2} \bar{u}^t \cdot \bar{v}^t \,\mathrm{d}s,
\end{aligned}$$

and where $\beta > 0$ is a penalty parameter. The bilinear forms $b_h^j(\cdot,\cdot)$ and $b_h^{I,j}(\cdot,\cdot)$ are given by

$$b_h^j(\mathbf{p}^j, v) := -\sum_{K\in\mathcal{T}^j} \int_K p \nabla \cdot v \,\mathrm{d}x + \sum_{K\in\mathcal{T}^j} \int_{\partial K} \bar{p}^j v \cdot n \,\mathrm{d}s, \tag{2.25a}$$
$$b_h^{I,j}(\bar{p}^j, \bar{v}) := -\int_{\Gamma^I} \bar{p}^j \bar{v} \cdot n^j \,\mathrm{d}s. \tag{2.25b}$$

Using the same technique as in [16, Section 3.3], we obtain for $t > 0$

$$-\nabla \cdot u_h = \chi^d \Pi_Q g^d \qquad \forall x \in K,\ \forall K \in \mathcal{T}, \tag{2.26a}$$

$$[\![u_h \cdot n]\!] = 0 \qquad \forall x \in F,\ \forall F \in \mathcal{F} \backslash \mathcal{F}^I, \tag{2.26b}$$

$$u_h^j \cdot n = \bar{u}_h \cdot n \qquad \forall x \in F,\ \forall F \in \mathcal{F}^I,\ j = s, d, \tag{2.26c}$$

where $[\![\cdot]\!]$ is the usual jump operator and n the unit normal vector on F. To discretize in time the space semi-discrete problem eq. (2.22), we introduce a partition of $(0, T)$ into N intervals of length Δt (the time step) so that $N\Delta t = T$; more generally, a variable time step can be considered. For $n \in \{0, \ldots, N\}$, a superscript n indicates the value of a function at the discrete time $t^n := n\Delta t$. For a sequence $\{v^n\}$, we introduce the first order difference operator d_t such that, for all $n \in \{1, \ldots, N\}$,

$$d_t v^n = \frac{v^n - v^{n-1}}{\Delta t}. \tag{2.27}$$

Then the fully discrete scheme is obtained by applying the backward Euler scheme to eq. (2.22): for $n \geq 1$, find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that

$$\begin{aligned}\sum_{K \in \mathcal{T}^s} \int_K d_t u_h^n \cdot v_h \,\mathrm{d}x &+ B_h((\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\ &= \sum_{K \in \mathcal{T}^s} \int_K f^s(t^n) \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}^d} \int_K g^d(t^n) q_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h,\end{aligned} \tag{2.28}$$

where the initial condition u_h^0 is obtained by a suitable projection of u_0 into V_h.

2.3.1 Theoretical results

In this section, we present the consistency and well-posedness properties of the numerical scheme and also prove a priori error estimates in the energy norm. However, first we need to recall a sequence of well-known inequalities fundamental to our proofs and introduce the interpolants to quantify the approximation error.

2.3.1.1 Useful inequalities

Discrete trace inequality [24, Lemma 1.46, Remark 1.47]:

$$\|v\|_{\partial K} \le C_{T,1} h_K^{-1/2} \|v\|_K \quad \forall v \in P_k(K). \tag{2.29}$$

From [9, Theorem 1.6.6], we also have

$$\|v\|_{0,\infty,\partial K} \le C \|v\|_{0,\infty,K} \quad \forall v \in W^1_\infty(K). \tag{2.30}$$

Continuous trace inequalities from [24, Lemma 1.49] and [9, Theorem 1.6.6]:

$$\|v\|^2_{\partial K} \le C_{T,2}(h_K^{-1}\|v\|^2_K + h_K \|v\|^2_{1,K}) \quad \forall v \in H^1(K), \tag{2.31}$$

$$\|v\|_{\partial \Omega^s} \le C \|v\|_{1,\Omega^s} \quad \forall v \in H^1(\Omega^s), \tag{2.32}$$

By the Poincaré inequality eq. (2.32) further implies

$$\|v\|_{\partial \Omega^s} \le C_{c,T} \|\nabla v\|_{\Omega^s} \quad \forall v \in \{v \in H^1(\Omega^s) \ : \ v = 0 \text{ on } \Gamma^s\}. \tag{2.33}$$

Here $K \in \mathcal{T}$, and $C_{T,1}$, $C_{T,2}$, $C_{c,T} > 0$ are constants that are independent of h_K.

Regarding the trace on the interface, we have [30, (1.24)], [9, Theorem 1.6.6]:

$$\|v\|_{\Gamma^I} \le C_{T,4} \|\nabla v\|_{\Omega^s} \quad \forall v \in \left\{v \in H^1(\Omega^s) \ : \ v = 0 \text{ on } \Gamma^s\right\}, \tag{2.34}$$

$$\|v\|_{\Gamma^I} \le C_{T,5} \|v\|_{1,\Omega^s} \quad \forall v \in H^1(\Omega^s). \tag{2.35}$$

Furthermore, by [30, Theorem 4.4], for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $k \ge 1$,

$$\|v_h^s\|_{\Gamma^I} \le C |||\boldsymbol{v}_h|||_{v,s} \le C |||\boldsymbol{v}_h|||_v. \tag{2.36}$$

By a slight modification of the proof in [30, Theorem 4.4], we have a similar result for the discrete concentration space in Ω^s. That is, for any $w_h \in C_h^s$, $\ell \ge 1$,

$$\|w_h\|_{\Gamma^I} \le C |||\boldsymbol{w}_h|||_{c,s} \le |||\boldsymbol{w}_h|||_c. \tag{2.37}$$

Inverse inequalities from [24, Lemma 1.44, Lemma 1.50]:

$$\|\nabla v\|_K \le C_{\text{inv}} h_K^{-1} \|v\|_K \quad \forall v \in P_k(K), \tag{2.38}$$

$$\|v\|_{0,\infty,K} \le C_{\text{inv}} h_K^{-\dim/2} \|v\|_K \quad \forall v \in P_k(K), \tag{2.39}$$

where $C_{\text{inv}} > 0$ is independent of h_K. From [30, Proposition 4.5], [7, Remark 1.1], for any $v \in H^1(\mathcal{T}_h^s)$, and for any $\mu \in \bar{V}_h$, there exists a positive constant C such that the following Poincaré-type inequality holds

$$\|v\|_{\Omega^s}^2 \le C \|\!|\!|(v, \bar{v})|\!|\!|_{v,s}^2. \tag{2.40}$$

The following version of Korn's first inequality is a consequence of [8, (1.19)], [45, p.100], [30, Proposition 4.7]:

$$|\!|\!|\mathbf{v}_h|\!|\!|_{v,s} \le C_K \sum_{K \in \mathcal{T}^s} (\|\varepsilon(v_h)\|_K^2 + h_K^{-1} \|v_h - \bar{v}_h\|_{\partial K}^2), \tag{2.41}$$

for some positive constant C_K.

We next state coercivity and continuity properties of a_h that will be useful in energy norm estimates.

Lemma 1 (Coercivity of a_h, [16, Lemma 2])**.** *There exists a constant $C_a > 0$, independent of h, and a constant $\beta_0 > 0$ such that for $\beta > \beta_0$ and for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,*

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \ge C_a |\!|\!|\boldsymbol{v}_h|\!|\!|_v^2. \tag{2.42}$$

Lemma 2 (Continuity of a_h, [16, Lemma 3])**.** *There exists a generic constant $C > 0$, independent of h, such that for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$,*

$$a_h(\boldsymbol{u}, \boldsymbol{v}) \le C |\!|\!|\boldsymbol{u}|\!|\!|_{v'} |\!|\!|\boldsymbol{v}|\!|\!|_{v'}. \tag{2.43}$$

In our analysis we will need the following inf-sup condition on the discrete spaces which can be found in [16, Theorem 1]. A proof is provided in the appendix for completeness.

Theorem 2 (inf-sup condition)**.** *There exists a constant $c^{*}_{\text{inf}} > 0$, independent of h, such that for any $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,*

$$c^{\star}_{\text{inf}} \| \boldsymbol{q}_h \|_p \leq \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \boldsymbol{v}_h \neq 0}} \frac{\sum_{j=s,d} \left(b^j_h(\boldsymbol{q}^j_h, v_h) + b^{I,j}_h(\bar{q}^j_h, \bar{v}_h) \right)}{\| \boldsymbol{v}_h \|_v}.$$

Now we are ready to proceed with our proofs.

2.3.1.2 Well-posedness and consistency To prove the existence and uniqueness of $(\boldsymbol{u}^n_h, \boldsymbol{p}^n_h)$ and related error estimates we rewrite eq. (2.28) as follows.

Theorem 3. *There exists a constant $\beta_0 > 0$ such that if $\beta > \beta_0$, there exists a unique solution $(\boldsymbol{u}^n_h, \boldsymbol{p}^n_h) \in \boldsymbol{X}_h$ to eq. (2.28) satisfying*

$$\| \boldsymbol{u}^n_h \|_v + \| \boldsymbol{p}^n_h \|_p \leq C\Big(\frac{1}{\Delta t} \| u^{n-1}_h \|_{\Omega^s} + \| f_s(t^{n-1}) \|_{\Omega^s} + \| g^d(t^{n-1}) \|_{\Omega^d} \Big), \quad (2.44)$$

where $C > 0$ is a constant independent of h.

Proof. The result is a straightforward consequence of the abstract theory for saddle point problems stated in Theorem A.1 which applies due to the coercivity of $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ [16, Lemma 2] in $\boldsymbol{V}_h$, and therefore of

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) := \frac{1}{\Delta t} \sum_{K \in \mathcal{T}^s} \int_K u_h \cdot v_h \, \mathrm{d}x + a_h(\boldsymbol{u}_h, \boldsymbol{v}_h),$$

and the inf-sup condition given in Theorem 2.

□

A trivial consequence of [16, Lemma 1] is the following result.

Lemma 3 (Consistency)**.** *If $(u, p) \in X$ solves the Stokes–Darcy system eq. (2.1), then letting $\boldsymbol{u} = (u, \gamma(u))$ and $\boldsymbol{p} = (p, \gamma(p^s), \gamma(p^d))$, there holds*

$$\sum_{K \in \mathcal{T}^s} \int_K \partial_t u \cdot v \, \mathrm{d}x + B_h((\boldsymbol{u}, \boldsymbol{p}), (\boldsymbol{v}, \boldsymbol{q})) = \int_{\Omega^s} f^s \cdot v \, \mathrm{d}x + \int_{\Omega^d} g^d q \, \mathrm{d}x, \quad (2.45)$$

for all $(\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}(h)$.

2.3.1.3 A priori estimates From this consistency result and eq. (2.1), we obtain the following error equation that will be used to derive the a priori error estimates.

Theorem 4 (Error equation)**.** *If* $(u, p) \in X$ *solves the Stokes–Darcy system eq.* (2.1) *and* $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ *is the solution of eq.* (2.28)*, then*

$$\sum_{K \in \mathcal{T}^s} \int_K (d_t u_h^n - \partial_t u^n) \cdot v_h \, \mathrm{d}x + B_h((\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) - B_h((\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = 0, \tag{2.46}$$

for any $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$.

Proof. Using the consistency equation in Lemma 3 at time $t = t^n$,

$$\sum_{K \in \mathcal{T}^s} \int_K \partial_t u^n \cdot v_h \, \mathrm{d}x + B_h((\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = \sum_{K \in \mathcal{T}^s} \int_K f^s(t^n) \cdot v_h \, \mathrm{d}x + \sum_{K \in \mathcal{T}^d} \int_K g^d(t^n) q_h \, \mathrm{d}x,$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$.

Subtracting this equation from eq. (2.28) yields

$$\sum_{K \in \mathcal{T}^s} \int_K (d_t u_h^n - \partial_t u^n) \cdot v_h \, \mathrm{d}x + B_h((\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) - B_h((\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = 0 \tag{2.47}$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$.

□

The standard approach in a priori error estimation is by measuring the error projected on the discrete space and then combining it with the approximation properties of the interpolant. For the velocity error, we use the BDM interpolation operator [32, Lemma 7] that satisfies the following properties:

Lemma 4 (BDM interpolation operator)**.** *There is an interpolation operator* $\Pi_V : [H^1(\Omega)]^{\dim} \to V_h$*, such that if* $u \in [H^{k+1}(K)]^{\dim}$*,* $K \in \mathcal{T}$*, then*

i. $\displaystyle\int_K q(\nabla \cdot u - \nabla \cdot \Pi_V u)\,\mathrm{d}x = 0$ *for all* $q \in P_{k-1}(K)$*,*

ii. $\displaystyle\int_F \bar{q}(u - \Pi_V u) \cdot n\,\mathrm{d}s = 0$ *for all* $\bar{q} \in P_k(F)$*, where* F *is an edge if* $\dim = 2$ *and a face if* $\dim = 3$ *on* ∂K*.*

iii. $[\![n \cdot \Pi_V u]\!] = 0$*, where* $[\![\cdot]\!]$ *is the usual jump operator.*

iv. $\|u - \Pi_V u\|_{m,K} \le C h_K^{\ell-m} \|u\|_{\ell,K}$ *with* $m = 0, 1, 2$ *and* $m \le \ell \le k+1$*.*

v. $\left\|\nabla \cdot (u - \Pi_V u)\right\|_{m,K} \le C h_K^{\ell-m} \|\nabla \cdot u\|_{\ell,K}$ *with* $m = 0, 1$ *and* $m \le \ell \le k$*.*

Furthermore, for $u \in W^{1,\infty}(K)$, [31, (2.33)],

$$\|u - \Pi_V u\|_{\infty,K} + h_K \|\nabla(u - \Pi_V u)\|_{\infty,K} \le C h_K \|u\|_{1,\infty,K}. \tag{2.48}$$

Let $P_{\bar{V}}$ be the L^2-projection into $\bar{V}_h$ and define the interpolant $\bar{\Pi}_V : L^2(\Omega^s) \to \bar{V}_h$ by tweaking $P_{\bar{V}}$ at the interface as follows:

$$\bar{\Pi}_V u = \begin{cases} (P_{\bar{V}} u)|_F & \text{if } F \in \mathcal{F}^s \backslash \mathcal{F}^I, \\ (\Pi_V u)^s|_F & \text{if } F \in \mathcal{F}^I. \end{cases}$$

It is straightforward to deduce the following estimate using eq. (2.31), the approximation properties of the L^2-projection, and Lemma 4(iv.): For $v \in [H^\ell(\Omega^s)]^{\dim}$, $1 \le \ell \le k+1$,

$$\|v - \bar{\Pi}_V v\|_{\partial K} \le C h_K^{\ell-1/2} \|v\|_{\ell,K}\,. \tag{2.49}$$

Next, we split the error into two parts:

$$\xi_u^n := u^n - \Pi_V u^n, \qquad \zeta_u^n := u_h^n - \Pi_V u^n, \tag{2.50a}$$

$$\xi_p^n := p^n - \Pi_Q p^n, \qquad \zeta_p^n := p_h^n - \Pi_Q p^n, \tag{2.50b}$$

$$\bar{\xi}_u^n := \gamma(u^{sn}) - \bar{\Pi}_V u^{sn}, \qquad \bar{\zeta}_u^n := \bar{u}_h^n - \bar{\Pi}_V u^{sn}, \tag{2.50c}$$

$$\bar{\xi}_p^{jn} := \gamma(p^{jn}) - \bar{\Pi}_Q^j p^{jn}, \qquad \bar{\zeta}_p^{jn} := \bar{p}_h^{jn} - \bar{\Pi}_Q^j p^{jn}, \tag{2.50d}$$

where $j = s, d$.

Then we can write

$$\begin{aligned} u^n - u_h^n &= \xi_u^n - \zeta_u^n, \quad \gamma(u^n) - \bar{u}_h^n = \bar{\xi}_u^n - \bar{\zeta}_u^n, \\ p^n - p_h^n &= \xi_p^n - \zeta_p^n, \quad \gamma(p^{jn}) - \bar{p}_h^{jn} = \bar{\xi}_p^{jn} - \bar{\zeta}_p^{jn},\ j = s,\ d, \end{aligned} \tag{2.51}$$

and we use the notation $\boldsymbol{\xi}_u^n := (\xi_u^n, \bar{\xi}_u^n)$, $\boldsymbol{\xi}_p^n := (\xi_p^n, \bar{\xi}_p^{sn}, \bar{\xi}_p^{dn})$, and $\boldsymbol{\xi}_p^{jn} := (\xi_p^n, \bar{\xi}_p^{jn})$, $j = s,\ d$.

Similarly, $\boldsymbol{\zeta}_u^n := (\zeta_u^n, \bar{\zeta}_u^n)$, $\boldsymbol{\zeta}_p^n := (\zeta_p^n, \bar{\zeta}_p^{sn}, \bar{\zeta}_p^{dn})$, and $\boldsymbol{\zeta}_p^{jn} := (\zeta_p^n, \bar{\zeta}_p^{jn})$, $j = s,\ d$ consistent with the notation introduced before for simplicity.

The following result is a consequence of [16, Lemma 7, Lemma 8].

Theorem 5. *Suppose that u is such that $u^{sn} \in (H^\ell(\Omega^s))^{\dim}$, and $u^{dn} \in (H^{\ell-1}(\Omega^d))^{\dim}$, $2 \le \ell \le k+1$. Then*

$$|||\boldsymbol{\xi}_u^n|||_{v',s} \le Ch^{\ell-1} \|u^n\|_{H^\ell(\Omega^s)}, \tag{2.52}$$

$$|||\boldsymbol{\xi}_u^n|||_{v'} \le Ch^{\ell-1} (\|u^n\|_{H^\ell(\Omega^s)} + \|u^n\|_{H^{\ell-1}(\Omega^d)}). \tag{2.53}$$

Furthermore, for $j = s, d$, if $p^{jn} \in H^\ell(\Omega^j)$, $0 \le \ell \le k$, then

$$|||\boldsymbol{\xi}_p^{jn}|||_{p,j} \le Ch^\ell \|p^n\|_{H^\ell(\Omega^j)}. \tag{2.54}$$

The following lemma demonstrates that the BDM interpolant simplifies the analysis by eliminating some terms from the error equation.

Lemma 5. *For any $n \geq 1$, $(\boldsymbol{\zeta}_u^n, \boldsymbol{\zeta}_p^n)$ and $(\boldsymbol{\xi}_u^n, \boldsymbol{\xi}_p^n)$ as defined in eq.* (2.50) *satisfy*

$$\sum_{j=s,d} (b_h^j(\boldsymbol{q}_h^j, \xi_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\xi}_u^n) = 0, \quad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h, \tag{2.55}$$

$$\sum_{j=s,d} (b_h^j(\boldsymbol{\xi}_p^{jn}, v_h) + b_h^{I,j}(\bar{\xi}_p^{jn}, \bar{v}_h)) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.56}$$

Proof. This result is a consequence of the mass conservation properties of the discrete velocity u_h eq. (2.26), the properties of the BDM projection Π_V in Lemma 4, the definition of $\bar{\Pi}_V$, and the properties of L^2-projections Π_Q, $\bar{\Pi}_Q^j$. Indeed, to prove eq. (2.55), let $\boldsymbol{q}_h = (q_h, \bar{q}_h^s, \bar{q}_h^d) \in \boldsymbol{Q}_h$. Then, noting that $q_h \in P_{k-1}(K), \bar{q}_h \in P_k(K)$, using Lemma 4(i.), Lemma 4(ii.), and the definition of $\bar{\Pi}_V$ on Γ^I,

$$\begin{aligned}
&\sum_{j=s,d} \left(b_h^j(\boldsymbol{q}_h^j, \xi_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\xi}_u^n)\right) \\
&\quad = \sum_{j=s,d} \Big(- \sum_{K \in \mathcal{T}^j} \int_K q_h \nabla \cdot (u^n - \Pi_V u^n) \,\mathrm{d}x \\
&\qquad + \sum_{K \in \mathcal{T}^j} \int_{\partial K} \bar{q}_h^j (u^n - \Pi_V u^n) \cdot n^j \,\mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^j (\gamma(u^{sn}) - \bar{\Pi}_V u^{sn}) \cdot n^j \,\mathrm{d}s \Big) \\
&\qquad = \sum_{j=s,d} \Big(- \int_{\Gamma^I} \bar{q}_h^j (\gamma(u^{sn}) - \Pi_V u^{sn}) \cdot n^j \,\mathrm{d}s \Big) = 0.
\end{aligned} \tag{2.57}$$

To prove eq. (2.56), let $\boldsymbol{v}_h \in \boldsymbol{V}_h$. Noting that $\nabla \cdot v_h \in P_{k-1}(K)$, $v_h \cdot n^j$, $\bar{v}_h \cdot n^j$ $\in P_k(F)$, and using the definition of L^2-projections Π_Q and $\bar{\Pi}_Q^j$, $j = s, d$, we have

$$\begin{aligned}
&\sum_{j=s,d} \left(b_h^j(\boldsymbol{\xi}_p^{jn}, v_h) + b_h^{I,j}(\bar{\xi}_p^{jn}, \bar{v}_h)\right) = - \sum_{j=s,d} \Big(- \sum_{K \in \mathcal{T}^j} \int_K \xi_p^n \nabla \cdot v_h \,\mathrm{d}x \\
&\qquad + \sum_{K \in \mathcal{T}^j} \int_{\partial K} \bar{\xi}_p^{jn} v_h \cdot n^j \,\mathrm{d}s - \int_{\Gamma^I} \bar{\xi}_p^{jn} \bar{v}_h \cdot n^j \,\mathrm{d}s \Big) \\
&\quad = \sum_{j=s,d} \Big(- \sum_{K \in \mathcal{T}^j} \int_K (p^n - \Pi_Q p^n) \nabla \cdot v_h \,\mathrm{d}x \\
&\quad + \sum_{K \in \mathcal{T}^j} \int_{\partial K} (\gamma(p^{jn}) - \bar{\Pi}_Q^j p^{jn}) v_h \cdot n^j \,\mathrm{d}s - \int_{\Gamma^I} (\gamma(p^{jn}) - \bar{\Pi}_Q^j p^{jn}) \bar{v}_h \cdot n^j \,\mathrm{d}s \Big) = 0.
\end{aligned}$$

We define the discrete-in-time norm to estimate the velocity error:

$$\|f\|_{\ell^2(0,T;X)} = \big(\Delta t \sum_{m=1}^{n} \|f^m\|_X^2\big)^{1/2}. \tag{2.58}$$

Theorem 6 (Velocity error)**.** *Let $(u,p) \in X$ be the solution of the Stokes–Darcy system eqs.* (2.1) *and* (2.2) *such that*

$$u^s \in L^2(0,T;[H^{k+1}(\Omega^s)]^{\mathrm{dim}}),\ \partial_t u^s \in L^2(0,T;[H^k(\Omega^s)]^{\mathrm{dim}}),$$
$$\partial_{tt} u^s \in L^2(0,T;[L^2(\Omega^s)]^{\mathrm{dim}}),\ u^d \in L^2(0,T;[H^k(\Omega^d)]^{\mathrm{dim}}),\quad k \geq 1.$$

Define $u_h^0 = \Pi_V u_0 \in V_h$ and let $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ solve eq. (2.28)*. Then $\boldsymbol{\zeta}_u^n$ satisfies the following estimate*

$$\|\zeta_u^n\|_{\Omega^s}^2 + (\Delta t)^2 \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 + C_a \Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_u^m|||_v^2 \quad \leq C'\big((\Delta t)^2 + h^{2k}\big). \tag{2.59}$$

Proof. We start splitting the error in eq. (2.46) by using eq. (2.50) as follows:

$$\sum_{K \in \mathcal{T}^s} \int_K d_t u_h^n \cdot v_h \,\mathrm{d}x + B_h((\boldsymbol{\zeta}_u^n, \boldsymbol{\zeta}_p^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = B_h((\boldsymbol{\xi}_u^n, \boldsymbol{\xi}_p^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)), \tag{2.60}$$

for any $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$.

Setting $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{\zeta}_u^n, -\boldsymbol{\zeta}_p^n)$ in eq. (2.60), using

$$a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$$

and the coercivity of a_h eq. (2.42) yields

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\zeta_u^n\|_{\Omega^s}^2 - \|\zeta_u^{n-1}\|_{\Omega^s}^2) + \frac{\Delta t}{2}\|d_t \zeta_u^n\|_{\Omega^s}^2 + C_a |||\boldsymbol{\zeta}_u^n|||_v^2 \\
&\qquad \leq \sum_{K \in \mathcal{T}^s} \int_K (d_t \zeta_u^n) \cdot \zeta_u^n \,\mathrm{d}x + a_h(\boldsymbol{\zeta}_u^n, \boldsymbol{\zeta}_u^n) \\
&\qquad = \sum_{K \in \mathcal{T}^s} \int_K (\partial_t u^n - d_t \Pi_V u^n) \cdot \zeta_u^n \,\mathrm{d}x + B_h((\boldsymbol{\xi}_u^n, \boldsymbol{\xi}_p^n), (\boldsymbol{\zeta}_u^n, -\boldsymbol{\zeta}_p^n)) \\
&\qquad =: I_1 + I_2.
\end{aligned} \tag{2.61}$$

Using eq. (2.40), and employing Young's inequality for some $\epsilon > 0$,

$$\begin{aligned}
I_1 &= \sum_{K\in\mathcal{T}^s} \int_K (\partial_t u^n - d_t \Pi_V u^n) \cdot \zeta_u^n \,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}^s} \int_K (\partial_t u^n - d_t u^n) \cdot \zeta_u^n \,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_K d_t \xi_u^n \cdot \zeta_u^n \,\mathrm{d}x \\
&\le \|\partial_t u^n - d_t u^n\|_{\Omega^s} \|\zeta_u^n\|_{\Omega^s} + \|d_t \xi_u^n\|_{\Omega^s} \|\zeta_u^n\|_{\Omega^s} \\
&\le C\|\partial_t u^n - d_t u^n\|_{\Omega^s} \|\!|\boldsymbol{\zeta}_u^n|\!\|_v + C\|d_t \xi_u^n\|_{\Omega^s} \|\!|\boldsymbol{\zeta}_u^n|\!\|_v \\
&\le C\big(\|\partial_t u^n - d_t u^n\|_{\Omega^s}^2 + \|d_t \xi_u^n\|_{\Omega^s}^2\big) + \epsilon \|\!|\boldsymbol{\zeta}_u^n|\!\|_v^2.
\end{aligned} \tag{2.62}$$

We have by eqs. (2.55) and (2.56), the continuity of a_h eq. (2.43), and Young's inequality,

$$\begin{aligned}
I_2 &= B_h((\boldsymbol{\xi}_u^n, \boldsymbol{\xi}_p^n), (\boldsymbol{\zeta}_u^n, -\boldsymbol{\zeta}_p^n)) \\
&= a_h(\boldsymbol{\xi}_u^n, \boldsymbol{\zeta}_u^n) + \sum_{j=s,d} \big(b_h^j(\boldsymbol{\xi}_p^{jn}, \zeta_u^n) + b_h^{I,j}(\bar{\xi}_p^{jn}, \bar{\zeta}_u^n)\big) \\
&\qquad - \sum_{j=s,d} \big(b_h^{jn}(\boldsymbol{\zeta}_p^{jn}, \xi_u^n) + b_h^{I,j}(\bar{\zeta}_p^{jn}, \bar{\xi}_u^n)\big) \\
&= a_h(\boldsymbol{\xi}_u^n, \boldsymbol{\zeta}_u^n) \\
&\le C\|\!|\boldsymbol{\zeta}_u^n|\!\|_v \|\!|\boldsymbol{\xi}_u^n|\!\|_{v'} \le C\|\!|\boldsymbol{\xi}_u^n|\!\|_{v'}^2 + \epsilon \|\!|\boldsymbol{\zeta}_u^n|\!\|_v^2.
\end{aligned} \tag{2.63}$$

Using eqs. (2.62) and (2.63) in eq. (2.61), we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\zeta_u^n\|_{\Omega^s}^2 - \|\zeta_u^{n-1}\|_{\Omega^s}^2\Big) + \frac{\Delta t}{2}\|d_t \zeta_u^n\|_{\Omega^s}^2 + C_a \|\!|\boldsymbol{\zeta}_u^n|\!\|_v^2 \\
&\qquad \le 2\epsilon \|\!|\boldsymbol{\zeta}_u^n|\!\|_v^2 + C\Big(\|\partial_t u^n - d_t u^n\|_{\Omega^s}^2 + \|d_t \xi_u^n\|_{\Omega^s}^2 + \|\!|\boldsymbol{\xi}_u^n|\!\|_{v'}^2\Big).
\end{aligned}$$

Letting $\epsilon = \dfrac{C_a}{4}$ (C_a is the coercivity constant), we get

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\zeta_u^n\|_{\Omega^s}^2 - \|\zeta_u^{n-1}\|_{\Omega^s}^2) + \frac{\Delta t}{2}\|d_t \zeta_u^n\|_{\Omega^s}^2 + \frac{C_a}{2}\|\!|\boldsymbol{\zeta}_u^n|\!\|_v^2 \\
&\qquad \le C\Big(\|\partial_t u^n - d_t u^n\|_{\Omega^s}^2 + \|d_t \xi_u^n\|_{\Omega^s}^2 + \|\!|\boldsymbol{\xi}_u^n|\!\|_{v'}^2\Big).
\end{aligned}$$

Then multiplying by $2\Delta t$, summing from 1 to n, noting that $\zeta_u^0 = 0$, and applying eqs. (C.2) and (C.3), we obtain

$$\|\zeta_u^n\|^2_{\Omega^s} + (\Delta t)^2 \sum_{m=1}^{n} \|d_t\zeta_u^m\|^2_{\Omega^s} + C_a\Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_u^m|||_v^2$$
$$\leq C\Bigg((\Delta t)^2 \|\partial_{tt}u\|^2_{L^2(0,T;[L^2(\Omega^s)]^{\dim})} + \|\partial_t \xi_u\|^2_{L^2(0,T;[L^2(\Omega^s)^{\dim})} + \Delta t \sum_{m=1}^{n} |||\boldsymbol{\xi}_u^m|||^2_{v'} \Bigg).$$

Note that using Lemma 4 iv., we have

$$\|\partial_t\xi_u\|_{L^2(0,T;[L^2(\Omega^s)]^{\dim})} \leq Ch^k \|\partial_t u\|_{L^2(0,T;[H^k(\Omega^s)]^{\dim})}.$$

Therefore using Theorem 5,

$$\begin{aligned}
\|\zeta_u^n\|^2_{\Omega^s} + (\Delta t)^2 \sum_{m=1}^{n} \|d_t\zeta_u^m\|^2_{\Omega^s} + C_a\Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_u^m|||_v^2 \\
\leq & C(\Delta t)^2 \|\partial_{tt}u\|^2_{L^2(0,T;[L^2(\Omega^s)]^{\dim}))} + Ch^{2k}\|\partial_t u\|^2_{L^2(0,T;[H^k(\Omega^s)]^{\dim})} \\
& + Ch^{2k}\Big(\|u\|^2_{\ell^2(0,T;[H^{k+1}(\Omega^s)]^{\dim})} + \|u\|^2_{\ell^2(0,T;[H^k(\Omega^d)]^{\dim})} \Big) \\
\leq & C\big((\Delta t)^2 + h^{2k}\big),
\end{aligned} \tag{2.64}$$

where the last constant C depends on the regularity of the solution but is independent of h.

$\square$

Note that using a suitable dual problem, it is possible to obtain optimal a priori estimates but it is beyond the scope of this thesis. The proof in the steady-state case can be found in [16].

2.3.2 Numerical experiments

In this section, we provide a numerical convergence analysis of the HDG method by demonstrating that we achieve optimal convergence using a manufactured solution. The NGSolve finite element library [49] is used for the implementation. We will use the unit square domain $\Omega = [0,1]^2$ consisting of two rectangular subdomains $\Omega^d = [0,1] \times [0,0.5]$ and $\Omega^s = [0,1] \times [0.5,1]$. Even though we fully discretized the scheme using the first order backward Euler method for simplicity in the analysis, we are not bound by that choice in our computations. We choose to use the second order backward difference method (BDF2) here but Crank-Nicolson method can also be used. We initialize BDF2 by using a single step of backward Euler method.

The fully discrete form using BDF2 is as follows: For a sequence $\{v^n\}$, we introduce the BDF2 operator $d_t^{(2)}$ such that, for all $n \in \{2, \ldots, N\}$,

$$d_t^{(2)} v^n = \frac{3v^n - 4v^{n-1} + v^{n-2}}{2\Delta t}. \tag{2.65}$$

Then the fully discrete scheme is obtained by applying the BDF2 scheme to eq. (2.22): for $n \geq 2$, find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that

$$\begin{aligned}\sum_{K \in \mathcal{T}^s} \int_K d_t^{(2)} u_h^n \cdot v_h \,\mathrm{d}x + B_h((\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\ = \sum_{K \in \mathcal{T}^s} \int_K f^s(t^n) \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}^d} \int_K g^d(t^n) q_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h,\end{aligned} \tag{2.66}$$

where the initial condition u_h^0 is a suitable projection of u_0 into V_h and u_h^1 is obtained by solving a single step of the backward Euler scheme in the beginning.

2.3.2.1 Example 1 Let $\alpha = \mu\sqrt{\kappa}(1 + 4\pi^2)/2$. We solve the Stokes-Darcy problem with the exact solution given by

$$\begin{aligned} u^s &= (-\frac{1}{2\pi^2}\sin(\pi x_1 + t)e^{(x_2+t)/2}, \frac{1}{\pi}\cos(\pi x_1 + t)e^{(x_2+t)/2})^T, \\ u^d &= (-2\sin(\pi x_1 + t)e^{(x_2+t)/2}, \frac{1}{\pi}\cos(\pi x_1 + t)e^{(x_2+t)/2})^T, \end{aligned}$$

$$p^s = \frac{\kappa\mu - 2}{\kappa\pi}\cos(\pi x_1 + t)e^{(x_2+t)/2},$$
$$p^d = -\frac{2}{\kappa\pi}\cos(\pi x_1 + t)e^{(x_2+t)/2}.$$

In the numerical experiments, we used sufficiently small time steps to observe the high order convergence rates even though BDF2 requires no restriction on the time step.

We present our numerical solutions for $k = 2$, $\ell = k - 1 = 1$ with different combinations of κ and μ to show robustness with respect to the parameters. Tables 2.1 to 2.4 presents computed errors in the velocity and the pressure on Ω^s and on Ω^d for different values of μ and κ given by $\kappa = \mu = 1$ in Table 2.1, $\kappa = 1, \mu = 10^{-6}$ in Table 2.2, $\kappa = 10^3, \mu = 1$ in Table 2.3, and $\kappa = 10^3, \mu = 10^{-6}$ in Table 2.4.

We observe optimal rates of convergence for all unknowns, independent of the value of the viscosity and permeability. It can also be seen that there is a variance in the error of the pressure when μ and κ changes but the error in the velocity is not affected by this change significantly due to the pressure robustness of our scheme even in the case of small viscosity. In addition, the last columns in the tables show that pointwise mass conservation is satisfied in both regions. Similar conclusions can be drawn from Tables 2.5 and 2.6 for $k = 3$ with $\kappa = 1, \mu = 10^{-6}$, and $\kappa = 10^3, \mu = 10^{-6}$, respectively.

2.3.2.2 Example 2 In this numerical example, we accept the same boundary conditions as in [16] which imposes a parabolic velocity profile on the left boundary of the Stokes region Γ^s_1. Let the boundary of the Stokes region be partitioned as $\Gamma^s = \Gamma^s_1 \cup \Gamma^s_2 \cup \Gamma^s_3$ where

$$\Gamma^s_1 := \left\{x \in \Gamma^s \;:\; x_1 = 0\right\}, \Gamma^s_2 := \left\{x \in \Gamma^s \;:\; x_1 = 1\right\}, \Gamma^s_3 := \left\{x \in \Gamma^s \;:\; x_2 = 1\right\}.$$

Table 2.1: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 1$, $\kappa = 1$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega_s}$	Order	$\|p - p_h\|_{\Omega_s}$	Order	$\|\nabla \cdot u_h\|_{\Omega_s}$
1/2	8	3.5e-03	–	8.9e-02	–	2.0e-07
1/4	28	6.4e-04	2.4	2.7e-02	1.7	2.9e-09
1/8	152	6.0e-07	5.2	5.7e-08	3.7	7.5e-15
1/16	576	4.4e-06	3.4	1.2e-03	2.1	7.0e-13
1/32	2348	5.2e-07	3.1	3.0e-04	2.0	7.1e-15
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.3e-02	–	2.3e-02	–	1.3e-06
1/4	28	3.1e-03	2.0	9.1e-03	1.3	5.4e-08
1/8	152	1.9e-04	4.0	1.4e-03	2.7	2.7e-10
1/16	576	2.5e-05	2.9	3.7e-04	1.9	5.5e-12
1/32	2348	2.7e-06	3.2	8.4e-05	2.1	3.8e-12

Table 2.2: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 10^{-6}$, $\kappa = 1$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega_s}$	Order	$\|p - p_h\|_{\Omega_s}$	Order	$\|\nabla \cdot u_h\|_{\Omega_s}$
1/2	8	1.0e-03	–	3.9e-02	–	2.0e-07
1/4	28	3.1e-03	2.1	9.1e-03	1.3	5.4e-08
1/8	152	2.0e-05	3.7	1.9e-03	2.6	4.5e-11
1/16	576	2.2e-06	3.2	4.5e-04	2.1	6.9e-13
1/32	2348	2.7e-07	3.1	1.1e-04	2.1	2.2e-14
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.3e-02	–	2.3e-02	–	1.3e-06
1/4	28	3.1e-03	2.0	9.1e-03	1.3	5.4e-08
1/8	152	1.9e-04	4.0	1.4e-03	2.7	2.7e-10
1/16	576	2.5e-05	2.9	3.7e-04	1.9	5.5e-12
1/32	2348	2.7e-06	3.2	8.4e-05	2.1	3.8e-12

Table 2.3: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 1$, $\kappa = 10^3$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\|u-u_h\|_{\Omega_s}$	Order	$\|p-p_h\|_{\Omega_s}$	Order	$\|\nabla\cdot u_h\|_{\Omega_s}$
1/2	8	3.4e-03	–	8.9e-02	–	2.0e-07
1/4	28	6.3e-04	2.4	2.7e-02	1.7	2.9e-09
1/8	152	4.8e-05	3.7	5.3e-03	2.3	4.5e-11
1/16	576	4.4e-06	3.4	1.2e-03	2.1	1.1e-12
1/32	2348	5.2e-07	3.1	3.0e-04	2.0	1.4e-12
			Darcy solution (Ω^d)			
h	Cells	$\|u-u_h\|_{\Omega_d}$	Order	$\|p-p_h\|_{\Omega_d}$	Order	$\|\nabla\cdot u_h+\Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.3e-02	–	2.3e-05	–	1.3e-06
1/4	28	3.1e-03	2.0	9.1e-06	1.3	5.4e-08
1/8	152	1.9e-04	4.0	1.4e-06	2.7	2.7e-10
1/16	576	2.5e-05	2.9	3.7e-07	1.9	2.7e-11
1/32	2348	2.7e-06	3.2	8.4e-08	2.1	1.1e-10

Table 2.4: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 10^{-6}$, $\kappa = 10^3$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\|u-u_h\|_{\Omega_s}$	Order	$\|p-p_h\|_{\Omega_s}$	Order	$\|\nabla\cdot u_h\|_{\Omega_s}$
1/2	8	9.1e-04	–	6.4e-05	–	2.0e-07
1/4	28	2.5e-04	1.9	1.3e-05	2.3	2.9e-09
1/8	152	2.0e-05	3.6	1.9e-06	2.8	4.5e-11
1/16	576	2.2e-06	3.2	4.5e-07	2.1	7.0e-13
1/32	2348	2.6e-07	3.1	1.1e-07	2.1	9.0e-14
			Darcy solution (Ω^d)			
h	Cells	$\|u-u_h\|_{\Omega_d}$	Order	$\|p-p_h\|_{\Omega_d}$	Order	$\|\nabla\cdot u_h+\Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.3e-02	–	2.3e-05	–	1.3e-06
1/4	28	3.1e-03	2.1	9.1e-06	1.3	5.4e-08
1/8	152	1.9e-04	4.0	1.4e-06	2.7	2.7e-10
1/16	576	2.5e-05	2.9	3.7e-07	1.9	5.5e-12
1/32	2348	2.7e-06	3.2	8.4e-08	2.1	3.8e-12

Table 2.5: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 3$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 10^{-6}$, $\kappa = 1$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\lVert u - u_h \rVert_{\Omega_s}$	Order	$\lVert p - p_h \rVert_{\Omega_s}$	Order	$\lVert \nabla \cdot u_h \rVert_{\Omega_s}$
1/2	8	2.5e-04	–	3.5e-03	–	5.7e-10
1/4	28	2.2e-05	3.5	6.9e-04	2.3	2.0e-12
1/8	152	6.1e-07	5.2	5.6e-05	3.6	3.8e-15
1/16	576	3.3e-08	4.2	6.4e-06	3.1	8.8e-14
1/32	2348	2.0e-09	4.1	7.7e-07	3.1	9.4e-14
			Darcy solution (Ω^d)			
h	Cells	$\lVert u - u_h \rVert_{\Omega_d}$	Order	$\lVert p - p_h \rVert_{\Omega_d}$	Order	$\lVert \nabla \cdot u_h + \Pi_Q g^d \rVert_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-03	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-04	3.0	2.2e-12
1/8	152	3.5e-06	5.1	4.0e-05	3.8	7.2e-13
1/16	576	2.2e-07	4.0	5.1e-06	3.0	3.0e-12
1/32	2348	1.3e-08	4.1	6.1e-07	3.1	1.2e-11

Table 2.6: Table of convergence by the HDG method for the solutions in the time-dependent Stokes-Darcy problem using $k = 3$, $\Delta t = 10^{-4}$, $T = 0.1$, $\mu = 10^{-6}$, $\kappa = 10^3$ in Section 2.3.2.1.

			Stokes solution (Ω^s)			
h	Cells	$\lVert u - u_h \rVert_{\Omega_s}$	Order	$\lVert p - p_h \rVert_{\Omega_s}$	Order	$\lVert \nabla \cdot u_h \rVert_{\Omega_s}$
1/2	8	2.4e-04	–	4.2e-06	–	5.7e-10
1/4	28	2.2e-05	3.5	7.3e-07	2.5	2.1e-12
1/8	152	6.0e-07	5.2	5.7e-08	3.7	7.5e-15
1/16	576	3.3e-08	4.2	6.4e-09	3.2	4.5e-15
1/32	2348	2.0e-09	4.1	7.9e-10	3.0	1.4e-13
			Darcy solution (Ω^d)			
h	Cells	$\lVert u - u_h \rVert_{\Omega_d}$	Order	$\lVert p - p_h \rVert_{\Omega_d}$	Order	$\lVert \nabla \cdot u_h + \Pi_Q g^d \rVert_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-06	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-07	3.0	2.2e-12
1/8	152	3.6e-06	5.1	4.0e-08	3.8	8.2e-13
1/16	576	2.2e-07	4.0	5.1e-09	3.0	3.1e-12
1/32	2348	1.3e-08	4.1	6.1e-10	3.1	1.2e-11

Similarly, let $\Gamma^d = \Gamma^d_1 \cup \Gamma^d_2$ where

$$\Gamma^d_1 := \left\{ x \in \Gamma^d \ : \ x_1 = 0 \text{ or } x_1 = 1 \right\}, \quad \Gamma^d_2 := \left\{ x \in \Gamma^d \ : \ x_2 = 0 \right\}.$$

We impose the following boundary conditions:

$$\begin{aligned}
u &= (x_2(3/2 - x_2)/5, 0) && \text{on } \Gamma^s_1, \\
(-2\mu\varepsilon(u) + p\mathbb{I})n &= 0 && \text{on } \Gamma^s_2, \\
u \cdot n &= 0 \text{ and } (-2\mu\varepsilon(u) + p\mathbb{I})^t = 0 && \text{on } \Gamma^s_3, \\
u \cdot n &= 0 && \text{on } \Gamma^d_1, \\
p &= -0.05 && \text{on } \Gamma^d_2.
\end{aligned}$$

We set the permeability to

$$\kappa = 500 \sin(5\pi x) \cos(8\pi y^2) + \cos(3\pi x) \cos(2\pi x) \sin(8\pi y) + 500.$$

Other parameters in eqs. (2.1) and (2.2) are set as $\mu = 0.1$, $\alpha = 0.5$, $k = 2$, $\beta = 10k^2$, $h = 1/32$, $\Delta t = 10^{-3}$, $T = 1$, $f^s = (0, 0)^T$ and $g^d = 0$. At the final time, $\|\nabla \cdot u_h\|_{\Omega^s} = 2.04e^{-15}$, $\|\nabla \cdot u_h\|_{\Omega^d} = 2.14e^{-10}$ which shows that the numerical solution is pointwise divergence-free.

Figure 2.2 shows the permeability, and the computed velocity field. High permeability regions are represented by red and low permeability regions are resresented by blue. As expected, in the Darcy region Ω^d the flow fluid flows much slower in the areas of low permeability and tries to avoid them by flowing towards highly permeable regions and flows more freely there.

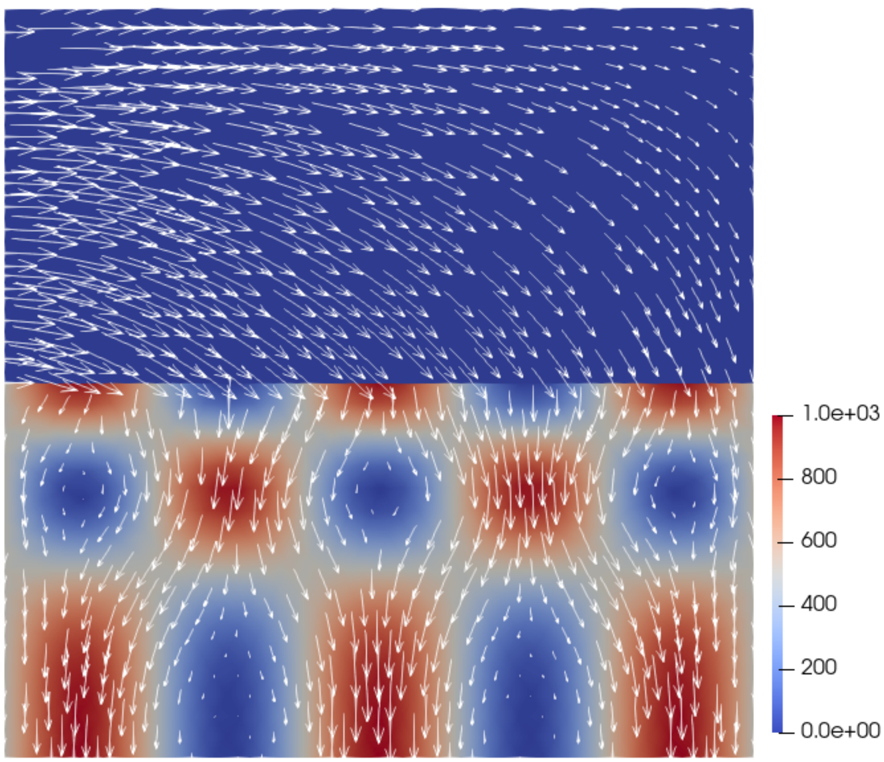


Figure 2.2: Solution at the final time $T = 1$ for the test case using $h = 1/32$, $\Delta t = 10^{-3}$ in Section 2.3.2.2

CHAPTER THREE

TIME-DEPENDENT TRANSPORT PROBLEM

In this chapter, we will focus on the time-dependent advection-diffusion equation which can be used to model the transport of a chemical species via a given velocity field. We will use an HDG method based on the one defined in [53] for the steady-state case. The HDG methods were introduced with the purpose of reducing the high number of degrees of freedom associated with DG methods, while retaining their desirable properties which make them an excellent choice for advection-diffusion problems. In fact, the HDG methods just like the DG methods satisfy local conservativity, high order accuracy, and achieve stabilization through the interelement jumps.

3.1 Time-dependent advection-diffusion equation

Let $\Omega \subset \mathbb{R}^{\text{dim}}$, $\text{dim} = 2, 3$ be a polygonal/polyhedral domain. On the boundary $\Gamma = \partial\Omega$, we denote by n the unit outward normal vector to the domain Ω. For a given finite time $T > 0$, we consider $J = (0, T]$. The advection-diffusion-reaction equation in the conservative form reads:

$$\partial_t c + \nabla \cdot (u\, c - D\nabla c) = f \text{ in } \Omega \times J, \tag{3.1}$$

where $c : \Omega \times J \to \mathbb{R}$ is the concentration of a contaminant, $D > 0$ is a constant, $u : \Omega \to \mathbb{R}^{\text{dim}}$ is a divergence-free vector field that is Lipschitz continuous on Ω and satisfies $\|u\|_{L^\infty(\Omega)} \leq 1$, and $f : \Omega \times J \to \mathbb{R}$ is a source term.

We denote the outflow portions of the boundary where $u \cdot n \geq 0$ and inflow portions where $u \cdot n < 0$ by Γ_+, and Γ_-, respectively, and denote by ζ the function that is defined on the boundary such that $\zeta = 0$ on Γ_+ and $\zeta = 1$ on Γ_-.

We assume that the boundary is split into two parts Γ_N and Γ_D such that $\Gamma_N \cap \Gamma_D = \emptyset$ and $\Gamma_N \cup \Gamma_D = \partial\Omega$ and prescribe the boundary conditions

$$(-\zeta c\, u + D\nabla c) \cdot n = 0 \text{ on } \Gamma_N \times J, \tag{3.2a}$$

$$c = 0 \text{ on } \Gamma_D \times J, \tag{3.2b}$$

Finally, we impose the following initial condition:

$$c(x, 0) = c_0(x) \text{ in } \Omega. \tag{3.3}$$

Remark 1. *It is possible to consider no diffusion case ($D = 0$) by assuming that $\Gamma_D = \emptyset$, $\Gamma_N = \partial\Omega = \Gamma_-$ and imposing the boundary condition*

$$c\, u \cdot n = 0 \textit{ on } \Gamma \times J. \tag{3.4}$$

We can also consider the nonhomogeneous boundary condition $(-\zeta c\, u + D\nabla c) \cdot n = g$ where $g : \Gamma_N \times J \to \mathbb{R}$ is a given suitably smooth function by adding an extra term $\int_{\Gamma_N} gw \,\mathrm{d}s$ to the right hand side of the weak formulation.

3.2 The weak formulation

Let $H^1_{\Gamma_D}(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D\}$. Multiplying equation (3.1) by $w \in H^1_{\Gamma_D}(\Omega)$, integrating on Ω and using integrating by parts, we get

$$\int_\Omega \partial_t c\, w \,\mathrm{d}x - \int_\Omega (cu - D\nabla c) \cdot \nabla w \,\mathrm{d}x + \int_{\partial\Omega} (cu - D\nabla c) \cdot nw \,\mathrm{d}s = \int_\Omega fw \,\mathrm{d}x. \tag{3.5}$$

Using the boundary conditions in eq. (3.2), we obtain the following weak formulation: Find $c \in L^2(0, T; H^1_{\Gamma_D}(\Omega))$ such that for a.e. $t \in J$,

$$\int_\Omega \partial_t c\, w \,\mathrm{d}x - \int_\Omega (cu - D\nabla c) \cdot \nabla w \,\mathrm{d}x + \int_{\Gamma_+} cu \cdot nw \,\mathrm{d}s = \int_\Omega fw \,\mathrm{d}x \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{3.6}$$

and $c(0) = c_0$ in Ω.

3.3 The numerical method

We will now describe the HDG method to solve the transport problem defined by equations eqs. (3.1) to (3.3). Let

$$C_h = \{c_h \in L^2(\Omega) : \; c_h \in P_k(K), \forall\, K \in \mathcal{T}\}, \tag{3.7}$$

$$\bar{C}_h = \{\bar{c}_h \in L^2(\Gamma_0) : \; \bar{c}_h \in P_k(F), \forall\, F \in \mathcal{F}\}. \tag{3.8}$$

We define $\boldsymbol{C}_h = C_h \times \bar{C}_h$ and an element in $\boldsymbol{C}_h$ will be denoted by $\boldsymbol{w}_h = (w_h, \bar{w}_h)$.

On $\boldsymbol{C}_h$, we define the following semi-norm:

$$|\!|\!| \boldsymbol{w}_h |\!|\!|_c^2 = \sum_{K \in \mathcal{T}} \left(\|\nabla w_h\|_K^2 + h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right) + \| |u \cdot n|^{1/2}\, \bar{w}_h \|_{\partial\Omega}^2, \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h.$$

In our HDG method, we discretize the advective term by

$$\begin{aligned} B_h^a(\boldsymbol{c}, \boldsymbol{w}) = &- \sum_{K \in \mathcal{T}} \int_K c u \cdot \nabla w \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} c u \cdot n (w - \bar{w}) \,\mathrm{d}s \\ &- \sum_{K \in \mathcal{T}} \int_{\partial K^{\mathrm{in}}} u \cdot n (c - \bar{c})(w - \bar{w}) \,\mathrm{d}s + \int_{\Gamma_+} u \cdot n\, \bar{c}\, \bar{w} \,\mathrm{d}s, \end{aligned} \tag{3.9}$$

where ∂K^{in} is the inflow part of boundary ∂K on which $u \cdot n < 0$. The diffusive term is discretized by

$$\begin{aligned} B_h^d(\boldsymbol{c}, \boldsymbol{w}) = &\sum_{K \in \mathcal{T}} \int_K D \nabla c \cdot \nabla w \,\mathrm{d}x - \sum_{K \in \mathcal{T}} \int_{\partial K} D \nabla c \cdot n (w - \bar{w}) \,\mathrm{d}s \\ &+ \sum_{K \in \mathcal{T}} \frac{\beta_c}{h_K} \int_{\partial K} D (c - \bar{c})(w - \bar{w}) \,\mathrm{d}s \\ &- \sum_{K \in \mathcal{T}} \int_{\partial K} D \nabla w \cdot n (c - \bar{c}) \,\mathrm{d}s, \end{aligned} \tag{3.10}$$

where $\beta_c > 0$ is a penalty parameter.

Together, B_h^a and B_h^d describe the bilinear form as

$$B_h(\boldsymbol{c}, \boldsymbol{w}) = B_h^a(\boldsymbol{c}, \boldsymbol{w}) + B_h^d(\boldsymbol{c}, \boldsymbol{w}). \tag{3.11}$$

The semi-discretization of the problem is defined as follows: For a.e. $t \in J$, find $\boldsymbol{c}_h(t) \in \boldsymbol{C}_h$ such that

$$\int_\Omega \partial_t c_h(t) w_h \, \mathrm{d}x + B_h(\boldsymbol{c}_h(t), \boldsymbol{w}_h) = \int_\Omega f(t) w_h \, \mathrm{d}x \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h. \tag{3.12}$$

To explain the bilinear form B_h, we consider $K \in \mathcal{T}$ and $w_h \in C_h$. Then $\boldsymbol{w}_h = (w_h \chi_K, 0)$ satisfies

$$\begin{aligned} B_h(\boldsymbol{c}_h, \boldsymbol{w}_h) = & \int_K [-c_h u + D\nabla c_h] \cdot \nabla w_h \, \mathrm{d}x - \int_{\partial K} (c_h - \bar{c}_h) D \nabla w_h \cdot n \, \mathrm{d}s \\ & - \int_{\partial K} \left[-c_h u + D\nabla c_h + \left(\zeta_K u - \frac{\beta_c}{h_K} n \right) (c_h - \bar{c}_h) \right] \cdot n w_h \, \mathrm{d}s, \end{aligned}$$

where $\zeta_K = 0$ on the outflow portion of ∂K and $\zeta_K = 1$ on the inflow portion ∂K^{in} of ∂K. Therefore eq. (3.12) on a single cell can be viewed as: given $\bar{c}_h \in \bar{C}_h$, find $c_h \in C_h$ such that for all $w_h \in C_h$:

$$\int_K \partial_t c_h(t) w_h \, \mathrm{d}x + \int_K \sigma(c_h) \cdot \nabla w_h \, \mathrm{d}x - \int_{\partial K} \hat{\sigma}(c_h) \cdot n w_h \, \mathrm{d}s = \int_K f(t) w_h \, \mathrm{d}x,$$

where $\sigma(c) = -cu + D\nabla c$ is the standard flux and the numerical flux is $\hat{\sigma}(c) = -cu + D\nabla c + \left(\zeta_K u - \frac{\beta_c}{h_K} n \right) (c - \bar{c})$. We note that the advective part of the numerical flux is upwinded as ζ_K vanishes on the outflow portions of the element boundaries and the interior penalty type term $\frac{\beta_c}{h_K} n(c_h - \bar{c}_h)$ penalizes the jumps [2]. For the time discretization, we use the same notation as in Section 2.3. Our fully discrete scheme searches for $\{\boldsymbol{c}_h^n\}_{n \geq 1}$ such that

$$\sum_{K \in \mathcal{T}} \int_K d_t c_h^n w_h \, \mathrm{d}x + B_h(\boldsymbol{c}_h^n, \boldsymbol{w}_h) = \sum_{K \in \mathcal{T}} \int_K f^n w_h \, \mathrm{d}x, \tag{3.13}$$

for all $\boldsymbol{w}_h \in \boldsymbol{C}_h$. We initialize by using the L^2-projection of c_0 onto C_h.

3.3.1 Theoretical results

First, we show stability and existence and uniqueness of the solution to the semi-discrete scheme eq. (3.12). The proof from [53] is reproduced here for completeness.

Lemma 6 (Coercivity)**.** *There exists a constant $\beta_0 > 0$ such that if $\beta_c > \beta_0$, then for all $\boldsymbol{w}_h \in \boldsymbol{C}_h$*

$$B_h(\boldsymbol{w}_h, \boldsymbol{w}_h) \geq C \|\!|\boldsymbol{w}_h|\!\|_c^2. \tag{3.14}$$

Proof. By definition of B_h^d eq. (3.10), we have

$$\begin{aligned}
B_h^d(\boldsymbol{w}_h, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}} \int_K D\nabla w_h \cdot \nabla w_h \, \mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} D\nabla w_h \cdot n(w_h - \bar{w}_h) \, \mathrm{d}s \\
&= \sum_{K\in\mathcal{T}} \int_K D\nabla w_h \cdot \nabla w_h \, \mathrm{d}x - 2 \sum_{K\in\mathcal{T}} \int_{\partial K} D\nabla w_h \cdot n(w_h - \bar{w}_h) \, \mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}} \frac{\beta_c}{h_K} \int_{\partial K} D(w_h - \bar{w}_h)(w_h - \bar{w}_h) \, \mathrm{d}s \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

We have

$$J_1 = \sum_{K\in\mathcal{T}} \int_K D\nabla w_h \cdot \nabla w_h \, \mathrm{d}x = \sum_{K\in\mathcal{T}} D\|\nabla w_h\|_K^2.$$

Next by the Cauchy-Schwarz, Young's, trace eq. (2.29) inequalities as follows:

$$\begin{aligned}
|J_2| &\leq 2 \sum_{K\in\mathcal{T}} \int_{\partial K} D\nabla w_h \cdot n(w_h - \bar{w}_h) \, \mathrm{d}s \\
&\leq \sum_{K\in\mathcal{T}} 2(Dh_K)^{1/2} \|\nabla w_h \cdot n\|_{\partial K} \left(\frac{D}{h_K}\right)^{1/2} \|w_h - \bar{w}_h\|_{\partial K} \\
&\leq \sum_{K\in\mathcal{T}} \left(\epsilon D h_K \|\nabla w_h \cdot n\|_{\partial K}^2 + \frac{D}{\epsilon h_K} \|w_h - \bar{w}_h\|_{\partial K}^2 \right) \\
&\leq \sum_{K\in\mathcal{T}} \left(\epsilon C D \|\nabla w_h\|_K^2 + \frac{D}{\epsilon h_K} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)
\end{aligned}$$

where $\epsilon > 0$ is the parameter for Young's inequality.

It is clear that

$$J_3 = \sum_{K\in\mathcal{T}} \frac{\beta_c}{h_K} \int_{\partial K} D(w_h - \bar{w}_h)(w_h - \bar{w}_h)\ \mathrm{d}s = \sum_{K\in\mathcal{T}} \frac{D\beta_c}{h_K} \|w_h - \bar{w}_h\|^2_{\partial K}.$$

Combining the bounds for $J_1, |J_2|$ and J_3, we obtain

$$B_h^d(\boldsymbol{w}_h, \boldsymbol{w}_h) \geq \sum_{K\in\mathcal{T}} (D - \epsilon CD)\|\nabla w_h\|^2_K + \sum_{K\in\mathcal{T}} (D\beta_c - \frac{D}{\epsilon})\frac{1}{h_K}\|w_h - \bar{w}_h\|^2_{\partial K}.$$

Choosing $\epsilon = \frac{1}{2C}$, we have

$$B_h^d(\boldsymbol{w}_h, \boldsymbol{w}_h) \geq \sum_{K\in\mathcal{T}} \frac{D}{2}\|\nabla w_h\|^2_K + \sum_{K\in\mathcal{T}} (D\beta_c - 2CD)\frac{1}{h_K}\|w_h - \bar{w}_h\|^2_{\partial K}.$$

If $\beta_0 = 2C$ and $\beta_c > \beta_0$, then we have

$$B_h^d(\boldsymbol{w}_h, \boldsymbol{w}_h) \geq C\left(\sum_{K\in\mathcal{T}} \|\nabla w_h\|^2_K + \sum_{K\in\mathcal{T}} h_K^{-1}\|w_h - \bar{w}_h\|^2_{\partial K}\right). \tag{3.15}$$

We now prove that

$$B_h^a(\boldsymbol{w}_h, \boldsymbol{w}_h) = \frac{1}{2}\sum_{K\in\mathcal{T}^s} \||u\cdot n|^{1/2}(w_h - \bar{w}_h)\|^2_{\partial K} + \frac{1}{2}\int_{\partial\Omega} |u\cdot n|\,\bar{w}_h^2\,\mathrm{d}s. \tag{3.16}$$

From eq. (3.9),

$$\begin{aligned} B_h^a(\boldsymbol{w}_h, \boldsymbol{w}_h) = &-\sum_{K\in\mathcal{T}} \int_K w_h u\cdot\nabla w_h\ \mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} w_h\,(u\cdot n)(w_h - \bar{w}_h)\ \mathrm{d}s \\ &- \sum_{K\in\mathcal{T}} \int_{\partial K^{\mathrm{in}}} u\cdot n\,(w_h - \bar{w}_h)^2\ \mathrm{d}s + \int_{\Gamma_+} u\cdot n\,\bar{w}_h^2\,\mathrm{d}s. \end{aligned} \tag{3.17}$$

Using integration by parts and noting that $\nabla\cdot u = 0$ in $K \in \mathcal{T}$, we can rewrite the first term on the right hand side of eq. (3.17) as

$$\begin{aligned} -\sum_{K\in\mathcal{T}} \int_K w_h u\cdot\nabla w_h\ \mathrm{d}x &= -\frac{1}{2}\sum_{K\in\mathcal{T}} \int_K u\cdot\nabla w_h^2\ \mathrm{d}x \\ &= \frac{1}{2}\sum_{K\in\mathcal{T}} \int_K \nabla\cdot u w_h^2\ \mathrm{d}x - \frac{1}{2}\sum_{K\in\mathcal{T}} \int_{\partial K} w_h^2 u\cdot n\ \mathrm{d}s \ = -\frac{1}{2}\sum_{K\in\mathcal{T}} \int_{\partial K} w_h^2 u\cdot n\ \mathrm{d}s. \end{aligned}$$

Using the algebraic identity $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$ with $a = w_h$, $b = \bar{w}_h$, the second term on the right hand side of eq. (3.17) simplifies as

$$\sum_{K\in\mathcal{T}} \int_{\partial K} w_h \, (u\cdot n)(w_h - \bar{w}_h) \, \mathrm{d}s = \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K} u \cdot n (w_h^2 - \bar{w}_h^2 + (w_h - \bar{w}_h)^2) \, \mathrm{d}s.$$

Hence, using the simplifications above, eq. (3.17) reduces to

$$\begin{aligned} B_h^a(\boldsymbol{w}_h, \boldsymbol{w}_h) = -\frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K} u \cdot n \bar{w}_h^2 \, \mathrm{d}s + \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K \backslash \partial K^{\text{in}}} u \cdot n (w_h - \bar{w}_h)^2 \, \mathrm{d}s \\ - \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K^{\text{in}}} u \cdot n (w_h - \bar{w}_h)^2 \, \mathrm{d}s + \int_{\Gamma_+} u \cdot n \, \bar{w}_h^2 \, \mathrm{d}s. \end{aligned}$$

Since $\bar{w}_h$ and $u \cdot n$ are single-valued on element boundaries, and $|u\cdot n| = u \cdot n$ on $\partial K \backslash \partial K^{\text{in}}$ and $|u \cdot n| = -u \cdot n$ on ∂K^{in}, we have

$$\begin{aligned} B_h^a(\boldsymbol{w}_h, \boldsymbol{w}_h) =& -\frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K} u \cdot n \bar{w}_h^2 \, \mathrm{d}s + \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K} |u \cdot n| (w_h - \bar{w}_h)^2 \, \mathrm{d}s \\ &+ \int_{\Gamma_+} u \cdot n \, \bar{w}_h^2 \, \mathrm{d}s \\ =& \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{\partial K} |u \cdot n| (w_h - \bar{w}_h)^2 \, \mathrm{d}s + \frac{1}{2} \int_{\partial\Omega} |u \cdot n| \, \bar{w}_h^2 \, \mathrm{d}s. \end{aligned}$$

This implies that eq. (3.16) holds, and hence using eq. (3.15), we obtain

$$B_h(\boldsymbol{w_h}, \boldsymbol{w_h}) \geq C |\!|\!| \boldsymbol{w}_h |\!|\!|_c^2. \tag{3.18}$$

□

Theorem 7 (Existence and uniqueness)**.** *The semi-discrete scheme eq.* (3.12) *has a unique solution* $\boldsymbol{c}_h \in \boldsymbol{C}_h$*.*

Proof. Let $t > 0$ and take $\boldsymbol{w}_h = \boldsymbol{c}_h(t)$ in eq. (3.12). Then

$$\sum_{K\in\mathcal{T}} (\partial_t c_h(t), c_h(t))_K + B_h(\boldsymbol{c}_h(t), \boldsymbol{c}_h(t)) = \sum_{K\in\mathcal{T}} (f(t), c_h(t))_K. \tag{3.19}$$

By eq. (3.14) and the Cauchy–Schwarz inequality,

$$\frac{1}{2}\frac{d}{dt}\left\|c_h(t)\right\|_\Omega^2 + C\|\!|\boldsymbol{c}_h(t)|\!\|_c^2 \leq \sum_{K\in\mathcal{T}} (f(t), c_h(t))_K \leq \left\|f(t)\right\|_\Omega \left\|c_h(t)\right\|_\Omega. \tag{3.20}$$

Integrating eq. (3.20) from 0 to t for some $0 < t \leq T$, and applying the Cauchy–Schwarz and Young's inequalities,

$$\begin{aligned}\frac{1}{2}\left\|c_h(t)\right\|_\Omega^2 + C\int_0^t \|\!|\boldsymbol{c}_h(s)|\!\|_c^2 \,\mathrm{d}s \leq \frac{1}{2}\left\|c_h(0)\right\|_\Omega^2 \\ + \frac{1}{2}\int_0^t \left\|f(s)\right\|_\Omega^2 \mathrm{d}s + \frac{1}{2}\int_0^t \left\|c_h(s)\right\|_\Omega^2 \mathrm{d}s.\end{aligned}$$

Using eq. (C.4) and recalling that $c_h(0)$ is the L^2-projection of c_0 into C_h, we have

$$\left\|c_h(t)\right\|_\Omega^2 + \int_0^t \|\!|\boldsymbol{c}_h(s)|\!\|_c^2 \,\mathrm{d}s \leq C\left(\|c_0\|_\Omega^2 + \int_0^t \left\|f(s)\right\|_\Omega^2 \mathrm{d}s\right) \quad \forall t \in [0, T].$$

The existence and uniqueness follows by setting $c_0 = 0$ and $f = 0$. □

To estimate convergence rates in the energy norm, we introduce the continuous interpolant $\mathcal{I}c \in C_h \cap \mathcal{C}^0(\bar{\Omega})$ of c from [9] and we set $\bar{\mathcal{I}}c(t) = \mathcal{I}c|_{\Gamma^0}(t) \in \bar{C}_h$. Note that we can use other interpolants but this choice makes the analysis simpler by eliminating the $\xi_c - \bar{\xi}_c$ terms. Let $\bar{c}$ be the restriction of c to Γ^0 (the union of all facets). We split the errors as follows:

$$c - c_h = \xi_c - \zeta_c \quad \text{and} \quad \bar{c} - \bar{c}_h = \bar{\xi}_c - \bar{\zeta}_c,$$

where

$$\begin{aligned}
&\xi_c = c - \mathcal{I}c, && \bar{\xi}_c = \bar{c} - \bar{\mathcal{I}}c, && \boldsymbol{\xi}_c = (\xi_c, \bar{\xi}_c),\\
&\zeta_c = c_h - \mathcal{I}c, && \bar{\zeta}_c, = \bar{c}_h - \bar{\mathcal{I}}c, && \boldsymbol{\zeta}_c = (\zeta_c, \bar{\zeta}_c).
\end{aligned}$$

The following interpolation estimates hold [9, Chapter 4]:

$$\|\xi_c\|_{r,K} \leq Ch_K^{s-r}\|c\|_{s,K}, \quad r = 0, 1, \quad 2 \leq s \leq k, \tag{3.21}$$

and

$$\|\xi_c\|_{r,\infty,K} \leq C h_K^{s-r} \|c\|_{s,\infty,K}\,, \quad r = 0, 1, \quad 1 \leq s \leq k. \tag{3.22}$$

Using $r = 1$ in eq. (3.21), and observing that $\xi_c - \bar{\xi}_c = 0$ on element boundaries ∂K, we obtain

$$\begin{aligned} |||\boldsymbol{\xi}_c(t)|||_c &= \Big(\sum_{K\in\mathcal{T}} \|\nabla \xi_c(t)\|_K^2 \Big)^{1/2} \\ &\leq C h^{s-1} \|c(t)\|_{s,\Omega}, \quad 2 \leq s \leq k. \end{aligned} \tag{3.23}$$

Lemma 7. *Let $c \in L^2(0, T; H^k(\Omega))$ be the solution to eq. (3.1). Then for any $0 \leq t \leq T$,*

$$B_h(\boldsymbol{\xi}_c(t), \boldsymbol{w}_h) \leq C h^{k-1} \, \|c(t)\|_{k,\Omega} \, |||\boldsymbol{w}_h|||_c, \tag{3.24}$$

for all $\boldsymbol{w}_h \in \boldsymbol{C}_h$ and $k > 1$.

Proof. By definition of B_h eq. (3.11),

$$B_h(\boldsymbol{\xi}_c, \boldsymbol{w}_h) = B_h^a(\boldsymbol{\xi}_c, \boldsymbol{w}_h) + B_h^d(\boldsymbol{\xi}_c, \boldsymbol{w}_h).$$

Since $\xi_c - \bar{\xi}_c$ vanishes on facets, we have by eq. (3.9)

$$\begin{aligned} B_h^a(\boldsymbol{\xi}_c, \boldsymbol{w}_h) = &- \sum_{K\in\mathcal{T}} \int_K \xi_c u \cdot \nabla w_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} \xi_c \,(u\cdot n)(w_h - \bar{w}_h)\,\mathrm{d}s \\ &+ \int_{\Gamma_+} (u \cdot n)\, \bar{\xi}_c \bar{w}_h \,\mathrm{d}s. \\ =&: J_1 + J_2 + J_3. \end{aligned}$$

Next, we estimate J_1, J_2 and J_3, respectively. We start by bounding J_1. By eq. (3.21), we have

$$\begin{aligned} J_1 &\leq C \|u\|_{0,\infty,\Omega} \|\xi_c\|_\Omega \|\nabla w_h\|_\Omega \\ &\leq C h^k \|u\|_{0,\infty,\Omega} \|c\|_{k,\Omega} \, |||\boldsymbol{w}_h|||_c. \end{aligned}$$

By eq. (3.21) and [9, Theorem 1.6.6],

$$
\begin{aligned}
J_2 &\le C\|u\|_{0,\infty,\Omega}\left(\sum_{K\in\mathcal{T}} h_K\|\xi_c\|_{\partial K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}} h_K^{-1}\|w_h-\bar{w}_h\|_{\partial K}^2\right)^{1/2}\\
&\le C\|u\|_{0,\infty,\Omega}\left(\sum_{K\in\mathcal{T}} h_K(h_K^{-1}\|\xi_c\|_K^2+h_K|\xi_c|_{1,K}^2)\right)^{1/2}\left(\sum_{K\in\mathcal{T}} h_K^{-1}\|w_h-\bar{w}_h\|_{\partial K}^2\right)^{1/2}\\
&\le Ch^k\|c\|_{k,\Omega}\,|\!|\!|\boldsymbol{w}_h|\!|\!|_c.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and eq. (2.31),

$$
\begin{aligned}
J_3 &= \int_{\Gamma_+}(u\cdot n)\,\bar{\xi}_c\bar{w}_h\,\mathrm{d}s\\
&\le \left\||u\cdot n|^{1/2}\,\bar{\xi}_c\right\|_{\partial\Omega}\left\||u\cdot n|^{1/2}\,\bar{w}_h\right\|_{\partial\Omega}\\
&\le \|u\|_{0,\infty,\Omega}^{1/2}\left(\sum_{K\in\mathcal{T}}(h_K^{-1}\|\xi_c\|_K^2+h_K|\xi_c|_{1,K}^2)\right)^{1/2}\left\||u\cdot n|^{1/2}\,\bar{w}_h\right\|_{\partial\Omega}\\
&\le Ch^{k-1}\|u\|_{0,\infty,\Omega}^{1/2}\|c\|_{k,\Omega}\,|\!|\!|\boldsymbol{w}_h|\!|\!|_c.
\end{aligned}
$$

Combining the bounds for $J_1 - J_3$ and using the boundedness assumption of u, we obtain

$$
B_h^a(\boldsymbol{\xi}_c,\boldsymbol{w}_h)\le Ch^{k-1}\|c\|_{k,\Omega}\,|\!|\!|\boldsymbol{w}_h|\!|\!|_c. \tag{3.25}
$$

Noting that $\xi_c-\bar{\xi}_c$ vanishes on facets, we have

$$
\begin{aligned}
B_h^d(\boldsymbol{\xi}_c,\boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}}\int_K D\nabla\xi_c\cdot\nabla w\,\mathrm{d}x-\sum_{K\in\mathcal{T}}\int_{\partial K}[D\nabla\xi_c]\cdot n(w_h-\bar{w}_h)\,\mathrm{d}s\\
&\quad+\sum_{K\in\mathcal{T}}\frac{\beta_c}{h_K}\int_{\partial K}D(\xi_c-\bar{\xi}_c)(w_h-\bar{w}_h)\,\mathrm{d}s\\
&\quad-\sum_{K\in\mathcal{T}}\int_{\partial K}D\nabla w\cdot n(\xi_c-\bar{\xi}_c)\,\mathrm{d}s\\
&=\sum_{K\in\mathcal{T}}\int_K D\nabla\xi_c\cdot\nabla w\,\mathrm{d}x-\sum_{K\in\mathcal{T}}\int_{\partial K}D\nabla w\cdot n(\xi_c-\bar{\xi}_c)\,\mathrm{d}s.
\end{aligned}
$$

Using [9, Theorem 1.6.6] again and eq. (3.21),

$$
\begin{aligned}
B_h^d(\boldsymbol{\xi}_c, \boldsymbol{w}_h) \leq & D\|\nabla \xi_c\|_\Omega \, \|\!|\boldsymbol{w}_h|\!\|_c \\
& + D \left(\sum_{K\in\mathcal{T}} h_K \|\nabla \xi_c\|_{\partial K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}} h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)^{1/2} \\
\leq & C h^{k-1} \|c\|_{k,\Omega} \, \|\!|\boldsymbol{w}_h|\!\|_c.
\end{aligned} \tag{3.26}
$$

The result eq. (3.24) follows by combining eqs. (3.25) and (3.26).

□

Lemma 8 (Consistency)**.** *If $c \in H^1(0,T; H^2(\Omega))$ solves eq. (3.1), then for all $t \in (0,T]$,*

$$
\sum_{K\in\mathcal{T}} \int_K \partial_t c(t) w_h \,\mathrm{d}x + B_h(\boldsymbol{c}(t), \boldsymbol{w}_h) = \sum_{K\in\mathcal{T}} \int_K f(t) w_h \,\mathrm{d}x \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h. \tag{3.27}
$$

Proof. Since c and u are smooth, $\bar{w}$ is single valued, and $c = \bar{c}$ on ∂K, the advective term simplifies to

$$
\begin{aligned}
B_h^a(\boldsymbol{c}, \boldsymbol{w}_h) = & - \sum_{K\in\mathcal{T}} \int_K c\, u \cdot \nabla w_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} c\, u \cdot n (w_h - \bar{w}_h) \,\mathrm{d}s \\
& - \sum_{K\in\mathcal{T}} \int_{\partial K^{\mathrm{in}}} u \cdot n\, (c - \bar{c})(w_h - \bar{w}_h) \,\mathrm{d}s + \int_{\Gamma_+} u \cdot n\, \bar{c}\, \bar{w}_h \,\mathrm{d}s \\
= & - \sum_{K\in\mathcal{T}} \int_K c\, u \cdot \nabla w_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} c\, u \cdot n (w_h - \bar{w}_h) \,\mathrm{d}s \\
& + \int_{\Gamma_+} u \cdot n\, c\, \bar{w}_h \,\mathrm{d}s \\
= & \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (c\, u) w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} (c\, u) \cdot n w_h \,\mathrm{d}s \\
& + \sum_{K\in\mathcal{T}} \int_{\partial K} c\, u \cdot n (w_h - \bar{w}_h) \,\mathrm{d}s + \int_{\Gamma_+} u \cdot n\, c\, \bar{w}_h \,\mathrm{d}s \\
= & \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (c\, u) w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} c\, u \cdot n \bar{w}_h \,\mathrm{d}s + \int_{\Gamma_+} u \cdot n\, c\, \bar{w}_h \,\mathrm{d}s,
\end{aligned}
$$

and the diffusive part simplifies to

$$
\begin{aligned}
B_h^d(\boldsymbol{c}, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}} \int_K D\nabla c \cdot \nabla w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla c] \cdot n(w_h - \bar{w}_h) \,\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}} \frac{\beta_c}{h_K} \int_{\partial K} D(c-\bar{c})(w_h - \bar{w}_h) \,\mathrm{d}s - \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla w_h] \cdot n(c - \bar{c}) \,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}} \int_K D\nabla c \cdot \nabla w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla c] \cdot n(w_h - \bar{w}_h) \,\mathrm{d}s \\
&= - \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (D\nabla c) w_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla c] \cdot n w_h \,\mathrm{d}s \\
&\quad - \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla c] \cdot n(w_h - \bar{w}_h) \,\mathrm{d}s \\
&= - \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (D\nabla c) w_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} [D\nabla c] \cdot n \bar{w}_h \,\mathrm{d}s.
\end{aligned}
$$

Therefore, combining the terms of B_h^d and B_h^a from above, using the smoothness of the solution, and since $\bar{w}_h$ is single valued and $\bar{w}_h = 0$ on Γ_D,

$$
\begin{aligned}
B_h(\boldsymbol{c}, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (c\,u - D\nabla c) w_h \,\mathrm{d}x \\
&\qquad + \int_{\Gamma_+} u \cdot n\, c\, \bar{w}_h \,\mathrm{d}s + \int_{\partial\Omega} [-cu + D\nabla c] \cdot n \bar{w}_h \,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}} \int_K \nabla \cdot (c\,u - D\nabla c) w_h \,\mathrm{d}x + \int_{\Gamma_N} [-\zeta c u + D\nabla c] \cdot n \bar{w}_h \,\mathrm{d}s.
\end{aligned}
$$

This, eq. (3.1), and eq. (3.2) imply eq. (3.27). □

Lemma 9. *Let $c \in L^2(0,T; H^k(\Omega))$ be the solution to eq. (3.1) such that $\partial_t c \in L^2(0,T; H^k(\Omega))$ and $c_0 \in H^k(\Omega)$. Then,*

$$
\|\zeta_c(t)\|_\Omega^2 + \int_0^t \|\!|\boldsymbol{\zeta}_c(s)|\!\|_c^2 \,\mathrm{d}s \le C h^{2(k-1)} \quad \forall t \in [0,T]. \tag{3.28}
$$

Proof. Subtracting eq. (3.27) from eq. (3.12), we have

$$
\int_\Omega \partial_t [c_h(t) - c(t)] w_h \,\mathrm{d}x + B_h(\boldsymbol{c}_h(t) - \boldsymbol{c}(t), \boldsymbol{w}_h) = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h.
$$

Splitting the error, we get

$$\sum_{K\in\mathcal{T}}\int_K \partial_t\zeta_c w_h \,\mathrm{d}x + B_h(\boldsymbol{\zeta}_c,\boldsymbol{w}_h) = \sum_{K\in\mathcal{T}}\int_K \partial_t\xi_c w_h \,\mathrm{d}x + B_h(\boldsymbol{\xi}_c,\boldsymbol{w}_h).$$

Setting $\boldsymbol{w}_h = \boldsymbol{\zeta}_c(t)$, using coercivity eq. (3.14),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|\zeta_c(t)\big\|_\Omega^2 + C|\!|\!|\boldsymbol{\zeta}_c(s)|\!|\!|_c^2 &\le \sum_{K\in\mathcal{T}}\int_K (\partial_t\zeta_c)\zeta_c\,\mathrm{d}x + B_h(\boldsymbol{\zeta}_c,\boldsymbol{\zeta}_c)\\ &= \sum_{K\in\mathcal{T}}\int_K (\partial_t\xi_c)\zeta_c\,\mathrm{d}x + B_h(\boldsymbol{\xi}_c,\boldsymbol{\zeta}_c).\end{aligned}$$

Integrating from 0 to t, where $0 \le t \le T$, we obtain

$$\begin{aligned}\frac{1}{2}\big\|\zeta_c(t)\big\|_\Omega^2 + C\int_0^t |\!|\!|\boldsymbol{\zeta}_c(s)|\!|\!|_c^2\,\mathrm{d}s \le \frac{1}{2}\big\|\zeta_c(0)\big\|_\Omega^2 + \int_0^t\sum_{K\in\mathcal{T}}\int_K \partial_t\xi_c(s)\zeta_c(s)\,\mathrm{d}x\,\mathrm{d}s\\ + \int_0^t B_h(\boldsymbol{\xi}_c(s),\boldsymbol{\zeta}_c(s))\,\mathrm{d}s =: I_1 + I_2 + I_3.\end{aligned}$$

Using eq. (3.21) and since $c_h(0)$ is the L^2-projection of c_0 into C_h,

$$I_1 = \frac{1}{2}\big\|\zeta_c(0)\big\|_\Omega^2 \le \|c_h(0) - c_0\|_\Omega^2 + \|c_0 - \mathcal{I}c_0\|_\Omega^2 \le Ch^{2k}\|c_0\|_{k,\Omega}^2.$$

Using eq. (3.21) and Cauchy–Schwarz and Young's inequalities,

$$\begin{aligned}I_2 &= \int_0^t\sum_{K\in\mathcal{T}}\int_K \partial_t\xi_c(s)\zeta_c(s)\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \big\|\partial_t\xi_c(s)\big\|_\Omega\big\|\zeta_c(s)\big\|_\Omega\,\mathrm{d}s\\ &\le \int_0^t Ch^k\|\partial_t c\|_{k,\Omega}\big\|\zeta_c(s)\big\|_\Omega\,\mathrm{d}s\\ &\le \frac{C}{2}h^{2k}\|\partial_t c\|_{L^2(0,t;H^k(\Omega))}^2 + \frac{C}{2}\int_0^t\big\|\zeta_c(s)\big\|_\Omega^2\,\mathrm{d}s.\end{aligned}$$

Next, by eq. (3.24) and Young's inequality,

$$\begin{aligned}I_3 &= \int_0^t B_h(\boldsymbol{\xi}_c(s),\boldsymbol{\zeta}_c(s))\,\mathrm{d}s \le C\int_0^t h^{k-1}\big\|c(s)\big\|_{k,\Omega}|\!|\!|\boldsymbol{\zeta}_c(s)|\!|\!|_c\,\mathrm{d}s\\ &\le Ch^{2(k-1)}\|c\|_{L^2(0,t;H^k(\Omega))}^2 + \delta\int_0^t |\!|\!|\boldsymbol{\zeta}_c(s)|\!|\!|_c^2\,\mathrm{d}s,\end{aligned}$$

with $\delta > 0$ a constant.

Choosing δ small enough and combining the above bounds, we obtain

$$\begin{aligned}
&\left\|\zeta_c(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{\zeta}_c(s)\right|\!\right\|_c^2 \,\mathrm{d}s \\
&\leq Ch^{2(k-1)}\Big(\|c_0\|_{k,\Omega}^2 + \|\partial_t c\|_{L^2(0,t;H^k(\Omega))}^2 + \|c\|_{L^2(0,t;H^k(\Omega))}^2\Big) + \int_0^t C\left\|\zeta_c(s)\right\|_{\Omega}^2 \,\mathrm{d}s.
\end{aligned}$$

The result eq. (3.28) follows by applying Gronwall's inequality eq. (C.4). □

Theorem 8. *Let $c \in L^2(0,T;H^k(\Omega))$ be the solution to eq.* (3.1) *such that $\partial_t c \in L^2(0,T;H^k(\Omega))$ and $c_0 \in H^k(\Omega)$. Then*

$$\left\|c(t) - c_h(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{c}(s) - \boldsymbol{c}_h(s)\right|\!\right\|_c^2 \,\mathrm{d}s \leq Ch^{2(k-1)} \quad \forall t \in [0,T].$$

Proof. Using eqs. (3.21) and (3.23), we obtain

$$\left\|\xi_c(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{\xi}_c(s)\right|\!\right\|_c^2 \leq Ch^{2k}\|c\|_{k,\Omega}^2 + Ch^{2(k-1)}\|c\|_{L^2(0,t;H^k(\Omega))}^2. \tag{3.29}$$

Noting that

$$\begin{aligned}
&\left\|c(t) - c_h(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{c}(s) - \boldsymbol{c}_h(s)\right|\!\right\|_c^2 \,\mathrm{d}s \\
&\leq 2\left(\left\|\xi_c(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{\xi}_c(s)\right|\!\right\|_c^2 \,\mathrm{d}s\right) + 2\left(\left\|\zeta_c(t)\right\|_{\Omega}^2 + \int_0^t \left\|\!\left|\boldsymbol{\zeta}_c(s)\right|\!\right\|_c^2 \,\mathrm{d}s\right),
\end{aligned}$$

the result follows by using eq. (3.28). □

It can be shown by using a similar analysis as in [15] that there exists a unique solution $\mathbf{c}_h \in \mathbf{C}_h$ to (3.12) and given that u, c and c_0 are sufficiently smooth,

$$\|c - c_h\|_{L^2(0,T;L^2(\Omega))} \leq Ch^{k+1},$$

for some constant $C > 0$ that is independent of the mesh size and where $k \geq 1$.

3.3.2 Numerical experiments

In this section, we use the NGSolve finite element library [49] to implement our method and apply it to demonstrate optimal convergence in space for different boundary conditions and in time using the backward Euler and Crank-Nicolson time-stepping schemes. Let $\Omega = [0,1]^2$ and $[0,1]$ be the time interval. We solve the transport equation with the exact solution given by

$$c(x,t) = \sin(2\pi(x-t))\cos(2\pi(y-t)).$$

For the transport equation with mixed boundary condition, we use
$\Gamma_D := \{x \in \partial\Omega \,:\, x_1 = 1 \text{ or } x_2 = 1\}$ and $\Gamma_N := \{x \in \partial\Omega \,:\, x_1 = 0 \text{ or } x_2 = 0\}$.

We also applied the second order Crank-Nicolson scheme for the time discretization given below:

$$\sum_{K\in\mathcal{T}} \int_K d_t c_h^{n+1} w_h \,\mathrm{d}x + B_h(\boldsymbol{c}_h^{n+1/2}, \boldsymbol{w}_h) = \sum_{K\in\mathcal{T}} \int_K f^{n+1/2} w_h \,\mathrm{d}x, \tag{3.30}$$

for all $\boldsymbol{w}_h \in \boldsymbol{C}_h$. Figure 3.1 shows the computed solution with $h = 1/128$,

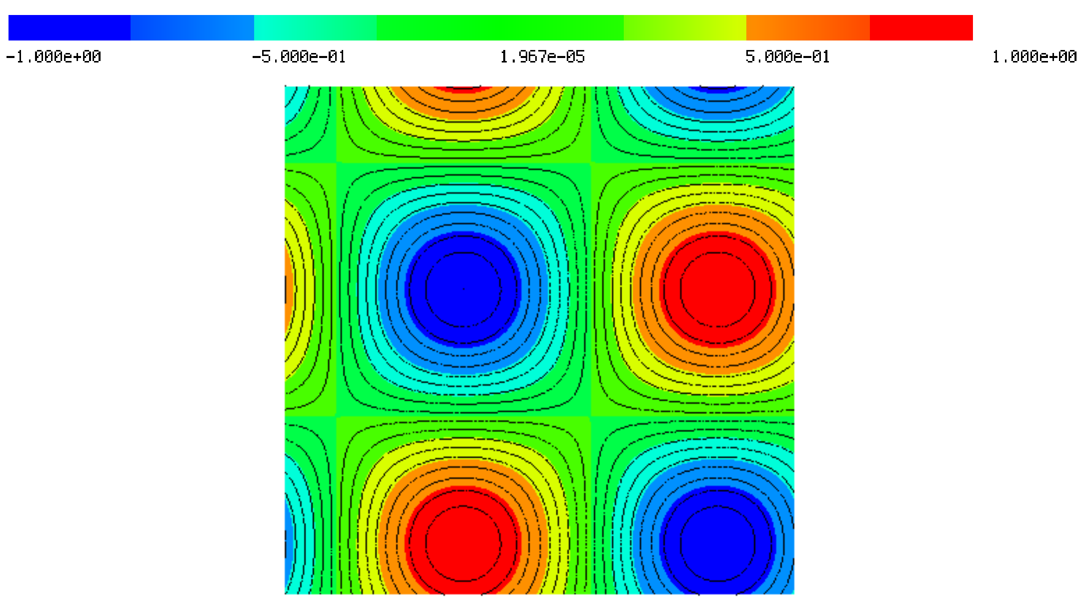


Figure 3.1: Solution at time $t = 0.1$ for the mixed boundary case with the Crank-Nicolson method using $h = 1/128$, $\Delta t = 0.0125$

Table 3.1: Table of convergence using the HDG method (3.13) for the transport problem with Dirichlet boundary condition ($\Gamma_N = \emptyset$) using polynomial degree $k = 2$, time step $\Delta t = h^3$.

h	DOFs	$\|\|c - c_h\|\|_\Omega$	Order
1/2	75	3.4e-01	–
1/4	381	2.3e-02	3.3
1/8	1560	1.9e-03	3.5
1/16	6501	2.1e-04	3.2
1/32	24993	2.6e-05	3.0

Table 3.2: Table of convergence using the HDG method (3.13) for the transport problem with Neumann boundary condition ($\Gamma_D = \emptyset$) using polynomial degree $k = 2$, time step $\Delta t = h^3$.

h	DOFs	$\|\|c - c_h\|\|_\Omega$	Order
1/2	75	2.9e-01	–
1/4	381	2.4e-02	3.6
1/8	1560	2.1e-03	3.5
1/16	6501	2.3e-04	3.2
1/32	24993	2.9e-05	3.0

$\Delta t = 0.0125$ to the transport problem with mixed boundary condition. The figures for the other cases look identical.

The observed spatial orders of convergence at final time $t = 1$ are presented in Tables 3.1 to 3.3. Optimal rate of convergence, which is three, for the concentration in each case of boundary conditions are observed as expected.

The convergence rates in time with the backward Euler and Crank-Nicolson schemes for time discretization are shown in Tables 3.4 and 3.5, respectively. The temporal convergence rate is observed to be 1 and 2 in Tables 3.4 and 3.5 as expected.

Table 3.3: Table of convergence using the HDG method (3.13) for the transport problem with the mixed boundary condition ($\Gamma_D, \Gamma_N \neq \emptyset$) using polynomial degree $k = 2$, time step $\Delta t = h^3$.

h	DOFs	$\|\|c - c_h\|\|_\Omega$	Order
1/2	75	2.8e-01	–
1/4	381	2.3e-02	3.6
1/8	1560	2.0e-03	3.5
1/16	6501	2.2e-04	3.2
1/32	24993	2.8e-05	3.0

Table 3.4: Table of temporal convergence using the HDG method (3.13) and the backward Euler method for the mixed boundary condition case using $k = 2$, $h = 1/128$ at time $t = 1$.

Δt	$\|\|c - c_h\|\|_\Omega$	Order
0.1	3.4e-02	–
0.05	1.9e-02	0.843
0.025	1.0e-02	0.915
0.0125	5.2e-03	0.955

Table 3.5: Table of temporal convergence by the Crank-Nicolson method for the mixed boundary condition case using $k = 2$, $h = 1/128$ at time $t = 1$.

Δt	$\|\|c - c_h\|\|_\Omega$	Order
0.1	1.1e-02	–
0.05	2.4e-03	2.2
0.025	5.8e-04	2.0
0.0125	1.4e-04	2.0

Table 3.6: Table of convergence using the HDG method (3.13) with the Crank-Nicolson method for the transport problem with the mixed boundary condition ($\Gamma_D, \Gamma_N \neq \emptyset$) using polynomial degree $k = 3$, time step $\Delta t = h^3$.

h	DOFs	$\|\|c - c_h\|\|_\Omega$	Order
1/2	112	2.7e-01	–
1/4	608	1.5e-03	4.2
1/8	2336	1.2e-04	3.6
1/16	9696	4.5e-06	4.8
1/32	38560	2.7e-07	4.1

Table 3.7: Table of temporal convergence by the Crank-Nicolson method for the mixed boundary condition case using $k = 3$, $h = 1/128$ at time $t = 1$.

Δt	$\|\|c - c_h\|\|_\Omega$	Order
0.1	1.1e-02	–
0.05	2.4e-03	2.2
0.025	5.8e-04	2.0
0.0125	1.4e-04	2.0

Tables 3.6 and 3.7 show our numerical results by using $k = 3$, and the Crank-Nicolson time-stepping. We observe optimal spatial (4) and temporal (2) convergence rates.

CHAPTER FOUR

TIME-DEPENDENT STOKES-DARCY-TRANSPORT PROBLEM

4.1 Stokes-Darcy-transport: fully coupled system

Let $\Omega \subset \mathbb{R}^{\dim}$, $\dim = 2, 3$ be the domain described in Section 1.4 consisting of two disjoint subdomains, Ω^s and Ω^d and $J = [0, T]$ be the time interval of interest.

The Stokes–Darcy–transport system for the velocity field $u : \Omega \times J \to \mathbb{R}^{\dim}$, pressure $p : \Omega \times J \to \mathbb{R}$ and concentration $c : \Omega \times J \to \mathbb{R}$ is given by

$$
\begin{aligned}
\partial_t u - \nabla \cdot (2\mu(c)\varepsilon(u)) + \nabla p &= f^s(c) && \text{in } \Omega^s \times J, && (4.1a)\\
\mathbb{K}^{-1}(c)u + \nabla p &= \mathbb{K}^{-1}(c) f^d(c) && \text{in } \Omega^d \times J, && (4.1b)\\
-\nabla \cdot u &= \chi_d(g_p - g_i) && \text{in } \Omega \times J, && (4.1c)\\
\phi \partial_t c + \nabla \cdot (cu - \widetilde{D}(u)\nabla c) &= \chi_d(c_I g_i - c g_p) && \text{in } \Omega \times J, && (4.1d)\\
u &= 0 && \text{on } \Gamma^s \times J, && (4.1e)\\
u \cdot n &= 0 && \text{on } \Gamma^d \times J, && (4.1f)\\
\widetilde{D}(u)\nabla c \cdot n &= 0 && \text{on } \partial\Omega \times J. && (4.1g)
\end{aligned}
$$

Compared to eq. (2.1), the Stokes-Darcy equations defined by eqs. (4.1a), (4.1c), (4.1e) and (4.1f) allow the fluid viscosity μ, and the body force terms f^s and f^d (which is now nonzero) to be concentration dependent. Furthermore, κ, the permeability of the porous medium is no longer assumed to be a constant but is a tensor and we set $\mathbb{K} = \frac{\kappa}{\mu}$. Compared to the advection diffusion equation eq. (3.1), eq. (4.1d) is now defined on a subdivided domain, instead of taking a constant divergence-free velocity field it now takes the unknown velocity determined by eqs. (4.1a), (4.1c), (4.1e) and (4.1f) as the convective velocity, the dispersion-diffusion tensor is now allowed to be a function of the unknown velocity

u, and ϕ, the porosity of the medium takes the value 1 in Ω^s and allowed to be between 0 and 1 in Ω^d. The tensor $\widetilde{D}(u)$ equals to $d\mathbb{I}$ in the Stokes region, where $\mathbb{I}$ is the identity matrix, and d is the diffusion coefficient in Ω^s and to $D(u)$ in the Darcy region where $D(u)$ denotes the diffusion dispersion tensor in Ω^d. All of these changes make the model strongly coupled and nonlinear and complicate the analysis.

The newly introduced functions g_i and g_p denote the source and sink terms related to injection and production wells and c_I is the injected concentration.

We will use u^j, p^j, and c^j, to denote the part of the velocity, pressure and concentration, respectively, in subdomain Ω^j, $j = s, d$.

For methods which do not satisfy mass conservation, skew-symmetry of the term $\nabla \cdot (cu)$ needs to be preserved in the discretization. This can be achieved by rewriting eq. (4.1d) in the following form as in [40, 47]:

$$\phi\partial_t c + \nabla \cdot (\tfrac{1}{2}cu - \widetilde{D}(u)\nabla c) + \tfrac{1}{2}u \cdot \nabla c + \tfrac{1}{2}\chi_d(g_i + g_p)c = \chi_d c_I g_i. \tag{4.2}$$

This way the term is skew-symmetric regardless of what $\nabla \cdot u$ is in Ω^d. The HDG scheme we present in this paper is mass conserving by construction, so we don't need to use this form.

On the interface Γ^I, we choose the unit normal vector, still denoted by n, to be the one pointing from Ω^s to Ω^d and prescribe the following interface conditions on $\Gamma^I \times J$:

$$u^s \cdot n = u^d \cdot n, \tag{4.3a}$$

$$p^s - (2\mu(c^s)\varepsilon(u^s)n) \cdot n = p^d, \tag{4.3b}$$

$$-2(\varepsilon(u^s)n) \cdot \tau^\ell = \gamma^\ell u^s \cdot \tau^\ell, \quad \ell = 1, \ldots, \dim - 1, \text{ on } \Gamma^I \times J, \tag{4.3c}$$

$$c^s = c^d, \tag{4.3d}$$

$$d\nabla c^s \cdot n = D(u^d)\nabla c^d \cdot n, \tag{4.3e}$$

where τ^ℓ, $\ell = 1, \ldots, \dim - 1$ denote the unit tangent vectors on Γ^I, $\gamma^\ell = \dfrac{\alpha}{\sqrt{\tau^\ell \cdot \kappa \tau^\ell}}$ and $\alpha > 0$ is a parameter determined by experimentation.

These conditions enforce continuity of the normal velocity (4.3a), normal component of the normal stress (4.3b), concentration (4.3d) and concentration flux (4.3e). Equation (4.3c), where $\alpha > 0$ is an experimentally determined parameter, is the called the Beavers–Joseph–Saffman law which enforces a condition on the tangential component of the normal stress [6, 48].

Finally, we assume the following initial conditions:

$$u^s(x,0) = u_0^s(x) \qquad \text{in } \Omega^s, \tag{4.4a}$$

$$c(x,0) = c_0(x) \qquad \text{in } \Omega. \tag{4.4b}$$

4.1.1 Assumptions

The dispersion-diffusion tensor $D(u)$ in Ω^d satisfies for $u, v \in \mathbb{R}^{\dim}$:

$$D_{\min}|\xi|^2 \le \xi^T D(u)\xi \quad \forall \xi \in \mathbb{R}^{\dim}, \tag{4.5a}$$

$$|D(u)| \le C(1 + |u|), \tag{4.5b}$$

$$|D(u) - D(v)| \le C|u - v|, \tag{4.5c}$$

$$D(0) = D_0, \tag{4.5d}$$

where $D_{\min}$, D_0, and C are positive constants and $|\cdot|$ denotes the Euclidean norm. We assume that $\phi \in L^\infty(\Omega)$ and $\mu(c)$ is bounded, monotone and Lipschitz continuous with Lipschitz constant μ_L for all $c \in \mathbb{R}$, and there exists $\phi_*, \phi^*, \mu_*, \mu^* > 0$ such that

$$\phi_* \le \phi(x) \le \phi^* \qquad \forall x \in \Omega, \tag{4.6a}$$

$$\mu_* \le \mu(c) \le \mu^* \qquad \forall c \in \mathbb{R}. \tag{4.6b}$$

Note also that in Ω^s, $\phi = 1$. The permeability matrix κ is symmetric, uniformly bounded and elliptic, that is, there exists positive constants $\kappa_* < \kappa^*$ such that

$$\kappa_*|\xi|^2 \le \xi^T \kappa(x)\xi \le \kappa^*|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega^d \cup \Gamma^I. \tag{4.7}$$

From (4.6b) and (4.7), we deduce that

$$K_*|\xi|^2 \le \xi^T \mathbb{K}(c,x)\xi \le K^*|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \quad \forall (c,x) \in [0,1] \times \Omega^d \cup \Gamma^I, \tag{4.8}$$

where $K_* = \dfrac{\kappa_*}{\mu^*}$ and $K^* = \dfrac{\kappa^*}{\mu_*}$. The body force functions f^s and f^d are assumed to be Lipschitz continuous in c with Lipschitz constants L_f^s and L_f^d. Note that f^s and f^d depend on x and c but do not depend explicitly on t. Finally we assume that $g_i, g_p \ge 0$, $g_i, g_p \in L^\infty(I; L^2(\Omega^d))$ and

$$\int_{\Omega^d} (g_i(x,t) - g_p(x,t))\, \mathrm{d}x = 0 \quad \forall t \in J,$$

and the injected concentration satisfies $0 \le c_I \le 1$ a.e. in Ω^d.

A weak formulation of the problem defined by (4.1), (4.3), (4.4) was presented in [47] and the analysis for a weak solution of a more complicated version of the problem where the free fluid flow is governed by the Navier-Stokes equations can be found in [17].

4.2 The numerical method

To avoid repetition, we point out that we reuse the notation from Section 2.3 to describe the mesh on Ω. The continuous and discrete function spaces and their associated norms and semi-norms also carry over from Section 2.3. We only update $k = k_f$ and the norm eq. (2.10) on $\mathbf{V}(h)$ slightly to accommodate for the tensor structure of the permeability κ. Here define

$$|||\mathbf{v}|||_v^2 := |||\mathbf{v}|||_{v,s}^2 + \|v\|_{\Omega^d}^2 + \sum_{j=1}^{\dim} \gamma^j \|\bar{v} \cdot \tau_j\|_{\Gamma^I}^2, \tag{4.9}$$

The discontinuous Galerkin finite element function spaces on Ω for the concentration and its trace are

$$\begin{aligned}
C_h &= \{c_h \in L^2(\Omega):\ c_h \in P_{k_c}(K),\ \forall\ K \in \mathcal{T}\}\,,\\
C_h^s &= \{c_h \in L^2(\Omega^s):\ c_h \in P_{k_c}(K),\ \forall\ K \in \mathcal{T}^s\}\,,\\
\bar{C}_h &= \{\bar{c}_h \in L^2(\Gamma_0):\ \bar{c}_h \in P_{k_c}(F)\ \forall\ F \in \mathcal{F}\}\,,\\
\bar{C}_h^s &= \{\bar{c}_h \in L^2(\Gamma_0^s):\ \bar{c}_h \in P_{k_c}(F)\ \forall\ F \in \mathcal{F}^s\}\,,
\end{aligned} \tag{4.10}$$

where $k_c = k_f - 1, k_f$. Let us briefly discuss the choice for k_c. We recall that for the flow and transport schemes to be compatible in the sense of [23], the approximate velocity field must be globally conservative and the method should be zeroth-order accurate, that is, it should preserve the constant concentration solution and this is crucial to avoid instabilities. The choice $k_f = k_c - 1$ was shown to produce a compatible scheme for the one-way coupled Stokes-Darcy-transport problem in [15]. Therefore, in our numerical experiments we will use $k_c = k_f - 1$.

We introduce $\boldsymbol{C}_h = C_h \times \bar{C}_h$, $C(h) := C_h + H^1(\Omega)$ and $\boldsymbol{C}_h^s = C_h^s \times \bar{C}_h^s$ and we use $\boldsymbol{c}_h = (c_h, \bar{c}_h)$ to denote elements in $\boldsymbol{C}_h$ or in $\boldsymbol{C}_h^s$. We define the following semi-norms on $\boldsymbol{C}_h$ and $\boldsymbol{C}_h^s$:

$$\begin{aligned}
|||\boldsymbol{w}_h|||_c^2 &= \sum_{K\in\mathcal{T}} (\|\nabla w_h\|_K^2 + h_K^{-1}\|w_h - \bar{w}_h\|_{\partial K}^2)\\
|||\boldsymbol{w}_h|||_{c,s}^2 &= \sum_{K\in\mathcal{T}^s} (\|\nabla w_h\|_K^2 + h_K^{-1}\|w_h - \bar{w}_h\|_{\partial K}^2).
\end{aligned}$$

Note that $|||\boldsymbol{w}_h|||_c$ is the same as in Section 3.3. The semi-discrete form for the Stokes–Darcy part of the problem in eqs. (4.1) and (4.3) is as follows: for $t > 0$, find $((\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)), \boldsymbol{c}_h(t)) \in \boldsymbol{X}_h \times \boldsymbol{C}_h$ such that

$$\begin{aligned}
\sum_{K\in\mathcal{T}^s} \int_K \partial_t u_h \cdot v_h \,\mathrm{d}x + B_h^{sd}(\boldsymbol{c}_h; (\boldsymbol{u}_h, \boldsymbol{p}_h), (\boldsymbol{v}_h, \boldsymbol{q}_h)) &= \int_{\Omega^s} f^s(c_h)\cdot v_h \,\mathrm{d}x \\
+ \int_{\Omega^d} \mathbb{K}^{-1}(c_h) f^d(c_h)\cdot v_h \,\mathrm{d}x &+ \sum_{K\in\mathcal{T}^d} \int_K (g_p - g_i)\, q_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h,
\end{aligned} \tag{4.11}$$

$$\sum_{K\in\mathcal{T}}\int_K \phi\,\partial_t c_h(t) w_h \,\mathrm{d}x + B_h^{tr}(u_h;\boldsymbol{c}_h,\boldsymbol{w}_h) + \int_{\Omega^d} g_p c_h w_h \,\mathrm{d}x$$
$$= \sum_{K\in\mathcal{T}^d}\int_K c_I\, g_i\, w_h \,\mathrm{d}x \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h, \qquad (4.12)$$

where the bilinear form B_h^{sd} in eq. (4.11) takes care of the discretization of the Stokes-Darcy momentum and mass conservation equations and is defined as follows:

$$B_h^{sd}(\boldsymbol{c}_h;(\boldsymbol{u}_h,\boldsymbol{p}_h),(\boldsymbol{v}_h,\boldsymbol{q}_h)) := a_h(\boldsymbol{c}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) + \sum_{j=s,d}\big(b_h^j(\boldsymbol{p}_h^j,v_h) + b_h^{I,j}(\bar{p}_h^j,\bar{v}_h)\big)$$
$$+ \sum_{j=s,d}\big(b_h^j(\boldsymbol{q}_h^j,u_h) + b_h^{I,j}(\bar{q}_h^j,\bar{u}_h)\big). \quad (4.13)$$

Note that B_h^{sd} is the same form as in [16] except that here μ depends on the approximate concentration wherever it appears. Therefore bilinear form $a_h(\cdot,\cdot)$ in eq. (4.13) is defined as

$$a_h(\boldsymbol{c};\boldsymbol{u},\boldsymbol{v}) := a_h^s(c;\boldsymbol{u},\boldsymbol{v}) + a_h^d(c;u,v) + a_h^I(c;\bar{u},\bar{v}), \qquad (4.14)$$

where

$$a_h^s(c;\boldsymbol{u},\boldsymbol{v}) := \sum_{K\in\mathcal{T}^s}\int_K 2\mu(c)\varepsilon(u):\varepsilon(v)\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta_s\mu(c)}{h_K}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \qquad (4.15a)$$
$$- \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot(v-\bar{v})\,\mathrm{d}s - \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu(c)\varepsilon(v)n^s\cdot(u-\bar{u})\,\mathrm{d}s,$$
$$a_h^d(c;u,v) := \int_{\Omega^d}\mathbb{K}^{-1}(c)u\cdot v\,\mathrm{d}x, \qquad (4.15b)$$
$$a_h^I(\bar{c};\bar{u},\bar{v}) := \sum_{j=1}^{\mathrm{dim}-1}\int_{\Gamma^I}\gamma^j\mu(\bar{c})(\bar{u}\cdot\tau^j)(\bar{v}\cdot\tau^j)\,\mathrm{d}s, \qquad (4.15c)$$

and where $\beta_s > 0$ is a penalty parameter. The bilinear forms $b_h^j(\cdot,\cdot)$ and $b_h^{I,j}(\cdot,\cdot)$ in eq. (4.13) are as defined in eq. (2.25). The term $B_h^{tr}(u;\boldsymbol{c}(t),\boldsymbol{w})$ in eq. (4.12) consists of the discretization of the advective and diffusive parts of the transport equation, that is,

$$B_h^{tr}(u(t);\boldsymbol{c}(t),\boldsymbol{w}) = B_h^a(u(t);\boldsymbol{c}(t),\boldsymbol{w}) + B_h^d(u(t);\boldsymbol{c}(t),\boldsymbol{w}), \tag{4.16}$$

where the advective part B_h^a is defined as in eq. (3.9), but the diffusive part B_h^d needs to modified to accommodate for the dependence of the dispersion/diffusion tensor D on the velocity u as follows:

$$\begin{aligned} B_h^d(u;\boldsymbol{c},\boldsymbol{w}) = \sum_{K\in\mathcal{T}} \int_K \tilde{D}(u)\nabla c\cdot\nabla w \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} [\tilde{D}(u)\nabla c]\cdot n(w-\bar{w}) \,\mathrm{d}s \\ + \sum_{K\in\mathcal{T}} \frac{\beta_{tr}}{h_K} \int_{\partial K} [\tilde{D}(u)n](c-\bar{c})(w-\bar{w})n \,\mathrm{d}s \\ - \sum_{K\in\mathcal{T}} \int_{\partial K} [\tilde{D}(u)\nabla w]\cdot n(c-\bar{c}) \,\mathrm{d}s. \end{aligned} \tag{4.17}$$

Here $\beta_{tr} > 0$ is a penalty parameter. To complete the discretization, we impose the initial conditions by appropriate projections of u_0 and c_0 into V_h and C_h, respectively.

Lemma 10 (Consistency)**.** *If $c \in L^2(0,T;H^2(\Omega))$, $u \in L^2(0,T;V)$ and $p \in L^2(0,T;Q)$ solve the Stokes–Darcy–transport system eqs.* (4.1) *and* (4.3)*, then letting $\boldsymbol{u} = (u,\gamma(u))$ and $\boldsymbol{p} = (p,\gamma(p^s),\gamma(p^d))$,*

$$\begin{aligned} \sum_{K\in\mathcal{T}^s} \int_K \partial_t u\cdot v \,\mathrm{d}x + B_h^{sd}(\boldsymbol{c};(\boldsymbol{u},\boldsymbol{p}),(\boldsymbol{v},\boldsymbol{q})) = \int_{\Omega^s} f^s(c)\cdot v \,\mathrm{d}x \\ + \int_{\Omega^d} \mathbb{K}^{-1}(c) f^d(c)\cdot v \,\mathrm{d}x + \sum_{K\in\mathcal{T}^d} \int_K (g_p - g_i) q \,\mathrm{d}x \quad \forall(\boldsymbol{v},\boldsymbol{q}) \in \boldsymbol{X}(h). \end{aligned} \tag{4.18}$$

$$\begin{aligned} \sum_{K\in\mathcal{T}} \int_K \phi\,\partial_t c(t) w \,\mathrm{d}x + B_h^{tr}(u;\boldsymbol{c},\boldsymbol{w}) + \int_{\Omega^d} g_p c w \,\mathrm{d}x \\ = \sum_{K\in\mathcal{T}^d} \int_K c_I\, g_i\, w \,\mathrm{d}x \quad \forall \boldsymbol{w} \in \boldsymbol{C}_h. \end{aligned} \tag{4.19}$$

where $\boldsymbol{c} = (c, \bar{c})$ *with* $\bar{c}$ *the restriction of* c *to* Γ^0.

Proof. Integrating by parts and noting that $u = \gamma(u)$ on cell boundaries in $\overline{\Omega}^s$, we have

$$\begin{aligned} a_h^s(c;\boldsymbol{u},\boldsymbol{v}) &= \sum_{K\in\mathcal{T}^s}\int_K 2\mu(c)\varepsilon(u):\varepsilon(v)\,\mathrm{d}x - \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot(v-\bar{v})\,\mathrm{d}s \\ &= -\sum_{K\in\mathcal{T}^s}\int_K \nabla\cdot(2\mu(c)\varepsilon(u))\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot\bar{v}\,\mathrm{d}s. \end{aligned} \tag{4.20}$$

Using smoothness of u and c, continuity of μ, single-valuedness of $\bar{v}$, $\bar{v} = 0$ on Γ^s, $v = (v\cdot n)n + \sum_{j=1}^{\dim-1}(v\cdot\tau^j)\tau^j$ and eq. (4.3c), we note that

$$\begin{aligned} \sum_{K\in\mathcal{T}^s}\int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot\bar{v}\,\mathrm{d}s &= \int_{\Gamma^I} 2\mu(c)\varepsilon(u)n^s\cdot\bar{v}\,\mathrm{d}s \\ &= \int_{\Gamma^I}\mu(c)((2\varepsilon(u)n^s\cdot n)n + \sum_{j=1}^{\dim-1}(2\varepsilon(u)n^s\cdot\tau^j)\tau^j)\cdot\bar{v}\,\mathrm{d}s \\ &= -\sum_{j=1}^{\dim-1}\int_{\Gamma^I}\gamma^j\mu(c)(u\cdot\tau^j)(\bar{v}\cdot\tau^j)\,\mathrm{d}s + \int_{\Gamma^I}(n^s\cdot 2\mu(c)\varepsilon(u)n^s)n^s\cdot\bar{v}\,\mathrm{d}s. \end{aligned} \tag{4.21}$$

Plugging eq. (4.21) into eq. (4.20),

$$\begin{aligned} a_h^s(c;\boldsymbol{u},\boldsymbol{v}) = &-\sum_{K\in\mathcal{T}^s}\int_K\nabla\cdot(2\mu(c)\varepsilon(u))\cdot v\,\mathrm{d}x - \sum_{j=1}^{\dim-1}\int_{\Gamma^I}\gamma^j\mu(c)(\bar{u}\cdot\tau^j)(\bar{v}\cdot\tau^j)\,\mathrm{d}s \\ &+ \int_{\Gamma^I}(n^s\cdot 2\mu(c)\varepsilon(u)n^s)n^s\cdot\bar{v}\,\mathrm{d}s. \end{aligned} \tag{4.22}$$

Next, integrating by parts and noting that $\gamma(p^j) = p^j$ on cell boundaries in $\overline{\Omega}^j$, with $j = s, d$, and $n^d = -n^s$ on Γ^I, we get

$$\sum_{j=s,d}(b_h^j(\boldsymbol{p}^j, v) + b_h^{I,j}(\gamma(p^j),\bar{v})) = \sum_{K\in\mathcal{T}}\int_K\nabla p\cdot v\,\mathrm{d}x + \int_{\Gamma^I}(p^d - p^s)\bar{v}\cdot n^s\,\mathrm{d}s. \tag{4.23}$$

Since $\bar{q}^s$, $\bar{q}^d$, $u^s \cdot n^s$ and $u^d \cdot n^d$ are single-valued on interior facets, and $u^s \cdot n^s = -u^d \cdot n^d$,

$$\begin{aligned}
&\sum_{j=s,d} (b_h^j(\boldsymbol{q}^j, u) + b_h^{I,j}(\bar{q}^j, \gamma(u^s))) \qquad (4.24)\\
&= -\sum_{K\in\mathcal{T}^s}\int_K q\nabla\cdot u \,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K} \bar{q}^s u\cdot n^s \,\mathrm{d}s - \int_{\Gamma^I} \bar{q}^s u^s\cdot n^s \,\mathrm{d}s\\
&\quad -\sum_{K\in\mathcal{T}^d}\int_K q\nabla\cdot u \,\mathrm{d}x + \sum_{K\in\mathcal{T}^d}\int_{\partial K} \bar{q}^d u\cdot n^d \,\mathrm{d}s - \int_{\Gamma^I} \bar{q}^d u^d\cdot n^d \,\mathrm{d}s\\
&= \sum_{K\in\mathcal{T}^d}\int_K (g_p - g_i) q \,\mathrm{d}x. \qquad (4.25)
\end{aligned}$$

Next, we have

$$a_h^d(c; u, v) = \int_{\Omega^d} \mathbb{K}^{-1}(c) u\cdot v \,\mathrm{d}x, \qquad (4.26)$$

and

$$a_h^I(\bar{c}; \gamma(u^s), \bar{v}) = \sum_{j=1}^{\dim-1}\int_{\Gamma^I} \gamma^j \mu(c)(\bar{u}\cdot\tau^j)(\bar{v}\cdot\tau^j) \,\mathrm{d}s. \qquad (4.27)$$

Therefore, combining equations eq. (4.20)-eq. (4.27),

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^s}\int_{\partial K} \partial_t u\cdot v \,\mathrm{d}x + B_h^{sd}(\boldsymbol{c}; (\boldsymbol{u}, \boldsymbol{p}), (\boldsymbol{v}, \boldsymbol{q}))\\
&\qquad = \sum_{K\in\mathcal{T}^s}\int_K (\partial_t u - 2\nabla\cdot(\mu(c)\varepsilon(u)) + \nabla p)\cdot v \,\mathrm{d}x\\
&\qquad\quad + \sum_{K\in\mathcal{T}^d}\int_K (\mathbb{K}^{-1}(c) u + \nabla p)\cdot v \,\mathrm{d}x\\
&+ \int_{\Gamma^I} (2\mu(n^s\cdot\varepsilon(u) n^s) n^s)\cdot\bar{v} \,\mathrm{d}s + \int_{\Gamma^I} (p^d - p^s) n^s\cdot\bar{v} \,\mathrm{d}s + \sum_{K\in\mathcal{T}^d}\int_K (g_p - g_i) q \,\mathrm{d}x.
\end{aligned}$$

Using eqs. (4.1a), (4.1b) and (4.3b) in the equation above, we obtain eq. (4.18).

Since c and u are smooth, $\bar{w}$ is single valued, $u\cdot n = 0$ on $\partial\Omega$, $c = \bar{c}$ on ∂K, using the interface conditions eq. (4.3d), and eq. (4.3a), the advective term simplifies to

$$
\begin{aligned}
B_h^a(u;\boldsymbol{c},\boldsymbol{w}) &= -\sum_{K\in\mathcal{T}}\int_K c\,u\cdot\nabla w \,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K} c\,u\cdot n(w-\bar{w})\,\mathrm{d}s \\
&\quad - \sum_{K\in\mathcal{T}}\int_{\partial K^{\mathrm{in}}} u\cdot n\,(c-\bar{c})(w-\bar{w})\,\mathrm{d}s \\
&= -\sum_{K\in\mathcal{T}}\int_K c\,u\cdot\nabla w \,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K} c\,u\cdot n(w-\bar{w})\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}}\int_K \nabla\cdot(c\,u)w\,\mathrm{d}x - \sum_{K\in\mathcal{T}}\int_{\partial K}(c\,u)\cdot nw\,\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}}\int_{\partial K} c\,u\cdot n(w-\bar{w})\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}}\int_K \nabla\cdot(c\,u)w\,\mathrm{d}x,
\end{aligned}
$$

and using eq. (4.3e), the diffusive part simplifies to

$$
\begin{aligned}
B_h^d(u;\boldsymbol{c},\boldsymbol{w}) &= \sum_{K\in\mathcal{T}}\int_K \tilde{D}(u)\nabla c\cdot\nabla w\,\mathrm{d}x - \sum_{K\in\mathcal{T}}\int_{\partial K}[\tilde{D}(u)\nabla c]\cdot n(w-\bar{w})\,\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}}\frac{\beta_{tr}}{h_K}\int_{\partial K}[\tilde{D}(u)n](c-\bar{c})(w-\bar{w})n\,\mathrm{d}s - \sum_{K\in\mathcal{T}}\int_{\partial K}[\tilde{D}(u)\nabla w]\cdot n(c-\bar{c})\,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}}\int_K \tilde{D}(u)\nabla c\cdot\nabla w\,\mathrm{d}x - \sum_{K\in\mathcal{T}}\int_{\partial K}[\tilde{D}(u)\nabla c]\cdot n(w-\bar{w})\,\mathrm{d}s \\
&= -\sum_{K\in\mathcal{T}}\int_K \nabla\cdot(\tilde{D}(u)\nabla c)w\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K}[\tilde{D}(u)\nabla c]\cdot nw\,\mathrm{d}s \\
&\quad - \sum_{K\in\mathcal{T}}\int_{\partial K}[\tilde{D}(u)\nabla c]\cdot n(w-\bar{w})\,\mathrm{d}s \\
&= -\sum_{K\in\mathcal{T}}\int_K \nabla\cdot(\tilde{D}(u)\nabla c)w\,\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
B_h^{\mathrm{tr}}(u;\boldsymbol{c},\boldsymbol{w}) = \sum_{K\in\mathcal{T}}\int_K \nabla\cdot(c\,u - \tilde{D}(u)\nabla c)w\,\mathrm{d}x. \tag{4.28}
$$

This and eq. (4.1d) imply (4.19). □

Let $K \in \mathcal{T}^s$ and suppose that $u_h = \bar{u}_h$ and $p_h = \bar{p}_h^s$ weakly on ∂K. Let $\bar{v}_h = 0$, $\boldsymbol{q}_h = 0$ and set $\boldsymbol{v}_h = 0$ in $\mathcal{T}\backslash K$. Then by integration by parts, we have

$$\begin{aligned}
B_h^{sd}(\boldsymbol{c}_h;(\mathbf{u}_h, \mathbf{p}_h), (\mathbf{v}_h, \mathbf{q}_h)) &= a_h^s(c_h; \mathbf{u}_h, \mathbf{v}_h) + b_h^s(\mathbf{p}_h^s, v_h) \\
&= \int_K 2\mu(c_h)\epsilon(u_h) : \epsilon(v_h)\ \mathrm{d}x - \int_{\partial K} 2\mu(c_h)\varepsilon(u_h)n^s \cdot v_h\,\mathrm{d}s \\
&\quad - \int_K p_h \nabla \cdot v_h\ \mathrm{d}x + \int_{\partial K} \bar{p}_h^s n \cdot v_h\ \mathrm{d}s \\
&= -\int_K \nabla \cdot (2\mu(c_h)\epsilon(u_h)) \cdot v_h\ \mathrm{d}x + \int_K \nabla p_h \cdot v_h\ \mathrm{d}x.
\end{aligned}$$

Thus, eq. (4.11) implies

$$\begin{aligned}
\int_K \partial_t u_h \cdot v_h\,\mathrm{d}x - \int_K \nabla \cdot (2\mu(c_h)\epsilon(u_h)) \cdot v_h\ \mathrm{d}x + \int_K \nabla p_h \cdot v_h\ \mathrm{d}x \\
= \int_K f^s(c_h) \cdot v_h\,\mathrm{d}x
\end{aligned}$$

for all $v_h \in V_h$.

It follows that eq. (4.1a) holds cell-wise.

Next, let $K \in \mathcal{T}^d$ and suppose that $p_h = \bar{p}_h^d$ weakly on ∂K. This time we let $\bar{v}_h = 0$, $\boldsymbol{q}_h = 0$ and $v_h = 0$ in $\mathcal{T}\backslash K$. Then by using integration by parts,

$$\begin{aligned}
B_h^{sd}(\boldsymbol{c}_h;(\mathbf{u}_h, \mathbf{p}_h), (\mathbf{v}_h, \mathbf{q}_h)) &= a_h^d(c_h; u_h, v_h) + b_h^s(\mathbf{p}_h^s, v_h) \\
&= \int_K \mathbb{K}^{-1}(c_h)u_h \cdot v_h\ \mathrm{d}x - \int_K p_h \nabla \cdot v_h\ \mathrm{d}x + \int_{\partial K} \bar{p}_h^d n^d \cdot v_h\ \mathrm{d}s \\
&= \int_K (\mathbb{K}^{-1}(c_h)u_h + \nabla p_h) \cdot v_h\ \mathrm{d}x.
\end{aligned}$$

Hence, eq. (4.11) implies

$$\int_K (\mathbb{K}^{-1}(c_h)u_h + \nabla p_h) \cdot v_h\ \mathrm{d}x = \int_K \mathbb{K}^{-1}(c_h)f^d(c_h) \cdot v_h\ \mathrm{d}x. \tag{4.29}$$

It follows that eq. (4.1b) holds cell-wise.

Setting $\boldsymbol{v}_h = 0$, $q_h = 0$ and $\bar{q}_h^s = 0$ in eq. (4.11), we have

$$\sum_{K\in\mathcal{T}^d} \int_{\partial K} \bar{q}_h^d u_h \cdot n^d \; \mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^d \bar{u}_h \cdot n^d \; \mathrm{d}s = 0 \tag{4.30}$$

for all $\bar{q}_h^d \in \bar{Q}_h^d$. Then,

$$\sum_{K\in\mathcal{T}^d\backslash\Gamma^I} \int_{\partial K} \bar{q}_h^d u_h \cdot n^d \; \mathrm{d}s = \sum_{F\in\mathcal{F}^d\backslash\mathcal{F}^I} \int_{F} \bar{q}_h^d [\![u_h \cdot n]\!] \; \mathrm{d}s = 0. \tag{4.31}$$

Since $[\![u_h \cdot n]\!] \in P_k(F)$ for each $F \in \mathcal{F}^d$, choosing $\bar{q}_h^d = [\![u_h \cdot n]\!]$, we get

$$[\![u_h \cdot n]\!] = 0 \text{ for all } F \in \mathcal{F}^d\backslash\mathcal{F}^I.$$

Similarly, we have

$$[\![u_h \cdot n]\!] = 0 \text{ for all } F \in \mathcal{F}^s\backslash\mathcal{F}^I \tag{4.32}$$

by letting $\boldsymbol{v}_h = 0$, $q_h = 0$ and $\bar{q}_h^d = 0$ in eq. (4.11).

Since $[\![u_h \cdot n]\!] = 0$ for all $F \in \mathcal{F}^j\backslash\mathcal{F}^I$, $j = s$, d, letting $\boldsymbol{v}_h = 0$, $q_h = 0$, and $\bar{q}_h^d = 0$ in eq. (4.11) again, we obtain

$$\sum_{j=s,d} \left(b_h^j(\boldsymbol{q}_h^j, u_h) + b_h^{I,j}(\bar{q}_h^j, \bar{u}_h)\right) = \int_{\Gamma^I} \bar{q}_h^s u_h \cdot n^s \; \mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^s \bar{u}_h \cdot n^s \; \mathrm{d}s = 0 \tag{4.33}$$

for all $\bar{q}_h^s \in \bar{Q}_h^s$.

Since $(u_h - \bar{u}_h) \cdot n \in P_k(F)$ for each $F \in \mathcal{F}^I$, choosing $\bar{q}_h^s = (u_h - \bar{u}_h) \cdot n$, we get

$$u_h \cdot n = \bar{u}_h \cdot n \;\text{ on }\; \Gamma^I. \tag{4.34}$$

Therefore,

$$[\![u_h \cdot n]\!] = 0 \qquad \forall x \in F, \quad \forall F \in \mathcal{F}, \tag{4.35a}$$

$$u_h \cdot n = \bar{u}_h \cdot n \qquad \forall x \in F, \quad \forall F \in \mathcal{F}^I. \tag{4.35b}$$

Setting $v_h = 0$ and $\boldsymbol{q}_h = 0$ in eq. (4.11), we have

$$
\begin{aligned}
&-\sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta_s\mu(c_h)}{h_K}(u_h^s-\bar{u}_h^s)\cdot\bar{v}_h\,\mathrm{d}s+\sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu(c_h)\varepsilon(u_h)n^s\cdot\bar{v}_h\,\mathrm{d}s\\
&+\sum_{j=1}^{\dim-1}\int_{\Gamma^I}\gamma^j\mu(\bar{c}_h)(\bar{u}_h\cdot\tau^j)(\bar{v}_h\cdot\tau^j)\,\mathrm{d}s-\int_{\Gamma^I}\bar{p}_h^s\bar{v}_h\cdot n^s\ \mathrm{d}s-\int_{\Gamma^I}\bar{p}_h^d\bar{v}_h\cdot n^d\ \mathrm{d}s=0.
\end{aligned}
\tag{4.36}
$$

This implies, by the single valuedness of $\bar{p}_h^s$ and $\bar{v}_h$ and since $\bar{v}_h = 0$ on Γ^s, that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^s}\int_{\partial K}\big(2\mu(c_h)\varepsilon(u_h)-\bar{p}_h^s\mathbb{I}-\frac{2\beta_s\mu(c_h)}{h_K}(u_h^s-\bar{u}_h^s)\otimes n\big)n\cdot\bar{v}_h\,\mathrm{d}s\\
&\qquad+\sum_{j=1}^{\dim-1}\int_{\Gamma^I}\gamma^j\mu(\bar{c}_h)(\bar{u}_h\cdot\tau^j)(\bar{v}_h\cdot\tau^j)\,\mathrm{d}s-\int_{\Gamma^I}\bar{p}_h^d\bar{v}_h\cdot n^d\ \mathrm{d}s=0.
\end{aligned}
\tag{4.37}
$$

Assuming that $-2(\varepsilon(u_h^s)n)\cdot\tau^j = \gamma^j\bar{u}_h\cdot\tau^j$, $j = 1,\dots\dim-1$ and $\bar{p}_h^s - n\cdot 2\mu(\bar{c}_h^s)\varepsilon(u_h^s)n = \bar{p}_h^d$ weakly on Γ^I,

$$
\begin{aligned}
&\sum_{j=1}^{\dim-1}\int_{\Gamma^I}\gamma^j\mu(\bar{c}_h)(\bar{u}_h\cdot\tau^j)(\bar{v}_h\cdot\tau^j)\,\mathrm{d}s-\int_{\Gamma^I}\bar{p}_h^d\bar{v}_h\cdot n^d\ \mathrm{d}s\\
&\qquad=\int_{\Gamma^I}(-2\mu(\bar{c}_h)\varepsilon(u_h^s)+\bar{p}_h^s\mathbb{I})n^s\cdot\bar{v}_h\ \mathrm{d}s
\end{aligned}
$$

This shows that the formulation imposes normal continuity weakly across facets of the 'numerical' Stokes stress tensor:

$$
\hat{\sigma}_h^s :=\begin{cases}2\mu(c_h)\varepsilon(u_h^s)-\bar{p}_h^s\mathbb{I}-\frac{2\beta_s\mu(c_h)}{h_K}(u_h^s-\bar{u}_h^s)\otimes n & \text{on } \Gamma_0^s\backslash\Gamma^I,\\ 2\mu(\bar{c}_h)\varepsilon(u_h^s)-\bar{p}_h^s\mathbb{I}-\frac{2\beta_s\mu(c_h)}{h_K}(u_h^s-\bar{u}_h^s)\otimes n & \text{on } \Gamma^I,\end{cases}
\tag{4.38}
$$

where $\mathbb{I}$ is the identity tensor, that is,

$$
\sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{v}\cdot\hat{\sigma}_h^s n^s\ \mathrm{d}s=\int_{\Gamma^I}\bar{v}\cdot\sigma^s n^s\ \mathrm{d}s,
$$

where $\sigma^s = 2\mu(c_h)\varepsilon(u_h^s) - p_h^s\mathbb{I}$.

To show pointwise mass conservation, we set $\boldsymbol{v}_h = 0$ and $\bar{q}_h^s = \bar{q}_h^d = 0$. Then, we have

$$B_h^{sd}(c_h; (\mathbf{u}_h, \mathbf{p}), (\mathbf{v}_h, \mathbf{q})) = \sum_{j=s,d} b_h^j(\boldsymbol{q}_h^j, u_h) = \sum_{j=s,d} \sum_{K \in \mathcal{T}^j} \int_K q_h \nabla \cdot u_h \, \mathrm{d}x.$$

Hence, by eq. (4.11),

$$\sum_{K \in \mathcal{T}} \int_K q_h \nabla \cdot u_h \, \mathrm{d}x = \sum_{K \in \mathcal{T}^d} \int_K (g_p - g_i)\, q_h \, \mathrm{d}x. \tag{4.39}$$

Since $\nabla \cdot u_h \in P_{k-1}(K)$ for each $K \in \mathcal{T}$, by choosing
$q_h = \chi^K(\Pi_Q(g_i - g_p) - \nabla \cdot u_h)$ where χ^K is the characteristic function of K, the numerical scheme imposes pointwise mass conservation, i.e.,

$$-\nabla \cdot u_h = \chi^d \Pi_Q(g_p - g_i) \quad \forall x \in K, \ \forall K \in \mathcal{T}, \tag{4.40}$$

where Π_Q is the standard L^2-projection operator onto Q_h.

4.2.1 Theoretical results

4.2.1.1 Properties of the associated forms

Lemma 11 (Coercivity of a_h)**.** *There exists a constant $C_a > 0$, independent of h but dependent on κ^*, μ_*, and μ^*, and a constant $\beta_0^s > 0$ such that if $\beta_s > \beta_s^0$, then*

$$a_h(c_h; \boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_a \|\!|\boldsymbol{v}_h|\!\|_v \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \forall c_h \in C_h. \tag{4.41}$$

Proof. Since K is symmetric and uniformly positive definite, we have

$$a_h^d(c; \mathbf{v}_h, \mathbf{v}_h) = \int_{\Omega^d} \mathbb{K}^{-1}(c) v_h \cdot v_h \, \mathrm{d}x \geq C \|v_h\|_{\Omega^d}^2. \tag{4.42}$$

Furthermore,

$$a_h^I(\bar{c}; \bar{v}_h, \bar{v}_h) = \sum_{j=1}^{\dim - 1} \int_{\Gamma^I} \gamma^j \mu(\bar{c})(\bar{v}_h \cdot \tau^j)(\bar{v}_h \cdot \tau^j) \, \mathrm{d}s \geq \mu_* \sum_{j=1}^{\dim - 1} \gamma^j \|\bar{v}_h \cdot \tau^j\|_{\Gamma^I}^2, \tag{4.43}$$

and

$$
\begin{aligned}
&a_h^s(c;\boldsymbol{v}_h,\boldsymbol{v}_h)\\
&= \sum_{K\in\mathcal{T}^s}\int_K 2\mu(c)\varepsilon(v_h):\varepsilon(v_h)\ \mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta_s\mu(c)}{h_K}(v_h-\bar{v}_h)\cdot(v_h-\bar{v}_h)\ \mathrm{d}s\\
&- \sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu(c)\varepsilon(v_h)n^s\cdot(v_h-\bar{v}_h)\ \mathrm{d}s - \sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu(c)\varepsilon(v_h)n^s\cdot(v_h-\bar{v}_h)\ \mathrm{d}s\\
&= \sum_{K\in\mathcal{T}^s}2\mu(c)\|\varepsilon(v_h)\|_K^2 + \sum_{K\in\mathcal{T}^s}\frac{2\beta_s\mu(c)}{h_K}\|v_h-\bar{v}_h\|_{\partial K}^2\\
&- \sum_{K\in\mathcal{T}^s}4\mu(c)\varepsilon(v_h)n^s\cdot(v_h-\bar{v}_h)\ \mathrm{d}s. \qquad (4.44)
\end{aligned}
$$

Using Cauchy-Schwarz inequality and eq. (2.29), we get

$$
\begin{aligned}
\Big|\sum_{K\in\mathcal{T}^s}4\mu(c)\varepsilon(v_h)n^s\cdot(v_h-\bar{v}_h)\,\mathrm{d}s\Big| &\leq 4\mu^*\|\varepsilon(v_h)n^s\|_{\partial K}\|v_h-\bar{v}_h\|_{\partial K}\\
&\leq 4\mu^* h_K^{1/2}\|\varepsilon(v_h)\|_{\partial K}\frac{1}{h_K^{1/2}}\|v_h-\bar{v}_h\|_{\partial K}\\
&\leq 4\mu^* C_{T,1}\|\varepsilon(v_h)\|_K\frac{1}{h_K^{1/2}}\|v_h-\bar{v}_h\|_{\partial K}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_h^s(c;\boldsymbol{v}_h,\boldsymbol{v}_h) &\geq \sum_{K\in\mathcal{T}^s}2\mu_*\|\varepsilon(v_h)\|_K^2 + \sum_{K\in\mathcal{T}^s}\frac{2\beta_s\mu_*}{h_K}\|v_h-\bar{v}_h\|_{\partial K}^2\\
&\quad - \sum_{K\in\mathcal{T}^s}4\mu^* C_{T,1}\|\varepsilon(v_h)\|_K\frac{1}{h_K^{1/2}}\|v_h-\bar{v}_h\|_{\partial K}\\
&\geq 2\mu_*\sum_{K\in\mathcal{T}^s}\Bigg(\|\varepsilon(v_h)\|_K^2 + \frac{\beta_s}{h_K}\|(v_h-\bar{v}_h)\|_{\partial K}^2\\
&\qquad - \frac{2\mu^*}{\mu_*}C_{T,1}\|\varepsilon(v_h)\|_K\frac{1}{h_K^{1/2}}\|v_h-\bar{v}_h\|_{\partial K}\Bigg).
\end{aligned}
$$

Recall from [24, p. 129] that $x^2-2\beta xy+\eta y^2 \geq \frac{\eta-\beta^2}{1+\eta}(x^2+y^2)$ for $\eta>\beta^2$. Then taking $x=\|\varepsilon(v_h)\|_K$, $y=h_K^{-1/2}\|v_h-\bar{v}_h\|_{\partial K}$, $\eta=\beta_s$, and $\beta=C_{T,1}\frac{\mu^*}{\mu_*}$, under the condition $\beta_s > (C_{T,1}\frac{\mu^*}{\mu_*})^2 =: \beta_0$, we obtain

$$a_h^s(c; \boldsymbol{v}_h, \boldsymbol{v}_h) \geq 2\mu_* \left(\frac{\beta_s - \beta_0}{1+\beta_s}\right) \sum_{K\in\mathcal{T}^s} (\|\varepsilon(v_h)\|_K^2 + h_K^{-1}\|v_h - \bar{v}_h\|_{\partial K}^2).$$

It follows by eq. (2.41) that

$$a_h^s(c; \boldsymbol{v}_h, \boldsymbol{v}_h) \geq \frac{\mu_*}{C_K}(\frac{\beta_s - \beta_0}{1+\beta_s}) \sum_{K\in\mathcal{T}^s} (\|\nabla v_h\|_K^2 + h_K^{-1}\|v_h - \bar{v}_h\|_{\partial K}^2). \tag{4.45}$$

Combining with eq. (4.42), eq. (4.43) and eq. (4.45), we have

$$a_h(\boldsymbol{c}; \boldsymbol{v}_h, \boldsymbol{v}_h) \geq C|\!|\!|\mathbf{v}_h|\!|\!|_v.$$

□

Theorem 9 (Continuity of a_h)**.** *There exists a constant $C > 0$, independent of h, such that for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$ and $\boldsymbol{c} \in C(h)$,*

$$a_h(c; \boldsymbol{u}, \boldsymbol{v}) \leq C|\!|\!|\boldsymbol{u}|\!|\!|_{v'}|\!|\!|\boldsymbol{v}|\!|\!|_{v'}.$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}(h)$, $c \in \mathbb{R}$. Then

$$\begin{aligned} a_h^s(c;\boldsymbol{u}, \boldsymbol{v}) = &\sum_{K\in\mathcal{T}^s} \int_K 2\mu(c)\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{2\beta_s\mu(c)}{h_K}(u - \bar{u}) \cdot (v - \bar{v})\,\mathrm{d}s \\ &- \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu(c)\varepsilon(u)n^s \cdot (v - \bar{v})\,\mathrm{d}s - \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu(c)\varepsilon(v)n^s \cdot (u - \bar{u})\,\mathrm{d}s, \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We will bound each term separately. Using eq. (4.6b) and the Cauchy-Schwarz inequality,

$$\begin{aligned} |J_1| = \Big|\sum_{K\in\mathcal{T}^s} \int_K 2\mu(c)\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x\Big| &\leq 2\mu^* \sum_{K\in\mathcal{T}^s} \|\varepsilon(u)\|_K \|\varepsilon(v)\|_K \\ &\leq 2\mu^* \sum_{K\in\mathcal{T}^s} \|\nabla u\|_K \|\nabla v\|_K \leq 2\mu^* |\!|\!|u|\!|\!|_v |\!|\!|v|\!|\!|_v, \end{aligned}$$

and

$$
\begin{aligned}
|J_2| &= \Big| \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{2\beta_s \mu(c)}{h_K}(u-\bar{u})\cdot(v-\bar{v})\ \mathrm{d}s \Big| \\
&\le 2\mu^*\beta_s \sum_{K\in\mathcal{T}^s} h_K^{-1/2}\|u-\bar{u}\|_{\partial K} h_K^{-1/2}\|v-\bar{v}\|_{\partial K}\ \mathrm{d}s \\
&\le 2\mu^*\beta_s \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|u-\bar{u}\|^2_{\partial K}\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|v-\bar{v}\|^2_{\partial K}\Big)^{1/2} \\
&\le C_2 \|\!|\boldsymbol{u}|\!\|_v \|\!|\boldsymbol{v}|\!\|_v .
\end{aligned}
$$

Next, using eq. (4.6b), the Cauchy-Schwarz inequality and the trace inequality eq. (2.31),

$$
\begin{aligned}
&|J_3| = \Big| \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot(v-\bar{v})\ \mathrm{d}s \Big| \\
&\le 2\mu^* \sum_{K\in\mathcal{T}^s} \|\varepsilon(u)n^s\|_{\partial K}\|v-\bar{v}\|_{\partial K} \\
&\le 2\mu^* \Big(\sum_{K\in\mathcal{T}^s} h_K\|\varepsilon(u)\|^2_{\partial K}\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|v-\bar{v}\|^2_{\partial K}\Big)^{1/2} \\
&\le 2\mu^* \Big(\sum_{K\in\mathcal{T}^s} h_K C_{T,2}(h_K^{-1}\|\varepsilon(u)\|^2_K + h_K\|\varepsilon(u)\|^2_{1,K})\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|v-\bar{v}\|^2_{\partial K}\Big)^{1/2} \\
&\le 2\mu^* \Big(\sum_{K\in\mathcal{T}^s} h_K C_{T,2}(h_K^{-1}C_3|u|^2_{1,K} + h_K C_4|u|^2_{2,K})\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|v-\bar{v}\|^2_{\partial K}\Big)^{1/2} \\
&\le 2\mu^* C \Big(\sum_{K\in\mathcal{T}^s} (\|\nabla u\|^2_K + h_K^2|u|^2_{2,K})\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \frac{1}{h_K}\|v-\bar{v}\|^2_{\partial K}\Big)^{1/2} \\
&\le C_5 \|\!|\boldsymbol{u}|\!\|_{v'} \|\!|\boldsymbol{v}|\!\|_v .
\end{aligned}
$$

Similarly, we get $J_4 \le C_5 \|\!|\boldsymbol{u}|\!\|_v \|\!|\boldsymbol{v}|\!\|_{v'}$ by symmetry. Collecting all bounds and noting that $\|\!|\boldsymbol{v}|\!\|_v \le \|\!|\boldsymbol{v}|\!\|_{v'}$, we have $a_h^s(c;\boldsymbol{u},\boldsymbol{v}) \le C_6 \|\!|\boldsymbol{u}|\!\|_{v'} \|\!|\boldsymbol{v}|\!\|_{v'}$.

Using the Cauchy-Schwarz and by the assumption on $\mathbb{K}$, we have

$$
a_h^d(c;u,v) = \int_{\Omega^d} \mathbb{K}^{-1}(c)u\cdot v\,\mathrm{d}x \le C_7\|u\|_{\Omega^d}\|v\|_{\Omega^d} \le C_7 \|\!|\boldsymbol{u}|\!\|_v \|\!|\boldsymbol{v}|\!\|_v .
$$

Applying the Cauchy-Schwarz inequality once again, we obtain

$$a_h^I(\bar{c};\bar{u},\bar{v}) = \sum_{j=1}^{\dim-1} \int_{\Gamma^I} \gamma^j \mu(\bar{c})(\bar{u}\cdot\tau^j)(\bar{v}\cdot\tau^j)\,\mathrm{d}s \le \mu^* \sum_{j=1}^{\dim-1} \gamma^j \|\bar{u}\cdot\tau^j\|_{\Gamma^I}\|\bar{v}\cdot\tau^j\|_{\Gamma^I}$$

$$\le \mu^* \|u\|_v \|v\|_v.$$

Combining all of the bounds above yields the result. □

4.2.1.2 Fully discrete numerical scheme   Solving the fully coupled problem using an implicit monolithic scheme is computationally expensive. Instead we decouple the Stokes-Darcy and transport problems by solving them sequentially as we describe below in Algorithm 1. This linearizes the problem and is acceptable since the Stokes-Darcy velocity-pressure evolution is expected to happen in a different time scale than the evolution of the concentration. In the first step, given an initial Stokes velocity defined only in the Stokes region, we solve the Stokes-Darcy problem and obtain a velocity that lives in the entire region. Then this velocity becomes an input for the concentration problem and we advance in time.

We partition the time interval $[0,T]$ as: $0 = t^0 < t^1 < \ldots < t^N = T$. For simplicity we assume that the partition is uniform and $t^{n+1} - t^n = \Delta t$ for $0 \le n \le N-1$. We denote a function $h(t)$ at time level t^n by $h^n = h(t^n)$. and for a sequence $\{u^n\}$ we denote by $d_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$ a first order difference operator.

In Algorithm 1, Π_C denotes the L^2-projection into C_h and Π_V^s denotes the BDM-interpolation into V_h^s. We note that $\Pi_V^s u_0$ preserves mass conservation and zero jump across the interface.

As in eq. (4.35) and eq. (4.40), the scheme enjoys strong mass conservation property

$$-\nabla\cdot u_h^n = \chi^d \Pi_Q(g_p^n - g_i^n) \quad \forall x \in K,\ \forall K \in \mathcal{T}, \tag{4.48}$$

Algorithm 1: Sequential approach

input: Set $u_h^0 = \Pi_V^s u_0$, $c_h^0 = \Pi_C c_0$ in Ω^s.

for $n = 1, \ldots, N$ **do**

1. Find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that

$$\sum_{K\in\mathcal{T}^s} \int_{\partial K} d_t u_h^n \cdot v_h \,\mathrm{d}x + B_h^{sd}(c_h^{n-1}; (\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = \int_{\Omega^s} f^s(c_h^{n-1}) \cdot v_h \,\mathrm{d}x \tag{4.46}$$

$$+ \int_{\Omega^d} \mathbb{K}^{-1}(c_h^{n-1}) f^d(c_h^{n-1}) q_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}^d} \int_K (g_p^n - g_i^n)\, q \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h.$$

2. Find $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ such that

$$\sum_{K\in\mathcal{T}} \int_K \phi\, d_t c_h^n w_h \,\mathrm{d}x + B_h^{tr}(u_h^n; \boldsymbol{c}_h^n, \boldsymbol{w}_h) + \sum_{K\in\mathcal{T}^d} \int_K g_p^n c_h^n w_h \,\mathrm{d}x \tag{4.47}$$

$$= \sum_{K\in\mathcal{T}^d} \int_K c_I g_i^n w_h \,\mathrm{d}x \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h.$$

end for

and normal continuity of the discrete velocity at each time step:

$$[\![u_h^n \cdot n]\!] = 0 \qquad \forall x \in F, \quad \forall F \in \mathcal{F}, \tag{4.49a}$$

$$u_h^n \cdot n = \bar{u}_h^n \cdot n \qquad \forall x \in F, \quad \forall F \in \mathcal{F}^I. \tag{4.49b}$$

Theorem 10. *Let $\beta > \beta_0$ be as in Lemma 11. Then given $c_h^{n-1} \in C_h$, $n \geq 1$, there exists a unique solution $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ to eq. (4.46) satisfying*

$$\begin{aligned}&|\!|\!| u_h^n |\!|\!|_v + |\!|\!| p_h^n |\!|\!|_p \\ &\leq C\Big(\frac{1}{\Delta t}\|u_h^{n-1}\|_{\Omega^s} + \|f^s(c_h^{n-1})\|_{\Omega^s} + \frac{\mu^*}{\kappa_*}\|f^d(c_h^{n-1})\|_{\Omega^d} + c_4\|g_p^n - g_i^n\|_{\Omega^d}\Big).\end{aligned} \tag{4.50}$$

Proof. The proof is the same as in Theorem 3 since a_h is coercive by Lemma 11 and the inf-sup condition Theorem 2 applies.

$\square$

Our existence proof for $\boldsymbol{c}_h^n$, $n \geq 1$ depends on an L^∞-bound on u_h^{n-1} and therefore it is essential to prove an L^2-convergence result for u_h^{n-1} for any $n \geq 1$. Therefore, assuming that $c_h^n \in C_h^n$ is given, we will derive error estimates for $u_h^n - u^n$ and $p_h^n - p^n$ for each $n \geq 1$.

Theorem 11 (Error equation for eq. (4.46))**.** *There holds*

$$\begin{aligned}
\sum_{K\in\mathcal{T}^s} \int_K (d_t u_h^n - \partial_t u^n) \cdot v_h \,\mathrm{d}x + a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{\zeta}_u^n, \boldsymbol{v}_h) \\
+ \sum_{j=s,d} \big(b_h^j(\boldsymbol{\zeta}_p^{jn}, v_h) + b_h^{I,j}(\bar{\zeta}_p^{jn}, \bar{v}_h)\big) + \sum_{j=s,d} \big(b_h^j(\boldsymbol{q}_h^j, \zeta_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\zeta}_u^n)\big) \\
= a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{\xi}_u^n, \boldsymbol{v}_h) + a_h(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) \\
+ \sum_{K\in\mathcal{T}^s} \int_K [f^s(c_h^{n-1}) - f^s(c^n)] \cdot v_h \,\mathrm{d}x \\
+ \sum_{K\in\mathcal{T}^d} \int_K [\mathbb{K}^{-1}(c_h^{n-1}) f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n) f^d(c^n)] \cdot v_h \,\mathrm{d}x. \quad (4.51)
\end{aligned}$$

Proof. By Lemma 10, at time $t = t^n$, for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$,

$$\begin{aligned}
\sum_{K\in\mathcal{T}^s} \int_K \partial_t u^n \cdot v_h \,\mathrm{d}x + B_h^{sd}(\boldsymbol{c}^n; (\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) = \sum_{K\in\mathcal{T}^s} \int_K f^s(c^n) \cdot v_h \,\mathrm{d}x \\
+ \sum_{K\in\mathcal{T}^d} \int_K \mathbb{K}^{-1}(c^n) f^d(c^n) \cdot v_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}^d} \int_K (g_p^n - g_i^n) q_h \,\mathrm{d}x.
\end{aligned}$$

Subtracting this equation from eq. (4.46), we obtain

$$\begin{aligned}
\sum_{K\in\mathcal{T}^s} \int_K (d_t u_h^n - \partial_t\, u^n) \cdot v_h \,\mathrm{d}x \\
+ B_h^{sd}(\boldsymbol{c}_h^{n-1}; (\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) - B_h^{sd}(\boldsymbol{c}^n; (\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\
= \sum_{K\in\mathcal{T}^s} \int_K [f^s(c_h^{n-1}) - f^s(c^n)] \cdot v_h \,\mathrm{d}x \\
+ \sum_{K\in\mathcal{T}^d} \int_K [\mathbb{K}^{-1}(c_h^{n-1}) f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n) f^d(c^n)] \cdot v_h \,\mathrm{d}x, \quad (4.52)
\end{aligned}$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$. Then, by eq. (2.51), the terms with B_h^{sd} can be rewritten as

$$\begin{aligned}
&B_h^{sd}(\boldsymbol{c}_h^{n-1}; (\boldsymbol{u}_h^n, \boldsymbol{p}_h^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) - B_h^{sd}(\boldsymbol{c}^n; (\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\
&= B_h^{sd}(\boldsymbol{c}_h^{n-1}; (\boldsymbol{u}_h^n - \boldsymbol{u}^n, \boldsymbol{p}_h^n - \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\
&\quad + B_h^{sd}(\boldsymbol{c}_h^{n-1}; (\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) - B_h^{sd}(\boldsymbol{c}^n; (\boldsymbol{u}^n, \boldsymbol{p}^n), (\boldsymbol{v}_h, \boldsymbol{q}_h)) \\
&= a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n - \boldsymbol{u}^n, \boldsymbol{v}_h) \\
&\quad + \sum_{j=s,d} \big(b_h^j(\boldsymbol{\zeta}_p^{jn}, v_h) + b_h^{I,j}(\bar{\zeta}_p^{jn}, \bar{v}_h) \big) - \sum_{j=s,d} \big(b_h^j(\boldsymbol{\xi}_p^{jn}, v_h) + b_h^{I,j}(\bar{\xi}_p^{jn}, \bar{v}_h) \big) \\
&\quad + \sum_{j=s,d} \big(b_h^j(\boldsymbol{q}_h^j, \zeta_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\zeta}_u^n) \big) - \sum_{j=s,d} \big(b_h^j(\boldsymbol{q}_h^j, \xi_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\xi}_u^n) \big) \\
&\quad + a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) \\
&= a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{\zeta}_u^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{\xi}_u^n, \boldsymbol{v}_h) + a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) \\
&\quad + \sum_{j=s,d} \big(b_h^j(\boldsymbol{\zeta}_p^{jn}, v_h) + b_h^{I,j}(\bar{\zeta}_p^{jn}, \bar{v}_h) \big) + \sum_{j=s,d} \big(b_h^j(\boldsymbol{q}_h^j, \zeta_u^n) + b_h^{I,j}(\bar{q}_h^j, \bar{\zeta}_u^n) \big).
\end{aligned} \tag{4.53}$$

where we applied eqs. (2.55) and (2.56). Combining this with eq. (4.52) yields the result. □

The velocity error at each time step depends on the error in concentration from the previous time step. Therefore, for the velocity error estimates, we will need some auxiliary results related to the concentration error. To estimate the error of the concentration, we use the continuous interpolant $\mathcal{I}c \in C_h \cap \mathcal{C}^0(\bar{\Omega})$ of c [9], and we set $\bar{\mathcal{I}}c(t) = \mathcal{I}c|_{\Gamma^0}(t) \in \bar{C}_h$. Denoting the restriction of c to Γ^0 by $\bar{c}$, we define

$$\begin{aligned}
\xi_c^n &= c^n - \mathcal{I}c^n, & \zeta_c^n &= c_h^n - \mathcal{I}c^n, & \boldsymbol{\xi}_c^n &= (\xi_c^n, \bar{\xi}_c^n), \\
\bar{\xi}_c^n &= \bar{c}^n - \bar{\mathcal{I}}c^n, & \bar{\zeta}_c^n &= \bar{c}_h^n - \bar{\mathcal{I}}c^n, & \boldsymbol{\zeta}_c^n &= (\zeta_c^n, \bar{\zeta}_c^n).
\end{aligned} \tag{4.54}$$

Furthermore, we split the error as follows.

$$c^n - c_h^n = \xi_c^n - \zeta_c^n, \quad \bar{c}^n - \bar{c}_h^n = \bar{\xi}_c^n - \bar{\zeta}_c^n. \tag{4.55}$$

The following interpolation estimates hold [9, Section 4.4].

$$\|\xi_c\|_{r,K} \le Ch_K^{\ell+1-r}\|c\|_{\ell+1,K}, \tag{4.56}$$

$$\|\xi_c\|_{r,\infty,K} \le Ch_K^{\ell+1-r}\|c\|_{\ell+1,\infty,K}, \tag{4.57}$$

where $r = 0, 1$ and $1 \le \ell \le k_c$.

Theorem 12. *Let $c_0^s \in H^{k_c+1}(\Omega^s)$, $c^s \in L^2(0,T;H^{k_c+1}(\Omega^s))$, $c^d \in H^1(0,T;L^2(\Omega^d))$ such that $\partial_t c^s \in L^2(0,T,H^1(\Omega^s))$, and $\bar{c} = \gamma(c)$ on Γ_0. Then we have the following estimates:*

$$\begin{aligned}&\Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}^s} h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \\ &\le C\Big(h^2(\Delta t)^2\|\partial_t c\|^2_{L^2(0,T;H^1(\Omega^s))} + \Delta t \sum_{m=1}^{n}\sum_{K\in\mathcal{T}^s} h_K^2\|c^{m-1} - c_h^{m-1}\|_{1,K}^2\Big),\end{aligned} \tag{4.58}$$

$$\begin{aligned}&\Delta t \sum_{m=1}^{n} \|\bar{c}^m - \bar{c}_h^{m-1}\|^2_{\Gamma^I} \le C\Big(\Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{\zeta}_c^{m-1}\|\!|_c^2 \\ &+(\Delta t)^2\|\partial_t c\|^2_{L^2(0,T;H^1(\Omega^s))} + \Delta t h^{2k_c+1}\|c_0\|^2_{k_c+1,\Omega^s} + h^{2k_c+1}\|c\|^2_{\ell^2(0,T;H^{k_c+1}(\Omega^s))}\Big),\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}&\Delta t \sum_{m=1}^{n} \Big(\sum_{K\in\mathcal{T}} (\|c^m - c_h^{m-1}\|_K^2\Big) \\ &\qquad\le C\Big((\Delta t)^2\|\partial_t c\|^2_{L^2(0,T;L^2(\Omega))} + \Delta t \sum_{m=1}^{n} \big(\sum_{K\in\mathcal{T}} \|c^{m-1} - c_h^{m-1}\|_K^2\big)\Big).\end{aligned} \tag{4.60}$$

Proof. We first estimate eq. (4.58). By the triangle inequality,

$$\begin{aligned}\sum_{K\in\mathcal{T}^s} h_K^2\|c^m - c_h^{m-1}\|_{1,K}^2 &\le \sum_{K\in\mathcal{T}^s} 2h_K^2(\|c^m - c^{m-1}\|_{1,K}^2 + \|c^{m-1} - c_h^{m-1}\|_{1,K}^2) \\ &= \sum_{K\in\mathcal{T}^s} 2h_K^2\Big((\Delta t)^2\|d_t c^m\|_{1,K}^2 + \|c^{m-1} - c_h^{m-1}\|_{1,K}^2\Big).\end{aligned}$$

Multiplying this by Δt, summing from $m = 1$ to n, and using eq. (C.3), we obtain eq. (4.58). Now we estimate eq. (4.59). We have

$$\|\bar{c}^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \leq 2(\|\bar{c}^m - \bar{c}^{m-1}\|_{\Gamma^I}^2 + \|\bar{c}^{m-1} - \bar{c}_h^{m-1}\|_{\Gamma^I}^2). \tag{4.61}$$

Since $\bar{c} = c^s|_{\Gamma_I}$ on Γ^I, and $(c^s)^m - (c^s)^{m-1} \in H^1(\Omega^s)$, by eq. (2.35), the first term on the right side of eq. (4.61) is bounded as follows.

$$\|\bar{c}^m - \bar{c}^{m-1}\|_{\Gamma^I}^2 \leq C\|c^m - c^{m-1}\|_{1,\Omega^s}^2 = C(\Delta t)^2\|d_t c^m\|_{1,\Omega^s}^2. \tag{4.62}$$

Splitting the second term on the right side of eq. (4.61) using $\mathcal{I}c = \bar{\mathcal{I}}c$ for any $F \in \mathcal{F}^I$ gives

$$\begin{aligned}\|\bar{c}^{m-1} - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 &\leq 2(\|c^{m-1} - \mathcal{I}c^{m-1}\|_{\Gamma^I}^2 + \|\bar{c}_h^{m-1} - \bar{\mathcal{I}}c^{m-1}\|_{\Gamma^I}^2) \\ &= 2(\|\xi_c^{m-1}\|_{\Gamma^I}^2 + \|\bar{\zeta}_c^{m-1}\|_{\Gamma^I}^2). \end{aligned}\tag{4.63}$$

The first term above is bounded by eq. (2.31) and eq. (4.56)

$$\begin{aligned}\|\xi_c^{m-1}\|_{\Gamma^I}^2 &\leq \sum_{K\in\mathcal{T}^s}\|\xi_c^{m-1}\|_{\partial K}^2 \leq \sum_{K\in\mathcal{T}^s} C_{T,2}(h_K^{-1}\|\xi_c^{m-1}\|_K^2 + h_K\|\xi_c^{m-1}\|_{1,K}^2) \\ &\leq \sum_{K\in\mathcal{T}^s} C(h_K^{-1}h_K^{2(k_c+1)}\|c^{m-1}\|_{k_c+1,K}^2 + h_K h_K^{2k_c}\|c^{m-1}\|_{k_c+1,K}^2) \\ &\leq Ch^{2k_c+1}\|c^{m-1}\|_{k_c+1,\Omega^s}^2. \end{aligned}\tag{4.64}$$

Using eq. (2.37) and the definition of $\|\!|\cdot|\!\|_c$,

$$\begin{aligned}\|\bar{\zeta}_c^{m-1}\|_{\Gamma^I}^2 &\leq 2\big(\|\bar{\zeta}_c^{m-1} - (\zeta_c^s)^{m-1}\|_{\Gamma^I}^2 + \|(\zeta_c^s)^{m-1}\|_{\Gamma^I}^2\big) \\ &\leq 2\Big(h\sum_{K\in\mathcal{T}^s} h_K^{-1}\|\bar{\zeta}_c^{m-1} - \zeta_c^{m-1}\|_{\partial K}^2 + \|\zeta_c^{m-1}\|_{\Gamma^I}^2\Big) \leq C|\!|\!|\zeta_c^{m-1}|\!|\!|_c^2. \end{aligned}\tag{4.65}$$

Collecting eqs. (4.61) to (4.65), we obtain

$$\|\bar{c}^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \;\leq\; C\Big((\Delta t)^2\|d_t c^m\|_{1,\Omega^s}^2 + h^{2k_c+1}\|c^{m-1}\|_{k_c+1,\Omega^s}^2 + |\!|\!|\zeta_c^{m-1}|\!|\!|_c^2\Big).$$

Multiplying this inequality by Δt, summing from 1 to n, and using eq. (C.3), we obtain

$$\Delta t \sum_{m=1}^{n} \|\bar{c}^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \leq C\Big(\Delta t \sum_{m=1}^{n} \|\!|\zeta_c^{m-1}|\!\|_c^2$$
$$+ (\Delta t)^2 \|\partial_t c\|_{L^2(0,T,H^1(\Omega^s))}^2 + \Delta t h^{2k_c+1} \|c_0\|_{k_c+1,\Omega^s}^2 + h^{2k_c+1} \|c\|_{\ell^2(0,T,H^{k_c+1}(\Omega^s))}^2\Big).$$

We next prove eq. (4.60). By the triangle inequality,

$$\begin{aligned}\sum_{K\in\mathcal{T}} \|c^m - c_h^{m-1}\|_K^2 &\leq \sum_{K\in\mathcal{T}} 2\big(\|c^m - c^{m-1}\|_K^2 + \|c^{m-1} - c_h^{m-1}\|_K^2\big)\\ &\leq \sum_{K\in\mathcal{T}} 2\big((\Delta t)^2 \|d_t c^m\|_K^2 + \|c^{m-1} - c_h^{m-1}\|_K^2\big). \qquad (4.66)\end{aligned}$$

Multiplying this by Δt, summing from $m = 1$ to n, and using eq. (C.3), we obtain eq. (4.60). $\square$

Now that the auxiliary result is established, we proceed with the proof of the velocity error.

Theorem 13. *Let c and u be the solutions of eqs.* (4.1) *and* (4.3) *such that*

$$u^s \in L^2(0,T;[H^{k_f+1}(\Omega^s)]^{\dim}) \cap L^\infty(0,T;[W^{1,\infty}(\Omega^s)]^{\dim}),$$
$$u_0 \in [H_0^1(\Omega^s)]^{\dim}, \nabla\cdot u^0 = 0,$$
$$\partial_t u^s \in L^2(0,T;[H^{k_f}(\Omega^s)]^{\dim}), \partial_{tt} u^s \in L^2(0,T;[L^2(\Omega^s)]^{\dim}),$$
$$u^d \in L^2(0,T;[H^{k_f}(\Omega^d)]^{\dim}) \cap L^\infty(0,T;[L^\infty(\Omega^d)]^{\dim}),$$
$$p \in L^2(0,T;L^2(\Omega)), \nabla p^d \in L^\infty(0,T;[L^\infty(\Omega^d)]^{\dim}),$$
$$c^s \in L^2(0,T;H^{k_c+1}(\Omega^s)), \partial_t c^s \in L^2(0,T;H^1(\Omega^s)), c_0^s \in H^{k_c+1}(\Omega^s),$$
$$c^d \in L^2(0,T;L^2(\Omega^d)), \partial_t c^d \in L^\infty(0,T;L^2(\Omega^d)),$$

and let $k_f, k_c \geq \dim - 1$, and $n \geq 1$. Suppose that $u_h^0, \ldots, u_h^{n-1} \in V_h$ and $\boldsymbol{c}_h^0, \ldots, \boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$, the solutions of eq. (4.47) *and eq.* (4.46) *with $k_f, k_c \geq \dim - 1$,* *respectively, are already obtained and satisfy for* $1 \le i \le n$*,*

$$\|c_h^{i-1} - c^{i-1}\|_{\Omega}^2 + \Delta t \sum_{m=1}^{i} \|\!|\boldsymbol{c}_h^{m-1} - \boldsymbol{c}^{m-1}\|\!|_c^2 \le C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \tag{4.67}$$

where C *is a generic constant that is independent of* h*. Then* $\boldsymbol{\zeta}_u^n$ *satisfies the following estimates:*

$$\|\zeta_u^n\|_{\Omega^s}^2 + (\Delta t)^2 \sum_{m=1}^{n} \|d_t\zeta_u^m\|_{\Omega^s}^2 + \Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{\zeta}_u^m\|\!|_v^2 \quad \le C'\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \tag{4.68}$$

$$\|\zeta_u^n\|_{\Omega^d}^2 \le C''\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \tag{4.69}$$

where the constants depend on $\mu^*, \kappa_*, \mu_L, L_f^s, L_f^d, \sum_{j=1}^{\dim-1}\gamma^j$*, and the regularity of* u_0, u, c_0*, and* c *but are independent of the mesh size.*

Proof. Proof of eq. (4.68)*:*

Setting $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{\zeta}_u^n, -\boldsymbol{\zeta}_p^n)$ in Theorem 11, and using $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$, and the coercivity of a_h in Lemma 11 yields

$$\begin{aligned}
\frac{1}{2\Delta t}(\|\zeta_u^n\|_{\Omega^s}^2 - \|\zeta_u^{n-1}\|_{\Omega^s}^2) &+ \frac{\Delta t}{2}\|d_t\zeta_u^n\|_{\Omega^s}^2 + C_a\|\!|\boldsymbol{\zeta}_u^n\|\!|_v^2 \\
&\le \sum_{K\in\mathcal{T}^s}\int_K (d_t\zeta_u^n)\cdot\zeta_u^n \,\mathrm{d}x + a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\zeta}_u^n,\boldsymbol{\zeta}_u^n) \\
&= \sum_{K\in\mathcal{T}^s}\int_K (\partial_t u^n - d_t\,\Pi_V u^n)\cdot\zeta_u^n \,\mathrm{d}x \\
&\quad + a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\xi}_u^n,\boldsymbol{\zeta}_u^n) + [a_h(\boldsymbol{c}^n;\boldsymbol{u}^n,\boldsymbol{\zeta}_u^n) - a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{u}^n,\boldsymbol{\zeta}_u^n)] \\
&\quad + \int_{\Omega^s}[f^s(c_h^{n-1}) - f^s(c^n)]\cdot\zeta_u^n \,\mathrm{d}x \\
&\quad + \int_{\Omega^d}[\mathbb{K}^{-1}(c_h^{n-1})f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n)f^d(c^n)]\cdot\zeta_u^n \,\mathrm{d}x \\
&=: I_1 + \ldots + I_5.
\end{aligned} \tag{4.70}$$

Using eq. (2.40), and employing Young's inequality for some $\epsilon > 0$,

$$\begin{aligned} I_1 &= \sum_{K\in\mathcal{T}^s} \int_K (\partial_t\, u^n - d_t\, \Pi_V u^n)\cdot \zeta_u^n \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}^s} \int_K (\partial_t\, u^n - d_t u^n)\cdot \zeta_u^n \,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_K d_t \xi_u^n \cdot \zeta_u^n \,\mathrm{d}x \\ &\le \|\partial_t\, u^n - d_t u^n\|_{\Omega^s} \|\zeta_u^n\|_{\Omega^s} + \|d_t \xi_u^n\|_{\Omega^s} \|\zeta_u^n\|_{\Omega^s} \\ &\le C\|\partial_t\, u^n - d_t u^n\|_{\Omega^s} |||\boldsymbol{\zeta}_u^n|||_v + C\|d_t \xi_u^n\|_{\Omega^s} |||\boldsymbol{\zeta}_u^n|||_v \\ &\le \frac{C^2}{2\epsilon}\big(\|\partial_t\, u^n - d_t u^n\|_{\Omega^s}^2 + \|d_t \xi_u^n\|_{\Omega^s}^2\big) + \epsilon |||\boldsymbol{\zeta}_u^n|||_v^2. \end{aligned} \tag{4.71}$$

It follows from Theorem 9 and Young's inequality, that

$$\begin{aligned} I_2 &= a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{\xi}_u^n, \boldsymbol{\zeta}_u^n) \\ &\le C |||\boldsymbol{\xi}_u^n|||_{v'} |||\boldsymbol{\zeta}_u^n|||_v \\ &\le \frac{C^2}{4\epsilon} |||\boldsymbol{\xi}_u^n|||_{v'}^2 + \epsilon |||\boldsymbol{\zeta}_u^n|||_v^2. \end{aligned} \tag{4.72}$$

To bound I_3, we first write out all of its terms.

$$\begin{aligned} I_3 &= a_h(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{\zeta}_u^n) - a_h(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{\zeta}_u^n) \\ &= \sum_{K\in\mathcal{T}^s} \int_K 2[\mu(c^n) - \mu(c_h^{n-1})]\varepsilon(u^n) : \varepsilon(\zeta_u^n)\,\mathrm{d}x \\ &\quad - \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2[\mu(c^n) - \mu(c_h^{n-1})]\varepsilon(u^n) n^s \cdot (\zeta_u^n - \bar{\zeta}_u^n)\,\mathrm{d}s \\ &\quad + \int_{\Omega^d} [\mathbb{K}^{-1}(c^n) - \mathbb{K}^{-1}(c_h^{n-1})] u^n \cdot \zeta_u^n \,\mathrm{d}x \\ &\quad + \sum_{j=1}^{\dim-1} \int_{\Gamma^I} \gamma^j [\mu(c^n) - \mu(\bar{c}_h^{n-1})](u^{sn}\cdot \tau^j)(\bar{\zeta}_u^n \cdot \tau^j)\,\mathrm{d}s \\ &=: I_{31} + \ldots + I_{34}. \end{aligned} \tag{4.73}$$

The first term on the right side of I_3 can be bounded as follows using the generalized

Hölder's inequality for integrals and sums and then the Young's inequality:

$$
\begin{aligned}
I_{31} &= \sum_{K\in\mathcal{T}^s} \int_K 2[\mu(c^n) - \mu(c_h^{n-1})]\varepsilon(u^n) : \varepsilon(\zeta_u^n)\,\mathrm{d}x \\
&\le 2 \sum_{K\in\mathcal{T}^s} \|\mu(c^n) - \mu(c_h^{n-1})\|_K \|\varepsilon(u^n)\|_{0,\infty,K} \|\varepsilon(\zeta_u^n)\|_K \\
&\le 2\mu_L \sum_{K\in\mathcal{T}^s} \|c^n - c_h^{n-1}\|_K \|\varepsilon(u^n)\|_{0,\infty,K} \|\nabla\zeta_u^n\|_K \\
&\le 2\mu_L \|\varepsilon(u^n)\|_{0,\infty,\Omega^s} \Big(\sum_{K\in\mathcal{T}^s} \|c^n - c_h^{n-1}\|_K^2\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} \|\nabla\zeta_u^n\|_K^2\Big)^{1/2} \\
&\le 2\mu_L \|\nabla u^n\|_{0,\infty,\Omega^s} \Big(\sum_{K\in\mathcal{T}^s} \|c^n - c_h^{n-1}\|_K^2\Big)^{1/2} |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v \\
&\le \frac{\mu_L^2}{\epsilon} \|\nabla u^n\|_{0,\infty,\Omega^s}^2 \sum_{K\in\mathcal{T}^s} \|c^n - c_h^{n-1}\|_K^2 + \epsilon |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2.
\end{aligned} \tag{4.74}
$$

Next, we bound I_{32}. By the Lipschitz continuity of μ, eq. (2.31), Lemma 4 (iv.), generalized Hölder's inequality and Young's inequality,

$$
\begin{aligned}
|I_{32}| &\le 2\mu_L \sum_{K\in\mathcal{T}^s} \|c^n - c_h^{n-1}\|_{\partial K} \|\nabla u^n\|_{0,\infty,K} \|\zeta_u^n - \bar{\zeta}_u^n\|_{\partial K} \\
&\le 2\mu_L \|\nabla u^n\|_{0,\infty,\Omega^s} \Big(\sum_{K\in\mathcal{T}^s} h_K \|c^n - c_h^{n-1}\|_{\partial K}^2\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}^s} h_K^{-1} \|\zeta_u^n - \bar{\zeta}_u^n\|_{\partial K}^2\Big)^{1/2} \\
&\le 2C\mu_L \|\nabla u^n\|_{0,\infty,\Omega^s} \Big(\sum_{K\in\mathcal{T}^s} (\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2)\Big)^{1/2} |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v \\
&\le \frac{C^2\mu_L^2}{\epsilon} \|\nabla u^n\|_{0,\infty,\Omega^s}^2 \sum_{K\in\mathcal{T}^s} (\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2) + \epsilon |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2.
\end{aligned} \tag{4.75}
$$

We bound I_{33} by using the assumption on $\mathbb{K}$ given in eq. (4.7).

$$
\begin{aligned}
I_{33} &\le \frac{\mu_L}{\kappa_*} \|c^n - c_h^{n-1}\|_{\Omega^d} \|u^n\|_{0,\infty,\Omega^d} |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v \\
&\le \frac{(\kappa_*^{-1}\mu_L)^2}{4\epsilon} \|u^n\|_{0,\infty,\Omega^d}^2 \|c^n - c_h^{n-1}\|_{\Omega^d}^2 + \epsilon |\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2.
\end{aligned} \tag{4.76}
$$

Again by the Lipschitz property of μ and Hölder's inequality,

$$\begin{aligned}
I_{34} &\le \mu_L \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \sum_{j=1}^{\dim-1} \Big(\gamma^j \|u^{sn} \cdot \tau^j\|_{0,\infty,\Gamma^I} \|\bar{\zeta}_u^n \cdot \tau^j\|_{\Gamma^I} \Big) \\
&\le \mu_L \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \big(\sum_{j=1}^{\dim-1} \gamma^j \big)^{1/2} \|u^{sn}\|_{0,\infty,\Gamma^I} \Big(\textstyle\sum_{j=1}^{\dim-1} \gamma^j \|\bar{\zeta}_u^n \cdot \tau^j\|_{\Gamma^I}^2 \Big)^{1/2} \\
&\le \mu_L \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \big(\sum_{j=1}^{\dim-1} \gamma^j \big)^{1/2} \|u^n\|_{0,\infty,\Omega^s} \|\boldsymbol{\zeta}_u^n\|_v \\
&\le \frac{\mu_L^2}{4\epsilon} \big(\sum_{j=1}^{\dim-1} \gamma^j \big) \|u^n\|_{0,\infty,\Omega^s}^2 \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}^2 + \epsilon \|\boldsymbol{\zeta}_u^n\|_v^2. \qquad (4.77)
\end{aligned}$$

Combining eqs. (4.73) to (4.77),

$$\begin{aligned}
I_3 \le & \frac{C^2\mu_L^2}{\epsilon} \|\nabla u^n\|_{0,\infty,\Omega^s}^2 \sum_{K\in\mathcal{T}^s} (\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2) \\
& + \frac{(\kappa_*^{-1}\mu_L)^2}{4\epsilon} \|u^n\|_{0,\infty,\Omega^d}^2 \|c^n - c_h^{n-1}\|_{\Omega^d}^2 \\
& + \frac{\mu_L^2}{4\epsilon} \big(\sum_{j=1}^{\dim-1} \gamma^j \big) \|u^n\|_{0,\infty,\Omega^s}^2 \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}^2 + 4\epsilon \|\boldsymbol{\zeta}_u^n\|_v^2. \qquad (4.78)
\end{aligned}$$

Since f^s is Lipschitz continuous in c with Lipschitz constant L_f^s, and recalling eq. (2.40),

$$\begin{aligned}
I_4 = \int_{\Omega^s} [f^s(c_h^{n-1}) - f^s(c^n)] \cdot \zeta_u^n \, \mathrm{d}x &\le L_f^s \|c_h^{n-1} - c^n\|_{\Omega^s} \|\zeta_u^n\|_{\Omega^s} \\
&\le L_f^s \|c_h^{n-1} - c^n\|_{\Omega^s} \|\boldsymbol{\zeta}_u^n\|_v \\
&\le \frac{(L_f^s)^2}{4\epsilon} \|c_h^{n-1} - c^n\|_{\Omega^s}^2 + \epsilon \|\boldsymbol{\zeta}_u^n\|_v^2. \qquad (4.79)
\end{aligned}$$

We have

$$\begin{aligned}
I_5 &= \int_{\Omega^d} [\mathbb{K}^{-1}(c_h^{n-1}) f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n) f^d(c^n)] \cdot \zeta_u^n \, \mathrm{d}x \\
&= \int_{\Omega^d} \big(\mathbb{K}^{-1}(c_h^{n-1})[f^d(c_h^{n-1}) - f^d(c^n)] + [\mathbb{K}^{-1}(c_h^{n-1}) - \mathbb{K}^{-1}(c^n)] f^d(c^n) \big) \cdot \zeta_u^n \, \mathrm{d}x \\
&= \int_{\Omega^d} \mathbb{K}^{-1}(c_h^{n-1})[f^d(c_h^{n-1}) - f^d(c^n)] + [\mu(c_h^{n-1}) - \mu(c^n)] \kappa^{-1} f^d(c^n) \big) \cdot \zeta_u^n \, \mathrm{d}x.
\end{aligned}$$

Since f^d and μ are Lipschitz continuous in c with Lipschitz continuity constants L_f^d and μ_L, respectively,

$$\begin{aligned}
I_5 &\leq \Big(\frac{1}{K_*}L_f^d + \frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d}\Big)\|c_h^{n-1} - c^n\|_{\Omega^d}\|\zeta_u^n\|_{\Omega^d} \\
&\leq \frac{1}{4\epsilon}\Big(\frac{1}{K_*}L_f^d + \frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d}\Big)^2\|c_h^{n-1} - c^n\|^2_{\Omega^d} + \epsilon|\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2. \qquad (4.80)
\end{aligned}$$

Adding eqs. (4.70) to (4.73), eqs. (4.78) and (4.80) we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\zeta_u^n\|^2_{\Omega^s} - \|\zeta_u^{n-1}\|^2_{\Omega^s}\Big) + \frac{\Delta t}{2}\|d_t\zeta_u^n\|^2_{\Omega^s} + C_a|\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2 \\
&\leq 8\epsilon|\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2 + C\Big(\|\partial_t u^n - d_t u^n\|^2_{\Omega^s} + \|d_t\xi_u^n\|^2_{\Omega^s} + |\!|\!|\boldsymbol{\xi}_u^n|\!|\!|_{v'}^2 \\
&\quad + \mu_L^2\|\nabla u^n\|^2_{0,\infty,\Omega^s}\sum_{K\in\mathcal{T}^s} h_K^2\|c^n - c_h^{n-1}\|^2_{1,K} \\
&\quad + \Big(\kappa_*^{-1}\mu_L\|u^n\|_{0,\infty,\Omega^d} + K_*^{-1}(L_f^d) + \frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d}\Big)^2\|c^n - c_h^{n-1}\|^2_{\Omega^d} \\
&\quad + \mu_L^2\Big(\sum_{j=1}^{\mathrm{dim}-1}\gamma^j\Big)\|u^n\|^2_{0,\infty,\Omega^s}\|\bar{c}^n - \bar{c}_h^{n-1}\|^2_{\Gamma^I} \\
&\quad + \big((L_f^s)^2 + \mu_L^2\|\nabla u^n\|^2_{0,\infty,\Omega^s}\big)\|c_h^{n-1} - c^n\|^2_{\Omega^s}.
\end{aligned}$$

Letting $\epsilon = \dfrac{C_a}{16}$ (C_a is the coercivity constant), we get

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\zeta_u^n\|^2_{\Omega^s} - \|\zeta_u^{n-1}\|^2_{\Omega^s}) + \frac{\Delta t}{2}\|d_t\zeta_u^n\|^2_{\Omega^s} + \frac{C_a}{2}|\!|\!|\boldsymbol{\zeta}_u^n|\!|\!|_v^2 \\
&\leq C\Big(\|\partial_t u^n - d_t u^n\|^2_{\Omega^s} + \|d_t\xi_u^n\|^2_{\Omega^s} + |\!|\!|\boldsymbol{\xi}_u^n|\!|\!|_{v'}^2 \\
&\quad + \mu_L^2\|\nabla u^n\|^2_{0,\infty,\Omega^s}\sum_{K\in\mathcal{T}^s} h_K^2\|c^n - c_h^{n-1}\|^2_{1,K} \\
&\quad + \Big(\kappa_*^{-1}\mu_L\|u^n\|_{0,\infty,\Omega^d} + K_*^{-1}(L_f^d) + \frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d}\Big)^2\|c^n - c_h^{n-1}\|^2_{\Omega^d} \\
&\quad + \mu_L^2\Big(\sum_{j=1}^{\mathrm{dim}-1}\gamma^j\Big)\|u^n\|^2_{0,\infty,\Omega^s}\|\bar{c}^n - \bar{c}_h^{n-1}\|^2_{\Gamma^I} \\
&\quad + \big((L_f^s)^2 + \mu_L^2\|\nabla u^n\|^2_{0,\infty,\Omega^s}\big)\|c_h^{n-1} - c^n\|^2_{\Omega^s}.
\end{aligned}$$

Then multiplying by $2\Delta t$, summing from 1 to n, noting that $\zeta_u^0 = 0$, and applying eqs. (C.2) and (C.3), we obtain

$$
\begin{aligned}
&\|\zeta_u^n\|_{\Omega^s}^2 + (\Delta t)^2 \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 + C_a \Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{\zeta}_u^m|\!\|_v^2 \\
&\leq C\Big[(\Delta t)^2 \|\partial_{tt} u\|_{L^2(0,T;L^2(\Omega^s))}^2 + \|\partial_t \xi_u\|_{L^2(0,T;L^2(\Omega^s))}^2 + 2\Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{\xi}_u^m|\!\|_{v'}^2 \\
&+ \Delta t \sum_{m=1}^{n} \Big(\|\nabla u^m\|_{0,\infty,\Omega^s}^2 \sum_{K \in \mathcal{T}^s} h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \Big) \\
&+ \Delta t \sum_{m=1}^{n} \Big(\kappa_*^{-1} \mu_L \|u^n\|_{0,\infty,\Omega^d} + K_*^{-1}(L_f^d) + \frac{\mu_L}{\kappa_*} \|f^d(c^n)\|_{0,\infty,\Omega^d} \Big)^2 \|c^m - c_h^{m-1}\|_{\Omega^d}^2 \\
&+ \Delta t \sum_{m=1}^{n} \|u^m\|_{0,\infty,\Omega^s}^2 \|\bar{c}^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \\
&+ \Delta t \sum_{m=1}^{n} \big((L_f^s)^2 + \mu_L^2 \|\nabla u^n\|_{0,\infty,\Omega^s}^2 \big) \|c_h^{m-1} - c^m\|_{\Omega^s}^2 \Big],
\end{aligned}
$$

where C depends on $\mu_L, \kappa_*, K_*, \gamma^j$, and L_f^s. Note that using Lemma 4 iv., we have

$$
\|\partial_t \xi_u^n\|_{L^2(0,T;L^2(\Omega^s))} \leq C h^{k_f} \|\partial_t u^n\|_{L^2(0,T;H^{k_f}(\Omega^s))}.
$$

Furthermore, $\|f^d(c^n)\|_{0,\infty,\Omega^d} \leq \|u^n\|_{0,\infty,\Omega^d} + K^* \|\nabla p^n\|_{0,\infty,\Omega^d}$. Therefore,

$$
\begin{aligned}
&\|\zeta_u^n\|_{\Omega^s}^2 + (\Delta t)^2 \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 + C_a \Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{\zeta}_u^m|\!\|_v^2 \leq C\Big[(\Delta t)^2 \|\partial_{tt} u\|_{L^2(0,T;L^2(\Omega^s))}^2 \\
&+ h^{2k_f} \Big(\|\partial_t u\|_{L^2(0,T;[H^{k_f}(\Omega^s)]^{\dim})}^2 + \|u\|_{\ell^2(0,T;H^{k_f+1}(\Omega^s))}^2 + \|u\|_{\ell^2(0,T;[H^{k_f}(\Omega^d)]^{\dim})}^2 \Big) \\
&+ \|\nabla u\|_{L^\infty(0,T;L^\infty(\Omega^s))}^2 \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}^s} \big(h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \big) \\
&+ \big(\|u\|_{L^\infty(0,T;L^\infty(\Omega^d))}^2 + \|\nabla p\|_{L^\infty(0,T;L^\infty(\Omega^d))}^2 + 1 \big) \Delta t \sum_{m=1}^{n} \|c^m - c_h^{m-1}\|_{\Omega^d}^2 \\
&+ \|u\|_{L^\infty(0,T;L^\infty(\Omega^s))}^2 \Delta t \sum_{m=1}^{n} \|\bar{c}^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \\
&+ \big(1 + \|\nabla u\|_{L^\infty(0,T;L^\infty(\Omega^s))} \big) \Delta t \sum_{m=1}^{n} \|c^m - c_h^{m-1}\|_{\Omega}^2 \Big].
\end{aligned}
$$

Equation (4.68) follows by eqs. (4.58) to (4.60), and assumption eq. (4.67).

Proof of eq. (4.69)*:*

Let $v_h^s = 0$, $\bar{v}_h = 0$, and $\boldsymbol{q}_h = \mathbf{0}$ in eq. (4.51). Then

$$\begin{aligned} a_h^d(c_h^{n-1}; \zeta_u^n, v_h) &+ b_h^d(\boldsymbol{\zeta}_p^{dn}, v_h) \\ &= a_h^d(c_h^{n-1}; \xi_u^n, v_h) + a_h^d(c^n; u^n, v_h) - a_h^d(c_h^{n-1}; u^n, v_h) \\ &\quad + \sum_{K \in \mathcal{T}^d} \int_K [\mathbb{K}^{-1}(c_h^{n-1}) f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n) f^d(c^n)] \cdot v_h \, \mathrm{d}x. \end{aligned} \tag{4.81}$$

On the other hand, letting $\boldsymbol{v}_h = \mathbf{0}$ and $\boldsymbol{q}_h^s = \mathbf{0}$ in eq. (4.51), we have

$$b_h^d(\boldsymbol{q}_h^d, \zeta_u^n) + b_h^{I,d}(\bar{q}_h^d, \bar{\zeta}_u^n) = 0.$$

This implies that

$$b_h^d(\boldsymbol{q}_h^d, \zeta_u^n) = -b_h^{I,d}(\bar{q}_h^d, \bar{\zeta}_u^n) = \int_{\Gamma^I} \bar{q}_h^d \bar{\zeta}_u^n \cdot n^d \, \mathrm{d}s \tag{4.82}$$

From [29, Lemma 3.2], since $u^s - u_h^s \in H^{\mathrm{div}}(\Omega^s)$, there exists $w \in H^{\mathrm{div}}(\Omega^d)$ such that

$$\nabla \cdot w = 0 \text{ in } \Omega^d, \quad w \cdot n = 0 \text{ on } \Gamma^d, \text{ and } w \cdot n^d = (\gamma(u^s) - u_h^s) \cdot n^d \text{ on } \Gamma^I.$$

With this choice of w, and using Lemma 4 i., ii., and iii. we observe that

$$b_h^d(\boldsymbol{q}_h^d, \Pi_V w) = \int_{\Gamma^I} \bar{q}_h^d (u^s - u_h^s) \cdot n^d \, \mathrm{d}s. \tag{4.83}$$

Adding eqs. (4.82) and (4.83), and recalling eq. (4.49), the definition of $\bar{\Pi}_V$, and Lemma 4 ii., we obtain

$$b_h^d(\boldsymbol{q}_h^d, \zeta_u^n + \Pi_V w) = 0. \tag{4.84}$$

This points us to the correct choice of v_h^d to test eq. (4.81) with. Letting $v_h^d = \zeta_u^n + \Pi_V w$ in eq. (4.81) and using eqs. (4.8) and (4.84) yields

$$
\begin{aligned}
\frac{1}{K^*}\|\zeta_u^n\|^2_{\Omega^d} &\le a_h^d(c_h^{n-1};\zeta_u^n,\zeta_u^n) = -a_h^d(c_h^{n-1};\zeta_u^n,\Pi_V w)\\
&+ a_h^d(c_h^{n-1};\xi_u^n,\zeta_u^n+\Pi_V w) + \Big(a_h^d(c^n;u^n,\zeta_u^n+\Pi_V w) - a_h^d(c_h^{n-1};u^n,\zeta_u^n+\Pi_V w)\Big)\\
&\quad + \sum_{K\in\mathcal{T}^d}\int_K [\mathbb{K}^{-1}(c_h^{n-1})f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n)f^d(c^n)]\cdot(\zeta_u^n+\Pi_V w)\,\mathrm{d}x\\
&=: A_1+A_2+A_3+A_4. \qquad (4.85)
\end{aligned}
$$

Below, we will bound A_1 to A_4 by a series of Cauchy-Schwarz, Hölder's, triangle, and Young's inequalities together with the properties of μ and κ.

First,

$$
\begin{aligned}
A_1 = -a_h^d(c_h^{n-1};\zeta_u^n,\Pi_V w) &\le \frac{1}{K_*}\|\zeta_u^n\|_{\Omega^d}\|\Pi_V w\|_{\Omega^d}\\
&\le \epsilon\|\zeta_u^n\|^2_{\Omega^d} + \frac{1}{4\epsilon K_*^2}\|\Pi_V w\|^2_{\Omega^d},
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 = a_h^d(c_h^{n-1};\xi_u^n,\zeta_u^n+\Pi_V w) &\le \frac{1}{K_*}\|\xi_u^n\|_{\Omega^d}\|\zeta_u^n+\Pi_V w\|_{\Omega^d}\\
&\le \frac{1}{4\epsilon K_*^2}\|\xi_u^n\|^2_{\Omega^d} + \epsilon\|\zeta_u^n\|^2_{\Omega^d} + \frac{1}{K_*}\|\xi_u^n\|_{\Omega^d}\|\Pi_V w\|_{\Omega^d}.
\end{aligned}
$$

Following the proof of eq. (4.76),

$$
\begin{aligned}
A_3 &= \sum_{K\in\mathcal{T}^d}\int_K [\mathbb{K}^{-1}(c^n) - \mathbb{K}^{-1}(c_h^{n-1})]u^n\cdot(\zeta_u^n+\Pi_V w)\,\mathrm{d}x\\
&\le \frac{\mu_L}{\kappa_*}\|c^n - c_h^{n-1}\|_{\Omega^d}\|u^n\|_{0,\infty,\Omega^d}\|\zeta_u^n+\Pi_V w\|_{\Omega^d}\\
&\le \frac{\mu_L}{\kappa_*}\|c^n - c_h^{n-1}\|_{\Omega^d}\|u^n\|_{0,\infty,\Omega^d}\big(\|\zeta_u^n\|_{\Omega^d} + \|\Pi_V w\|_{\Omega^d}\big)\\
&\le \frac{\mu_L^2}{4\epsilon\kappa_*^2}\|c^n - c_h^{n-1}\|^2_{\Omega^d}\|u^n\|^2_{0,\infty,\Omega^d} + \epsilon\|\zeta_u^n\|^2_{\Omega^d}\\
&\quad + \frac{\mu_L}{\kappa_*}\|c^n - c_h^{n-1}\|_{\Omega^d}\|u^n\|_{0,\infty,\Omega^d}\|\Pi_V w\|_{\Omega^d}.
\end{aligned}
$$

As in the proof of eq. (4.80),

$$\begin{aligned}
A_4 = &\sum_{K\in\mathcal{T}^d}\int_K [\mathbb{K}^{-1}(c_h^{n-1})f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n)f^d(c^n)]\cdot(\zeta_u^n + \Pi_V w)\,\mathrm{d}x\\
\leq&\Big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L_f^d\Big)\|c_h^{n-1} - c^n\|_{\Omega^d}\|\zeta_u^n + \Pi_V w\|_{\Omega^d}\\
\leq&\Big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L_f^d\Big)\|c_h^{n-1} - c^n\|_{\Omega^d}\big(\|\zeta_u^n\|_{\Omega^d} + \|\Pi_V w\|_{\Omega^d}\big)\\
\leq&\frac{1}{4\epsilon}\Big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L_f^d\Big)^2\|c_h^{n-1} - c^n\|_{\Omega^d}^2 + \epsilon\|\zeta_u^n\|_{\Omega^d}^2\\
&+\Big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L_f^d\Big)\|c_h^{n-1} - c^n\|_{\Omega^d}\|\Pi_V w\|_{\Omega^d}.
\end{aligned}$$

To conclude the proof we need a bound for $\|\Pi_V w\|_{\Omega^d}$ which is straightforward to deduce from Lemma 4 iv. and [16, Lemma 10].

$$\|\Pi_V w\|_{\Omega^d} \leq C\|w\|_{\Omega^d} \leq Ch^{k_f+1}\big(\|u^n\|_{k_f+1,\Omega^s} + \|u\|_{k_f,\Omega^d} + \|g_p - g_i\|_{k_f,\Omega^d}\big). \tag{4.86}$$

Combining this with the bounds of A_1 to A_4, choosing $\epsilon = \dfrac{1}{8K^*}$, and applying another set of Young's inequalities, we obtain

$$\|\zeta_u^n\|_{\Omega^d}^2 \leq C\big(\|\Pi_V w\|_{\Omega^d}^2 + \|\xi_u^n\|_{\Omega^d}^2 + \|c^n - c_h^{n-1}\|_{\Omega^d}^2\big),$$

where C depends on the problem parameters $\kappa_*, K_*, K^*, \mu_L, L_f^d$, and the regularity of u, p and c. Then eq. (4.86), Lemma 4 iv., and eq. (4.66) implies

$$\begin{aligned}
\|\zeta_u^n\|_{\Omega^d}^2 \leq C\Big(&h^{2k_f+2}\big(\|u^n\|_{k_f+1,\Omega^s}^2 + \|u^n\|_{k_f,\Omega^d}^2 + \|g_p - g_i\|_{k_f,\Omega^d}^2\big)\\
&+ h^{2k_f}\|u^n\|_{k_f,\Omega^d}^2 + (\Delta t)^2\|d_t c^n\|_{\Omega^d}^2 + \|c^{n-1} - c_h^{n-1}\|_{\Omega^d}^2\Big).
\end{aligned}$$

The result is a consequence of the assumptions on the regularity of the exact solution and eq. (4.67).

□

The following is a straightforward consequence of Theorem 13.

Corollary 1. *Let u^n and u_h^n be as defined in Theorem 13. Then for all $n \geq 1$,*

$$\|u^n - u_h^n\|_\Omega \leq C(\Delta t + h^{k_f} + h^{k_c}) \tag{4.87}$$

$$\sum_{K\in\mathcal{T}_h} h_K \|u^n - u_h^n\|_{\partial K}^2 \leq C((\Delta t)^2 + h^{2k_f} + h^{2k_c}). \tag{4.88}$$

Proof. By the triangle inequality, Lemma 4 iv., and eqs. (4.68) and (4.69),

$$\begin{aligned}\|u^n - u_h^n\|_\Omega &\leq \|\xi_u^n\|_\Omega + \|\zeta_u^n\|_\Omega \\ &\leq Ch^{k_f}\|u^n\|_{k_f,\Omega} + C((\Delta t) + h^{k_f} + h^{k_c}) \quad \leq C((\Delta t) + h^{k_f} + h^{k_c}).\end{aligned}$$

Now we prove eq. (4.88). Using eqs. (2.29), (2.31), (2.51), (4.68) and (4.69) together with Lemma 4 iv., we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} h_K \|u^n - u_h^n\|_{\partial K}^2 &\leq C \sum_{K\in\mathcal{T}_h} h_K (\|\xi_u^n\|_{\partial K}^2 + \|\zeta_u^n\|_{\partial K}^2) \\ &\leq C \sum_{K\in\mathcal{T}_h} (\|\xi_u^n\|_K^2 + h_K^2 \|\xi_u^n\|_{1,K}^2 + \|\zeta_u^n\|_K^2) \\ &\leq C((\Delta t)^2 + h^{2k_f} + h^{2k_c}).\end{aligned}$$

This completes the proof.

□

Before moving on to the next section, we note another consequence of eqs. (4.68) and (4.69) that we will use to prove the results about the discrete concentration solution. The proof is the same as in [15, Lemma 1].

Corollary 2. *Let u denote the velocity solution to eqs.* (4.1) *and* (4.3) *satisfying the assumptions in Theorem 13 with $k_f, k_c \geq \dim - 1$. Suppose $\Delta t \leq Ch_{k_f}^{\dim/2}$. Then for each $n \geq 1$, the discrete velocity u_h^n that solves eq.* (4.46) *satisfies*

$$\|u_h^n\|_{0,\infty,\Omega} \leq C, \tag{4.89}$$

where C depends on $\mu^*, \kappa_*, \mu_L, L_f^s, L_f^d, \sum_{j=1}^{\dim-1} \gamma^j$, and the regularity of u_0, u, c_0, and c but is independent of h, n or Δt.

Proof. Using eqs. (4.68) and (4.69), we have

$$\|\zeta_u^n\|_\Omega^2 \leq C((\Delta t)^2 + h^{2k_f} + h^{2k_c}). \tag{4.90}$$

Combining the above estimate with eq. (2.39) and Lemma 4 iv., for each $K \in \mathcal{T}$, we have

$$\begin{aligned}
\|u_h^n\|_{0,\infty,K} &\leq \|\zeta_u^n\|_{0,\infty,K} + \|\Pi_V u^n\|_{0,\infty,K} \\
&\leq C h_K^{-\dim/2} \|\zeta_u^n\|_K + \|\Pi_V u^n\|_{0,\infty,K} \\
&\leq C h_K^{-\dim/2} \big((\Delta t) + h^{k_f} + h^{k_c}\big) + \|u^n\|_{0,\infty,K} \\
&\leq C h_K^{-\dim/2} \big((\Delta t) + h^{k_f} + h^{k_c}\big) + \|u\|_{L^\infty(0,T;L^\infty(K))}.
\end{aligned}$$

The result eq. (4.89) follows by using the assumption on Δt and by taking the maximum over $K \in \mathcal{T}$. □

Remark 2. *We can achieve order k_f convergence in the energy norm if we restrict $k_c = k_f \geq \dim - 1$. However, this is not a big restriction since $k_f = 2$ coincides with the standard Taylor-Hood approximation for the velocity-pressure and for DG methods, higher order polynomial approximations are easy to implement. Furthermore, this restriction is not necessary if we assume that $|D(u)| \leq \bar{D}$ for some positive constant $\bar{D}$ as in [46, 2.12]. It is also possible to avoid it by using an approach involving a cutoff operator on the velocity solution as in [50, 51] if one is interested in lower order approximations. On the other hand, compatibility, as defined in [23], can be achieved by choosing $k_c = k_f - 1$, $k_f \geq \dim$ while the order of convergence in the energy norm reduces to $k_f - 1$. We note that, when $\dim = 3$, our theory supports compatibility only for $k_f \geq 3$ due to the restriction $k_c \geq \dim - 1$.*

4.2.1.3 Error estimate for the pressure In this section, we briefly discuss the a priori estimate for the pressure approximation.

Lemma 12. *Suppose that the assumptions in Theorem 13 holds and p and $\{\boldsymbol{p}_h^n\}$ are the pressure components of the solution of eqs.* (4.1), (4.3), (4.4) *and* (4.46), *respectively. Then*

$$\Delta t\sum_{m=1}^{n}|||\boldsymbol{\zeta}_p^m|||_p^2 \le C\Bigg(\Delta t\sum_{m=1}^{n}|||\boldsymbol{\zeta}_u^m|||_v^2+\Delta t\sum_{m=1}^{n}\|d_t\zeta_u^m\|_{\Omega^s}^2+\Delta t\sum_{m=1}^{n}|||\boldsymbol{\zeta}_c^{m-1}|||_c^2$$
$$+\Delta t\sum_{m=1}^{n}\sum_{K\in\mathcal{T}}\|c^{m-1}-c_h^{m-1}\|_K^2+\Delta t\sum_{m=1}^{n}\sum_{K\in\mathcal{T}^s}h_K^2\|c^{m-1}-c_h^{m-1}\|_{1,K}^2$$
$$+\Delta t\sum_{m=1}^{n}\|c^{m-1}-\bar{c}_h^{m-1}\|_{\Gamma^I}^2+(\Delta t)^2+h^{2k_f}+h^{2k_c+1}\Bigg). \tag{4.91}$$

Proof. Setting $\boldsymbol{q}_h=0$ in the error equation in Theorem 11, we obtain

$$\sum_{j=s,d}\big(b_h^j(\zeta_p^{jn},v_h)+b_h^{I,j}(\bar{\zeta}_p^{jn},\bar{v}_h)\big)$$
$$=\big(a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\xi}_u^n,\boldsymbol{v}_h)-a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\zeta}_u^n,\boldsymbol{v}_h)\big)$$
$$+\big(a_h(\boldsymbol{c}^n;\boldsymbol{u}^n,\boldsymbol{v}_h)-a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{u}^n,\boldsymbol{v}_h)\big)+\int_{\Omega^s}[f^s(c_h^{n-1})-f^s(c^n)]\cdot v_h\,\mathrm{d}x$$
$$+\int_{\Omega^d}[\mathbb{K}^{-1}(c^{n-1})f^d(c_h^{n-1})-\mathbb{K}^{-1}(c^n)f^d(c_h^n)]\cdot v_h\,\mathrm{d}x$$
$$-\sum_{K\in\mathcal{T}^s}\int_K(d_tu_h^n-\partial_t\,u^n)\cdot v_h\,\mathrm{d}x=:H_1+\ldots+H_5. \tag{4.92}$$

By Theorem 9 and Young's inequality, and noting that $|||\cdot|||_v$ and $|||\cdot|||_{v'}$ are equivalent on $\boldsymbol{V}_h$, we have

$$H_1=\big(a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\xi}_u^n,\boldsymbol{v}_h)-a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{\zeta}_u^n,\boldsymbol{v}_h)\big)$$
$$\le C(|||\boldsymbol{\zeta}_u^n|||_v+|||\boldsymbol{\xi}_u^n|||_{v'})|||\boldsymbol{v}_h|||_v. \tag{4.93}$$

Following the proof of eq. (4.78), we can show that

$$
\begin{aligned}
H_2 &= \big(a_h(\boldsymbol{c}^n;\boldsymbol{u}^n,\boldsymbol{v}_h) - a_h(\boldsymbol{c}_h^{n-1};\boldsymbol{u}^n,\boldsymbol{v}_h)\big) \\
&\le C\Bigg(\mu_L\|\nabla u^n\|_{0,\infty,\Omega^s}\Big(\sum_{K\in\mathcal{T}^s}(\|c^n - c_h^{n-1}\|_K^2 + h_K^2\|c^n - c_h^{n-1}\|_{1,K}^2)\Big)^{1/2} \\
&\quad + \kappa_*^{-1}\mu_L\|u^n\|_{0,\infty,\Omega^d}\|c^n - c_h^{n-1}\|_{\Omega^d} \\
&\quad + \mu_L\Big(\sum_{j=1}^{\dim-1}\gamma^j\Big)^{1/2}\|u^n\|_{0,\infty,\Omega^s}\|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}\Bigg)|||\boldsymbol{v}_h|||_v. \qquad (4.94)
\end{aligned}
$$

As in eq. (4.79), we have

$$
\begin{aligned}
H_3 &= \int_{\Omega^s}[f^s(c_h^{n-1}) - f^s(c^n)]\cdot v_h \,\mathrm{d}x \\
&\le L_f^s\|c_h^{n-1} - c^n\|_{\Omega^s}|||\boldsymbol{v}_h|||_v. \qquad (4.95)
\end{aligned}
$$

Next from eq. (4.80), it follows that

$$
\begin{aligned}
H_4 &= \int_{\Omega^d}[\mathbb{K}^{-1}(c_h^{n-1})f^d(c_h^{n-1}) - \mathbb{K}^{-1}(c^n)f^d(c^n)]\cdot v_h \,\mathrm{d}x \\
&\le \Big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L_f^d\Big)\|c_h^{n-1} - c^n\|_{\Omega^d}|||\boldsymbol{v}_h|||_v. \qquad (4.96)
\end{aligned}
$$

Using Cauchy-Schwarz inequality and Lemma 4 iv.,

$$
\begin{aligned}
H_5 &= -\sum_{K\in\mathcal{T}^s}\int_K(d_t u_h^n - \partial_t\, u^n)\cdot v_h \,\mathrm{d}x \\
&= -\sum_{K\in\mathcal{T}^s}\int_K d_t\zeta_u^n\cdot v_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}^s}\int_K(d_t\Pi_V u^n - \partial_t\Pi_V u^n)\cdot v_h \,\mathrm{d}x \\
&\quad + \sum_{K\in\mathcal{T}^s}\int_K\partial_t\xi_u^n\cdot v_h \,\mathrm{d}x \\
&\le \big(\|d_t\zeta_u^n\|_{\Omega^s} + \|d_t\Pi_V u^n - \partial_t\Pi_V u^n\|_{\Omega^s} + \|\partial_t\xi_u^n\|_{\Omega^s}\big)\|v_h\|_{\Omega^s} \\
&\le \big(\|d_t\zeta_u^n\|_{\Omega^s} + \|d_t\Pi_V u^n - \partial_t\Pi_V u^n\|_{\Omega^s} + Ch^{k_f}\|\partial_t u^n\|_{k_f,\Omega^s}\big)|||\boldsymbol{v}_h|||_v. \qquad (4.97)
\end{aligned}
$$

Therefore, combining (4.92)-(4.97), dividing both sides by $|||\boldsymbol{v}_h|||_v$, taking the supremum over $\boldsymbol{v}_h$, and using Theorem 2, we get

$$
\begin{aligned}
c^{\star}_{\inf}|\!|\!|\boldsymbol{\zeta}^n_p|\!|\!|_p \leq & \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq\boldsymbol{0}}} \frac{\sum_{j=s,d}(b^j_h(\boldsymbol{\zeta}^{jn}_p, v_h)+b^{I,j}_h(\bar{\zeta}^{jn}_p,\bar{v}_h))}{|\!|\!|\boldsymbol{v}_h|\!|\!|_v} \\
\leq & C\Bigg(|\!|\!|\boldsymbol{\zeta}^n_u|\!|\!|_v + |\!|\!|\boldsymbol{\xi}^n_u|\!|\!|_{v'} \\
& + \mu_L\|\nabla u^n\|_{0,\infty,\Omega^s}\Big(\sum_{K\in\mathcal{T}^s}(\|c^n-c^{n-1}_h\|^2_K + h^2_K\|c^n-c^{n-1}_h\|^2_{1,K})\Big)^{1/2} \\
& + \mu_L\big(\sum_{j=1}^{\dim-1}\gamma^j\big)^{1/2}\|u^n\|_{0,\infty,\Omega^s}\|c^n-\bar{c}^{n-1}_h\|_{\Gamma^I} + L^s_f\|c^{n-1}_h-c^n\|_{\Omega^s} \\
& + \big(\frac{\mu_L}{\kappa_*}\|f^d(c^n)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L^d_f + \kappa^{-1}_*\mu_L\|u^n\|_{0,\infty,\Omega^d}\big)\|c^{n-1}_h-c^n\|_{\Omega^d} \\
& + \|d_t\zeta^n_u\|_{\Omega^s} + \|d_t\Pi_V u^n - \partial_t\Pi_V u^n\|_{\Omega^s} + Ch^{k_f}\|\partial_t u^n\|_{k_f,\Omega^s}\Bigg).
\end{aligned}
$$

Squaring both sides, multiplying by $(c^*_{inf})^{-2}\Delta t$, summing from 1 to n, using eq. (C.2),

$$
\begin{aligned}
\Delta t\sum_{m=1}^n |\!|\!|\boldsymbol{\zeta}^m_p|\!|\!|^2_p \leq & C\Bigg(\Delta t\sum_{m=1}^n(|\!|\!|\boldsymbol{\zeta}^m_u|\!|\!|^2_v + |\!|\!|\boldsymbol{\xi}^m_u|\!|\!|^2_{v'}) \\
& + \mu^2_L\Delta t\sum_{m=1}^n\|\nabla u^m\|^2_{0,\infty,\Omega^s}\sum_{K\in\mathcal{T}^s}(\|c^m-c^{m-1}_h\|^2_K + h^2_K\|c^m-c^{m-1}_h\|^2_{1,K}) \\
& + \Delta t\sum_{m=1}^n\mu^2_L\big(\sum_{j=1}^{\dim-1}\gamma^j\big)\|u^m\|^2_{0,\infty,\Omega^s}\|c^m-\bar{c}^{m-1}_h\|^2_{\Gamma^I} + \Delta t\sum_{m=1}^n(L^s_f)^2\|c^{m-1}_h-c^m\|^2_{\Omega^s} \\
& + \Delta t\sum_{m=1}^n\big(\frac{\mu_L}{\kappa_*}\|f^d(c^m)\|_{0,\infty,\Omega^d} + \frac{1}{K_*}L^d_f + \kappa^{-1}_*\mu_L\|u^m\|_{0,\infty,\Omega^d}\big)^2\|c^{m-1}_h-c^m\|^2_{\Omega^d} \\
& + \Delta t\sum_{m=1}^n\|d_t\zeta^m_u\|^2_{\Omega^s} + (\Delta t)^2\|\partial_{tt}\Pi_V u\|^2_{L^2(0,T;[L^2(\Omega^s)]^{\dim})} \\
& + \Delta t h^{2k_f}\|\partial_t u\|^2_{\ell^2(0,T;[H^{k_f}(\Omega^s)]^{\dim}}\Bigg).
\end{aligned}
$$

This implies by eq. (2.53), stability of Π_V and under the assumption that $u^s\in L^\infty(0,T;[W^{1,\infty}(\Omega^s)]^{\dim})$, $u^d\in L^\infty(0,T;[L^\infty(\Omega^d)]^{\dim})$,

$\nabla p^d \in L^\infty(0,T;[L^\infty(\Omega^d)]^{\dim})$ that

$$\begin{aligned}
\Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_p^m|||_p^2 \le C \Bigg(& \Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_u^m|||_v^2 + \Delta t \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 \\
& + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}} \|c^m - c_h^{m-1}\|_K^2 + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}^s} h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \\
& + \Delta t \sum_{m=1}^{n} \|c^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \\
& + h^{2k_f} \left(\|u\|^2_{\ell^2(0,T;[H^{k_f+1}(\Omega^s)]^{\dim})} + \|u\|^2_{L^2(0,T;[H^{k_f}(\Omega^d)]^{\dim})} \right) \\
& + (\Delta t)^2 \|\partial_{tt} u\|^2_{L^2(0,T;[L^2(\Omega^s)]^{\dim})} + \Delta t h^{2k_f} \|\partial_t u\|^2_{\ell^2(0,T;[H^{k_f}(\Omega^s)]^{\dim}} \Bigg).
\end{aligned}$$

Then by eqs. (4.58) to (4.60),

$$\begin{aligned}
\Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_p^m|||_p^2 \le C \Bigg(& \Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_u^m|||_v^2 + \Delta t \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 + \Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_c^{m-1}|||_c^2 \\
& + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}} \|c^{m-1} - c_h^{m-1}\|_K^2 + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}^s} h_K^2 \|c^{m-1} - c_h^{m-1}\|_{1,K}^2 \\
& + h^{2k_f} \left(\|u\|^2_{\ell^2(0,T;[H^{k_f+1}(\Omega^s)]^{\dim})} + \|u\|^2_{L^2(0,T;[H^{k_f}(\Omega^d)]^{\dim})} \right) \\
& + (\Delta t)^2 \left(\|\partial_{tt} u\|^2_{L^2(0,T;[L^2(\Omega^s)]^{\dim})} + \|\partial_t c\|^2_{L^2(0,T;H^1(\Omega^s))} + \|\partial_t c\|^2_{L^2(0,T;L^2(\Omega))} \right) \\
& + \Delta t h^{2k_f} \|\partial_t u\|^2_{\ell^2(0,T;[H^{k_f}(\Omega^s)]^{\dim})} \\
& + \Delta t h^{2k_c+1} \|c_0\|^2_{k_c+1,\Omega^s} \\
& + h^{2k_c+1} \|c\|^2_{L^2(0,T;H^{k_c+1}(\Omega^s))} \Bigg).
\end{aligned}$$

Therefore, the result follows under the assumptions on the exact solution given in Theorem 13.

$\square$

Remark 3. *An immediate consequence of this lemma and Theorem 13 is*

$$
\begin{aligned}
\Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_p^m|||_p^2 \leq C\Big(&\Delta t + (\Delta t)^{-1}(h^{2k_f} + h^{2k_c}) + \Delta t \sum_{m=1}^{n} |||\boldsymbol{\zeta}_c^{m-1}|||_c^2 \\
&+ \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}} \|c^{m-1} - c_h^{m-1}\|_K^2 + \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}^s} h_K^2 \|c^{m-1} - c_h^{m-1}\|_{1,K}^2\Big).
\end{aligned}
$$

This loss of Δt is due to $\Delta t \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2$ in eq. (4.91). There is a need to bound this term which can be obtained by testing eq. (4.51) with $\boldsymbol{v}_h = d_t \boldsymbol{\zeta}_u^m$ if one needs an improved estimate. One can then proceed by multiplying by Δt, summing from $m = 1$ to n and employing a summation by parts on the terms on the right hand side that contains $a_h(\cdot;\cdot, d_t \boldsymbol{\zeta}_u^m)$ to transfer the discrete time derivative on $d_t \boldsymbol{\zeta}_u^m$ to the other terms. This and assuming that the exact solution is sufficiently smooth in time would provide the much needed bound for $\Delta t \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2$, that is,

$$
\Delta t \sum_{m=1}^{n} \|d_t \zeta_u^m\|_{\Omega^s}^2 \leq C(h^{2k_f} + h^{2k_c} + (\Delta t)^2).
$$

We do not provide the details of this proof here and refer to [18, p.42] for a similar discussion.

4.2.1.4 Existence and uniqueness of $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ In this section, we will prove existence and uniqueness of the discrete concentration solution. First observe that, the assumption eq. (4.5b) on D implies that for $v \in [L^\infty(\Omega^d)]^{\dim}$,

$$
\|D(v)\|_{0,\infty,\Omega^d} \leq C(1 + \|v\|_{0,\infty,\Omega^d}). \tag{4.98}
$$

and for $v \in [W^{1,\infty}(K)]^{\dim}$, $K \in \mathcal{T}$, by eq. (2.30)

$$
\|D(v)\|_{0,\infty,\partial K} \leq C(1 + \|v\|_{0,\infty,\partial K}) \leq C, \tag{4.99}
$$

where C depends on $\|v\|_{1,\infty,K}$.

We have

$$\|\tilde{D}(u_h^n)\|_{0,\infty,\Omega} = \begin{cases} \|d\mathbb{I}\|_{0,\infty,\Omega^s} & \text{in } \Omega^s \\ \|D(u_h^n)\|_{0,\infty,\Omega^d} & \text{in } \Omega^d \end{cases}$$
$$\leq \max\{d, \|D(u_h^n)\|_{0,\infty,\Omega^d}\}. \tag{4.100}$$

Similarly,

$$\|\tilde{D}(u_h^n)\|_{0,\infty,\partial K} \leq \max\{d, \|D(u_h^n)\|_{0,\infty,(\partial K\cap\Omega^d)}\} \tag{4.101}$$

Therefore by eqs. (4.89) and (4.98) to (4.101), there exists a constant $\tilde{D}_{\max} > 0$ that depends on d and the upper bound in eq. (4.89) such that

$$\|\tilde{D}(u_h^n)\|_{0,\infty,\Omega} \leq \tilde{D}_{\max}, \quad \|\tilde{D}(u_h^n)\|_{0,\infty,\partial K} \leq \tilde{D}_{\max}. \tag{4.102}$$

With eq. (4.102), the following coercivity result can be proved following the same steps in [15, Lemma 2, Lemma 3].

Theorem 14 (Coercivity of $B_h^{tr}(u_h^n; \boldsymbol{w}_h, \boldsymbol{w}_h)$)**.** *There exists a constant $\beta_0 > 0$ such that if $\beta_{tr} > \beta_0^{tr}$, then for all $\boldsymbol{w}_h \in \boldsymbol{C}_h$,*

$$B_h^{tr}(u_h^n; \boldsymbol{w_h}, \boldsymbol{w_h}) \geq C_{tr}|||\boldsymbol{w}_h|||_c^2 + \frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n w_h^2 \,\mathrm{d}x, \tag{4.103}$$

where $C_{tr} > 0$ is a constant that depends on $d, D_{\min}$, and the upper bound in eq. (4.89)*.*

Now that we have coercivity, we proceed with the existence and stability proof for the discrete concentration.

Theorem 15. *Let $c_0 \in L^2(\Omega), g_i \in L^2(0,T;L^2(\Omega^d))$, $g_i - g_p \in L^\infty(0,T;L^\infty(\Omega^d))$ (or $\nabla\cdot u \in L^\infty(0,T;L^\infty(\Omega^d))$). Let $n \geq 1$ and assume that $\boldsymbol{u}_h^n$ is the solution of eq.* (4.46) *satisfying eqs.* (4.68) *and* (4.69)*. If $d_n\Delta t < 1$, where*

$d_n =: \frac{1}{\phi}(1 + C\|g_i^n - g_p^n\|_{0,\infty,\Omega^d})$, *then there exists a unique solution* $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ *to eq.* (4.47)*. Furthermore,*

$$\phi_* \|c_h^n\|_\Omega^2 + C_{tr}\Delta t \sum_{m=1}^{n} |||\boldsymbol{c}_h^m|||_c^2 \leq e^{K\Delta t}\big(\phi_*\|c_0\|_\Omega^2 + \|g_i\|^2_{\ell^2(0,T;L^2(\Omega^d))}\big), \tag{4.104}$$

where $K := \sum\limits_{m=1}^{n} \frac{d_m}{1-\Delta t d_m}$.

Proof. Let $\boldsymbol{w}_h = \boldsymbol{c}_h^n$ in eq. (4.47). From the algebraic inequality $(a-b)a \geq \frac{1}{2}(a^2 - b^2)$, eq. (4.103), and the assumption that $0 \leq c_I \leq 1$ a.e., we have

$$\begin{aligned}
\frac{\phi_*}{2\Delta t}\left(\|c_h^n\|_\Omega^2 - \|c_h^{n-1}\|_\Omega^2\right) + C_{tr}|||\boldsymbol{c}_h^n|||_c^2 + \int_{\Omega^d} g_p^n (c_h^n)^2 \,\mathrm{d}x& \\
\leq \sum_{K\in\mathcal{T}^d}\int_K c_I g_i^n c_h^n \,\mathrm{d}x - \frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n (c_h^n)^2 \,\mathrm{d}x& \\
\leq \|g_i^n\|_{\Omega^d}\|c_h^n\|_{\Omega^d} + \frac{1}{2}\|\nabla\cdot u_h^n\|_{0,\infty,\Omega^d}\|c_h^n\|^2_{\Omega^d}.&
\end{aligned}$$

Multiplying this inequality by $2\Delta t$, summing from $m = 1$ to n, noting that $g_p \geq 0$, and recalling eq. (4.48) with stability of the L^2-projections Π_C, and Π_Q, we obtain

$$\begin{aligned}
&\phi_*\|c_h^n\|_\Omega^2 + 2C_{tr}\Delta t\sum_{m=1}^{n}|||\boldsymbol{c}_h^m|||_c^2 \qquad (4.105)\\
&\leq \phi_*\|c_h^0\|_\Omega^2 + 2\Delta t\sum_{m=1}^{n}\|g_i^m\|_{\Omega^d}\|c_h^m\|_{\Omega^d} + \Delta t\sum_{m=1}^{n}\|\nabla\cdot u_h^m\|_{0,\infty,\Omega^d}\|c_h^m\|^2_{\Omega^d}\\
&\leq \phi_*\|c_h^0\|_\Omega^2 + \Delta t\sum_{m=1}^{n}(\|g_i^m\|^2_{\Omega^d} + \|c_h^m\|^2_{\Omega^d}) + \Delta t\sum_{m=1}^{n}\|\Pi_Q(g_i^m - g_p^m)\|_{0,\infty,\Omega^d}\|c_h^m\|^2_{\Omega^d}\\
&\leq \phi_*\|c_0\|_\Omega^2 + \|g_i^m\|^2_{\ell^2(0,T;L^2(\Omega^d))} + \Delta t\sum_{m=1}^{n}(1 + C\|g_i^m - g_p^m\|_{0,\infty,\Omega^d})\|c_h^m\|^2_\Omega.
\end{aligned}$$

Applying Lemma C.2, we obtain eq. (4.104). This stability bound implies the existence of a unique solution since the system is finite dimensional and linear. Indeed, let $n \geq 1$ and assume that $\boldsymbol{c}_h^n$ and $\tilde{\boldsymbol{c}}_h^n \in \boldsymbol{C}_h$ both solve eq. (4.47). Then, $\{\boldsymbol{e}_h^n\}_n$, where $\boldsymbol{e}_h^n = \boldsymbol{c}_h^n - \tilde{\boldsymbol{c}}_h^n$ solves eq. (4.47) with $c_h^0 = 0, g_i^n = 0$. Therefore, by eq. (4.104),

$$\phi_* \|e_h^n\|_\Omega^2 + \Delta t \sum_{m=1}^{n} \|\!|\boldsymbol{e}_h^m|\!\|_c^2 \leq 0. \tag{4.106}$$

Hence, owing to the fact that $\phi_* > 0$, $\boldsymbol{e}_h^n = 0$, i.e, $\boldsymbol{c}_h^n = \tilde{\boldsymbol{c}}_h^n$.

□

4.2.1.5 Error estimate for the discrete concentration

We start by deriving an error equation.

Lemma 13 (Error equation for eq. (4.47))**.**

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \int_K \phi\, d_t \zeta_c^n w_h \,\mathrm{d}x + B_h^{tr}(u_h^n; \boldsymbol{\zeta}_c^n, \boldsymbol{w}_h) + \int_{\Omega^d} g_p^n \zeta_c^n w_h dx \qquad (4.107)\\
&= \sum_{K\in\mathcal{T}} \int_K \phi\, d_t \xi_c^n w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_K \phi\,(d_t c^n - \partial_t c^n) w_h \,\mathrm{d}x + B_h^{tr}(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{w}_h)\\
&+ B_h^a(u^n - u_h^n; \boldsymbol{c}^n, \boldsymbol{w}_h) + B_h^d(u^n; \boldsymbol{c}^n, \boldsymbol{w}_h) - B_h^d(u_h^n; \boldsymbol{c}^n, \boldsymbol{w}_h) + \int_{\Omega^d} g_p^n \xi_c^n w_h dx.
\end{aligned}$$

Proof. By Lemma 10, for $t = t^n$,

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \int_K \phi\, \partial_t c^n w_h \;\mathrm{d}x + B_h^{tr}(u^n; \boldsymbol{c}^n, \boldsymbol{w}_h) + \int_{\Omega^d} g_p^n c^n w_h \;\mathrm{d}x\\
&\qquad = \sum_{K\in\mathcal{T}^d} \int_K c_I\, g_i^n\, w_h \,\mathrm{d}x \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h, \qquad (4.108)
\end{aligned}$$

where $\boldsymbol{c}^n = (c^n, \bar{c}^n)$.

Subtracting eq. (4.108) from eq. (4.47) yields

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \int_K \phi\,(d_t c_h^n - \partial_t c^n) w_h \,\mathrm{d}x + B_h^{tr}(u_h^n; \boldsymbol{c}_h^n, \boldsymbol{w}_h) - B_h^{tr}(u^n; \boldsymbol{c}^n, \boldsymbol{w}_h)\\
&\qquad + \int_{\Omega^d} g_p^n (c_h^n - c^n) w_h \,\mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{C}_h. \qquad (4.109)
\end{aligned}$$

Next, we rewrite the terms with B_h^{tr} in eq. (4.109) by observing that B_h^{tr} is linear in the second slot and also its advective part is linear with respect to the first slot.

$$\begin{aligned}
B_h^{tr}(u_h^n;\boldsymbol{c}_h^n,\boldsymbol{w}_h) - B_h^{tr}(u^n;\boldsymbol{c}^n,\boldsymbol{w}_h) & \\
=& B_h^{tr}(u_h^n;\boldsymbol{c}_h^n-\boldsymbol{c}^n,\boldsymbol{w}_h) + \left[B_h^{tr}(u_h^n;\boldsymbol{c}^n,\boldsymbol{w}_h) - B_h^{tr}(u^n;\boldsymbol{c}^n,\boldsymbol{w}_h)\right] \\
=& B_h^{tr}(u_h^n;\boldsymbol{c}_h^n-\boldsymbol{c}^n,\boldsymbol{w}_h) + B_h^{a}(u_h^n-u^n;\boldsymbol{c}^n,\boldsymbol{w}_h) \\
&+ \left[B_h^{d}(u_h^n;\boldsymbol{c}^n,\boldsymbol{w}_h) - B_h^{d}(u^n;\boldsymbol{c}^n,\boldsymbol{w}_h)\right].
\end{aligned} \tag{4.110}$$

Using eq. (4.55), the linearity of B_h^{tr} in the second slot, and eqs. (4.109) and (4.110) completes the proof. □

Theorem 16. *In addition to the assumptions in Theorem 13, suppose that*

$$c_0 \in H^{k_c}(\Omega), \quad c \in L^2(0,T;H^{k_c+1}(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega)),$$

$$\partial_t c \in L^2(0,T;H^{k_c}(\Omega)), \quad \partial_{tt} c \in L^2(0,T;L^2(\Omega)), \quad g_i, g_p \in L^\infty(0,T;L^\infty(\Omega^d)).$$

Then

$$\|\zeta_c^n\|_\Omega^2 + \Delta t \sum_{m=1}^n \|\!|\boldsymbol{\zeta}_c^m|\!\|_c^2 \le C(h^{2k_f} + h^{2k_c} + (\Delta t)^2), \tag{4.111}$$

where C depends on ϕ_*, ϕ^*, d, D and the regularity of the solution but is independent of h or Δt.

Proof. Setting $\boldsymbol{w}_h = \boldsymbol{\zeta}_c^n$ in Lemma 13, using inequality $a(a-b) \ge \frac{a^2-b^2}{2}$ and the coercivity in Theorem 14,

$$\begin{aligned}
&\frac{\phi_*}{2\Delta t}(\|\zeta_c^n\|_\Omega^2 - \|\zeta_c^{n-1}\|_\Omega^2) + C_{tr}\|\!|\zeta_c^n|\!\|_c^2 + \int_{\Omega^d} g_p^n(\zeta_c^n)^2 dx \\
&\le \sum_{K\in\mathcal{T}}\int_K \phi\, d_t\xi_c^n\zeta_c^n\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_K \phi\,(\partial_t c^n - d_t c^n)\zeta_c^n\,\mathrm{d}x + B_h^{tr}(u_h^n;\boldsymbol{\xi}_c^n,\boldsymbol{\zeta}_c^n) \\
&\quad + B_h^a(u^n-u_h^n;\boldsymbol{c}^n,\boldsymbol{\zeta}_c^n) + \left[B_h^d(u^n;\boldsymbol{c}^n,\boldsymbol{\zeta}_c^n) - B_h^d(u_h^n;\boldsymbol{c}^n,\boldsymbol{\zeta}_c^n)\right] \\
&\quad - \frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n(\zeta_c^n)^2\,\mathrm{d}x + \int_{\Omega^d} g_p^n\xi_c^n\zeta_c^n\,\mathrm{d}x \\
&:= I_1 + \ldots + I_7.
\end{aligned} \tag{4.112}$$

Using eq. (4.6a), Cauchy–Schwarz inequality, Young's inequality with constant γ, and eq. (4.56),

$$
\begin{aligned}
I_1 = \sum_{K\in\mathcal{T}} \int_K \phi\, d_t\xi_c^n \zeta_c^n \,\mathrm{d}x &\le \phi^* \Big\| \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \partial_t \xi_c \,\mathrm{d}t \Big\|_\Omega \|\zeta_c^n\|_\Omega \qquad (4.113)\\
&\le \phi^* \frac{1}{\sqrt{\Delta t}} \|\partial_t \xi_c\|_{L^2(t^{n-1},t^n,L^2(\Omega))} \|\zeta_c^n\|_\Omega\\
&\le \frac{\phi^*}{4\phi_*\gamma\Delta t} \|\partial_t\xi_c\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))} + \gamma(\phi_*\|\zeta_c^n\|^2_\Omega)\\
&\le \frac{\phi^* h^{2k_c}}{4\phi_*\gamma\Delta t} \|\partial_t c\|^2_{L^2(t^{n-1},t^n,H^{k_c}(\Omega))} + \gamma(\phi_*\|\zeta_c^n\|^2_\Omega).
\end{aligned}
$$

Again by eq. (4.6a), this time integrating by parts in terms of t and applying Young's inequality,

$$
\begin{aligned}
I_2 = \sum_{K\in\mathcal{T}} \int_K \phi\,(\partial_t c^n - d_t c^n)\zeta_c^n \,\mathrm{d}x &\le \phi^* \Big\| \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} (t - t^{n-1}) \partial_{tt} c \,\mathrm{d}t \Big\|_\Omega \|\zeta_c^n\|_\Omega\\
&\le \frac{\phi^*\Delta t}{12\phi_*\gamma} \|\partial_{tt} c\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))} + \gamma(\phi_*\|\zeta_c^n\|^2_\Omega). \qquad (4.114)
\end{aligned}
$$

The following series of inequalities is dedicated to finding an upper bound for I_3. By definition of B_h^{tr},

$$
I_3 = B_h^{tr}(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n) = B_h^a(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n) + B_h^d(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n). \qquad (4.115)
$$

We will bound B_h^a and B_h^d separately. We start by B_h^a. Noting that $\xi_c^n - \bar{\xi}_c^n$ vanishes on facets, we have

$$
\begin{aligned}
B_h^a(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n) = &- \sum_{K\in\mathcal{T}} \int_K \xi_c^n\, u_h^n \cdot \nabla \zeta_c^n \;\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} \xi_c^n\,(u_h^n - u^n)\cdot n(\zeta_c^n - \bar{\zeta}_c^n)\;\mathrm{d}s\\
&+ \sum_{K\in\mathcal{T}} \int_{\partial K} \xi_c^n\, u^n \cdot n(\zeta_c^n - \bar{\zeta}_c^n)\;\mathrm{d}s =: I_{311} + I_{312} + I_{313}. \qquad (4.116)
\end{aligned}
$$

The term I_{311} is bounded easily by Hölder's inequality, eqs. (4.56) and (4.89):

$$
I_{311} \le C\|u_h^n\|_{0,\infty,\Omega} \|\xi_c^n\|_\Omega \|\nabla\zeta_c^n\|_\Omega \le C h^{k_c} \|c^n\|_{k_c,\Omega} \|\!|\boldsymbol{\zeta}_c^n\|\!|_c. \qquad (4.117)
$$

By Hölder's inequality, assuming $c \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and using eqs. (2.30), (4.56) and (4.88),

$$
\begin{aligned}
I_{312} &= \sum_{K\in\mathcal{T}} \int_{\partial K} \xi_c^n \,(u_h^n - u^n)\cdot n(\zeta_c^n - \bar{\zeta}_c^n)\ \mathrm{d}s \\
&\leq \sum_{K\in\mathcal{T}} \|\xi_c^n\|_{0,\infty,\partial K} \|u_h^n - u^n\|_{\partial K} \|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K} \\
&\leq C\Big(\sum_{K\in\mathcal{T}} h_K \|u_h^n - u^n\|_{\partial K}^2\Big)^{1/2} \big(\sum_{K\in\mathcal{T}} h_K^{-1} \|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K}^2\Big)^{1/2} \\
&\leq C\Big(\sum_{K\in\mathcal{T}} h_K \|u_h^n - u^n\|_{\partial K}^2\Big)^{1/2} |\!|\!|\boldsymbol{\zeta}_c^n|\!|\!|_c.
\end{aligned} \tag{4.118}
$$

Using Hölder's inequality, assuming $u \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and this time employing eqs. (2.31) and (4.56),

$$
\begin{aligned}
I_{313} &= \sum_{K\in\mathcal{T}} \|\xi_c^n\|_{\partial K} \|u^n\|_{0,\infty,\partial K} \|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K} \\
&\leq C\Big(\sum_{K\in\mathcal{T}} (\|\xi_c^n\|_K^2 + h_K^2 \|\xi_c^n\|_{1,K}^2)\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} h_K^{-1} \|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K}^2\Big)^{1/2} \qquad (4.119) \\
&\leq C h^{k_c} \|c^n\|_{k_c,\Omega} |\!|\!|\boldsymbol{\zeta}_c^n|\!|\!|_c.
\end{aligned}
$$

Putting eqs. (4.116) to (4.119) together, we get

$$
\begin{aligned}
|B_h^a(u_h^n;\boldsymbol{\xi}_c^n, \boldsymbol{w}_h)| &\leq C\Big(\big(\sum_{K\in\mathcal{T}} h_K \|u_h^n - u^n\|_{\partial K}^2\big)^{1/2} + h^{k_c}\|c^n\|_{k_c,\Omega}\Big) |\!|\!|\boldsymbol{\zeta}_c^n|\!|\!|_c \\
&\leq \frac{C^2}{4\epsilon}\Big(\sum_{K\in\mathcal{T}} h_K \|u_h^n - u^n\|_{\partial K}^2 + h^{2k_c}\|c^n\|_{k_c,\Omega}^2\Big) + \epsilon |\!|\!|\boldsymbol{\zeta}_c^n|\!|\!|_c^2
\end{aligned} \tag{4.120}
$$

Now we focus on B_h^d in eq. (4.115). Since $\xi_c^n - \bar{\xi}_c^n = 0$ on ∂K,

$$
\begin{aligned}
&B_h^d(u_h^n;\boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n) \\
&= \sum_{K\in\mathcal{T}} \int_K \tilde{D}(u_h^n)\nabla\xi_c^n \cdot \nabla\zeta_c^n\ \mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} [\tilde{D}(u_h^n)\nabla\xi_c^n]\cdot n(\zeta_c^n - \bar{\zeta}_c^n)\ \mathrm{d}s \\
&=: I_{321} + I_{322}.
\end{aligned} \tag{4.121}
$$

By Hölder's inequality, eq. (4.102), and eq. (4.56),

$$I_{321} \leq \tilde{D}_{\max} \|\nabla \xi_c^n\|_\Omega \|\nabla \zeta_c^n\|_\Omega \leq C h^{k_c} \|c\|_{k_c+1,\Omega} |||\boldsymbol{\zeta}_c^n|||_c. \tag{4.122}$$

Again by Hölder's inequality and this time using eqs. (2.31), (4.56) and (4.102),

$$\begin{aligned}
|I_{322}| &\leq \tilde{D}_{\max} \Big(\sum_{K \in \mathcal{T}} h_K \|\nabla \xi_c^n\|_{\partial K}^2 \Big)^{1/2} \Big(\sum_{K \in \mathcal{T}} h_K^{-1} \|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K}^2 \Big)^{1/2} \\
&\leq \tilde{D}_{\max} \Big(\sum_{K \in \mathcal{T}} C_{T,2} (\|\nabla \xi_c^n\|_K^2 + h_K^2 \|\nabla \xi_c^n\|_{1,K}^2) \Big)^{1/2} |||\boldsymbol{\zeta}_c^n|||_c \\
&\leq C \Big(\sum_{K \in \mathcal{T}} h_K^{2k_c} \|c^n\|_{k_c+1,K}^2 \Big)^{1/2} |||\boldsymbol{\zeta}_c^n|||_c \\
&\leq C h^{k_c} \|c^n\|_{k_c+1,\Omega} |||\boldsymbol{\zeta}_c^n|||_c.
\end{aligned} \tag{4.123}$$

Hence, a combination of eqs. (4.121) to (4.123) with Young's inequality gives

$$B_h^d(u_h^n; \boldsymbol{\xi}_c^n, \boldsymbol{\zeta}_c^n) \leq C h^{k_c} \|c^n\|_{k_c+1,\Omega} |||\boldsymbol{\zeta}_c^n|||_c \leq \frac{C^2}{4\epsilon} h^{2k_c} \|c^n\|_{k_c+1,\Omega}^2 + \epsilon |||\boldsymbol{\zeta}_c^n|||_c^2. \tag{4.124}$$

Therefore, from eqs. (4.120) and (4.124)

$$I_3 \leq C h^{2k_c} \|c^n\|_{k_c+1,\Omega}^2 + 2\epsilon |||\boldsymbol{\zeta}_c^n|||_c^2. \tag{4.125}$$

Since $c = \bar{c}$ on ∂K,

$$\begin{aligned}
I_4 &= B_h^a(u^n - u_h^n; \boldsymbol{c}^n, \boldsymbol{\zeta}_c^n) \\
&= -\sum_{K \in \mathcal{T}} \int_K c^n \, (u^n - u_h^n) \cdot \nabla \zeta_c^n \, \mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} c^n \, (u^n - u_h^n) \cdot n (\zeta_c^n - \bar{\zeta}_c^n) \, \mathrm{d}s \\
&=: I_{41} + I_{42}.
\end{aligned} \tag{4.126}$$

Hölder's and Young's inequalities give

$$\begin{aligned}
I_{41} &\leq \|c^n\|_{0,\infty,\Omega} \|u^n - u_h^n\|_\Omega \|\nabla \zeta_c^n\|_\Omega \leq \|u^n - u_h^n\|_\Omega \|c^n\|_{0,\infty,\Omega} |||\boldsymbol{\zeta}_c^n|||_c \\
&\leq \frac{1}{4\epsilon} \|c^n\|_{0,\infty,\Omega}^2 \|u^n - u_h^n\|_\Omega^2 + \epsilon |||\boldsymbol{\zeta}_c^n|||_c^2,
\end{aligned} \tag{4.127}$$

and

$$
\begin{aligned}
I_{42} &\le C\|c^n\|_{0,\infty,\Omega}\Big(\sum_{K\in\mathcal{T}_h} h_K\|u^n-u_h^n\|_{\partial K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\zeta_c^n-\bar{\zeta}_c^n\|_{\partial K}^2\Big)^{1/2} \\
&\le C\|c^n\|_{0,\infty,\Omega}\Big(\sum_{K\in\mathcal{T}_h} h_K\|u^n-u_h^n\|_{\partial K}^2\Big)^{1/2}|||\boldsymbol{\zeta}_c^n|||_c \\
&\le \frac{C^2}{4\epsilon}\|c^n\|_{0,\infty,\Omega}^2\sum_{K\in\mathcal{T}_h} h_K\|u^n-u_h^n\|_{\partial K}^2+\epsilon|||\boldsymbol{\zeta}_c^n|||_c^2. \qquad (4.128)
\end{aligned}
$$

Collecting eqs. (4.126) to (4.128) leads to

$$
I_4 \le 2\epsilon|||\boldsymbol{\zeta}_c^n|||_c^2+C\|c^n\|_{0,\infty,\Omega}^2\Big(\|u^n-u_h^n\|_\Omega^2+\sum_{K\in\mathcal{T}_h} h_K\|u^n-u_h^n\|_{\partial K}^2\Big).
$$

Since $c=\bar{c}$ on element boundaries, and $\tilde{D}(u^n)-\tilde{D}(u_h^n)=0$ in Ω^s,

$$
\begin{aligned}
I_5 = &\sum_{K\in\mathcal{T}^d}\int_K [D(u^n)-D(u_h^n)]\nabla c^n\cdot\nabla\zeta_c^n \,\mathrm{d}x \\
&-\sum_{K\in\mathcal{T}^d}\int_{\partial K}[(D(u^n)-D(u_h^n))\nabla c^n]\cdot n\,(\zeta_c^n-\bar{\zeta}_c^n)\,\mathrm{d}s \\
=:&I_{51}+I_{52}. \qquad (4.129)
\end{aligned}
$$

Using Lipschitz property eq. (4.5c) of D, Hölder's and Young's inequalities,

$$
\begin{aligned}
I_{51} &\le C\|u^n-u_h^n\|_{\Omega^d}\|\nabla c^n\|_{0,\infty,\Omega^d}\|\nabla\zeta_c^n\|_{\Omega^d} \\
&\le \frac{C^2}{4\epsilon}\|\nabla c^n\|_{0,\infty,\Omega^d}^2\|u^n-u_h^n\|_{\Omega^d}^2+\epsilon|||\boldsymbol{\zeta}_c^n|||_c^2. \qquad (4.130)
\end{aligned}
$$

To bound I_{52}, we first rearrange it as follows using eq. (4.54):

$$
\begin{aligned}
I_{52} = &\sum_{K\in\mathcal{T}^d}\int_{\partial K}[D(u_h^n)\nabla\xi_c^n]\cdot n\,(\zeta_c^n-\bar{\zeta}_c^n)\,\mathrm{d}s \\
&+\sum_{K\in\mathcal{T}^d}\int_{\partial K}[(D(u_h^n)-D(u^n))\nabla\mathcal{I}c^n]\cdot n\,(\zeta_c^n-\bar{\zeta}_c^n)\,\mathrm{d}s \\
&-\sum_{K\in\mathcal{T}^d}\int_{\partial K}[D(u^n)\nabla\xi_c^n)]\cdot n\,(\zeta_c^n-\bar{\zeta}_c^n)\,\mathrm{d}s =: I_{521}+I_{522}+I_{523}.
\end{aligned}
$$

From eq. (4.123),

$$
\begin{aligned}
I_{521} &= \sum_{K\in\mathcal{T}^d} \int_{\partial K} [D(u_h^n)\nabla \xi_c^n]\cdot n\,(\zeta_c^n - \bar{\zeta}_c^n)\ \mathrm{d}s \\
&\leq Ch^{k_c}\|c^n\|_{k_c+1,\Omega^d} |\!|\!| \boldsymbol{\zeta}_c^n |\!|\!|_c. \qquad (4.131)
\end{aligned}
$$

Using Lipschitz property eq. (4.5c) of D, Hölder's inequality, eq. (4.57), and eq. (4.88),

$$
\begin{aligned}
I_{522} &= \sum_{K\in\mathcal{T}^d} \int_{\partial K} [(D(u_h^n) - D(u^n))\nabla \mathcal{I}c^n]\cdot n(\zeta_c^n - \bar{\zeta}_c^n)\ \mathrm{d}s \\
&\leq \sum_{K\in\mathcal{T}^d} C\|u^n - u_h^n\|_{\partial K}\|\nabla\mathcal{I}c^n\|_{0,\infty,K}\|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K} \\
&\leq C\|c^n\|_{1,\infty,\Omega^d}\Big(\sum_{K\in\mathcal{T}^d} h_K\|u^n - u_h^n\|_{\partial K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}^d} h_K^{-1}\|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K}^2\Big)^{1/2} \\
&\leq C\|c^n\|_{1,\infty,\Omega^d}\Big(\sum_{K\in\mathcal{T}^d} h_K\|u^n - u_h^n\|_{\partial K}^2\Big)^{1/2} |\!|\!| \boldsymbol{\zeta}_c^n |\!|\!|_c. \qquad (4.132)
\end{aligned}
$$

Using Hölder's inequality, eqs. (2.31), (4.56) and (4.102),

$$
\begin{aligned}
I_{523} &= \sum_{K\in\mathcal{T}^d} \int_{\partial K} [D(u^n)\nabla \xi_c^n]\cdot n(\zeta_c^n - \bar{\zeta}_c^n)\ \mathrm{d}s \\
&\leq \tilde{D}_{\max}\Big(\sum_{K\in\mathcal{T}^d} h_K\|\nabla\xi_c^n\|_{\partial K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}^d} h_K^{-1}\|\zeta_c^n - \bar{\zeta}_c^n\|_{\partial K}^2\Big)^{1/2} \\
&\leq \tilde{D}_{\max}\Big(\sum_{K\in\mathcal{T}^d} C_{T,2}\big(\|\nabla\xi_c^n\|_K^2 + h_K^2\|\nabla\xi_c^n\|_{1,K}^2\big)\Big)^{1/2} |\!|\!| \boldsymbol{\zeta}_c^n |\!|\!|_c \\
&\leq Ch^{k_c}\|c^n\|_{k_c+1,\Omega^d} |\!|\!| \boldsymbol{\zeta}_c^n |\!|\!|_c. \qquad (4.133)
\end{aligned}
$$

Combining eqs. (4.131) to (4.133) and using Young's inequality, we have

$$
I_{52} \leq C\Big(h^{2k_c}\|c^n\|_{k_c+1,\Omega^d}^2 + \|c^n\|_{1,\infty,\Omega^d}^2 \sum_{K\in\mathcal{T}^d} h_K\|u^n - u_h^n\|_{\partial K}^2\Big) + \epsilon |\!|\!| \boldsymbol{\zeta}_c^n |\!|\!|_c^2. \qquad (4.134)
$$

Therefore, substituting eqs. (4.130) and (4.134) in eq. (4.129) yields

$$
\begin{aligned}
I_5 \leq 2\epsilon \|\!|\boldsymbol{\zeta}_c^n\|\!|_c^2 + C\Big(h^{2k_c}\|c^n\|_{k_c+1,\Omega^d}^2 + \|c^n\|_{1,\infty,\Omega^d}^2 \sum_{K\in\mathcal{T}^d} h_K \|u^n - u_h^n\|_{\partial K}^2 \\
+ \|\nabla c^n\|_{0,\infty,\Omega^d}^2 \|u^n - u_h^n\|_{\Omega^d}^2\Big). \qquad (4.135)
\end{aligned}
$$

By eq. (4.48), the stability of the L^2-projection Π_Q and Hölder's inequality,

$$
\begin{aligned}
I_6 &= -\frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n(\zeta_c^n)^2 \,\mathrm{d}x \\
&= -\frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \Pi_Q(g_i^n - g_p^n)(\zeta_c^n)^2 \,\mathrm{d}x \\
&\leq \frac{1}{2}\|g_i^n - g_p^n\|_{0,\infty,\Omega^d}\|\zeta_c^n\|_{\Omega^d}^2 \leq \frac{1}{2\phi_*}\|g_i^n - g_p^n\|_{0,\infty,\Omega^d}(\phi_*\|\zeta_c^n\|_\Omega^2). \qquad (4.136)
\end{aligned}
$$

Finally, using Hölder's inequality, eq. (4.56), and Young's inequality,

$$
\begin{aligned}
I_7 &\leq \|g_p^n\|_{0,\infty,\Omega^d}\|\xi_c^n\|_{\Omega^d}\|\zeta_c^n\|_{\Omega^d} \\
&\leq Ch^{k_c}\|g_p^n\|_{0,\infty,\Omega^d}\|c^n\|_{k_c,\Omega^d}\|\zeta_c^n\|_{\Omega^d} \\
&\leq \frac{C^2h^{2k_c}}{4\gamma\phi_*}\|g_p^n\|_{0,\infty,\Omega^d}^2\|c^n\|_{k_c,\Omega^d}^2 + \gamma(\phi_*\|\zeta_c^n\|_\Omega^2). \qquad (4.137)
\end{aligned}
$$

Collecting all bounds, choosing $\epsilon = \dfrac{C_{tr}}{12}$ (C_{tr} is the coercivity constant), $\gamma = \dfrac{1}{6}$, and recalling $g_p^n \geq 0$, we get

$$
\begin{aligned}
&\frac{\phi_*}{2\Delta t}(\|\zeta_c^n\|_\Omega^2 - \|\zeta_c^{n-1}\|_\Omega^2) + \frac{C_{tr}}{2}\|\!|\zeta_c^n\|\!|_c^2 \\
&\quad\leq C\Bigg(\frac{\phi^* h^{2k_c}}{\phi_*\Delta t}\|\partial_t c\|_{L^2(t^{n-1},t^n;H^{k_c}(\Omega))}^2 + \frac{\phi^*\Delta t}{\phi_*}\|\partial_{tt}c\|_{L^2(t^{n-1},t^n;L^2(\Omega))}^2 \qquad (4.138) \\
&\qquad + h^{2k_c}\|c^n\|_{k_c+1,\Omega}^2 + \|c^n\|_{1,\infty,\Omega}^2\Big(\|u^n - u_h^n\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K\|u^n - u_h^n\|_{\partial K}^2\Big) \\
&\qquad + \frac{h^{2k_c}}{\phi_*}\|g_p^n\|_{0,\infty,\Omega^d}^2\|c^n\|_{k_c,\Omega^d}^2\Bigg) + \frac{1}{2}\Big(\frac{1}{\phi_*}\|g_i^n - g_p^n\|_{0,\infty,\Omega^d} + 1\Big)(\phi_*\|\zeta_c^n\|_\Omega^2).
\end{aligned}
$$

Multiplying by $2\Delta t$, summing over m, and using Corollary 1,

$$
\begin{aligned}
&\phi_*\|\zeta_c^n\|_\Omega^2 + C_{tr}\Delta t \sum_{m=1}^{n} \|\!|\zeta_c^m|\!\|_c^2 \\
\leq &\phi_*\|\zeta_c^0\|_\Omega^2 + C\Big[h^{2k_c}\big(\frac{\phi^*}{\phi_*}\|\partial_t c\|^2_{L^2(0,T;H^{k_c}(\Omega))} + \|c\|^2_{\ell^2(0,T;H^{k_c+1}(\Omega))}\big) \\
&+ \frac{\phi^*}{\phi_*}(\Delta t)^2\|\partial_{tt}c\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \Delta t \sum_{m=1}^{n}\Big(\frac{1}{\phi_*}\|g_i^m - g_p^m\|_{0,\infty,\Omega^d} + 1\Big)(\phi_*\|\zeta_c^m\|_\Omega^2) \\
&+ \Delta t \sum_{m=1}^{n}\|c^m\|^2_{1,\infty,\Omega}\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big) \\
&+ h^{2k_c}\Delta t \sum_{m=1}^{n}\frac{1}{\phi_*}\|g_p^m\|^2_{0,\infty,\Omega^d}\|c^m\|^2_{k_c,\Omega^d}\Big].
\end{aligned}
$$

Note that using [24, Lemma 1.58] and eq. (4.56),

$$
\|\zeta_c^0\|_\Omega = \|c_h^0 - \mathcal{I}c_0\|_\Omega \leq \|\Pi_C c_0 - c_0\|_\Omega + \|c_0 - \mathcal{I}c_0\|_\Omega \leq Ch^{k_c}\|c_0\|_{k_c,\Omega}. \tag{4.139}
$$

Therefore, assuming that Δt is sufficiently small and using Lemma C.2 with

$$
a_n = \phi_*\|\zeta_c^n\|_\Omega^2, \quad b_n = C_{tr}\|\!|\zeta_c^n|\!\|_c^2, \quad d_n = \big(\phi_*^{-1}\|g_i^m - g_p^m\|_{0,\infty,\Omega^d} + 1\big),
$$

$$
c_n = \|c^m\|^2_{1,\infty,\Omega}\big((\Delta t)^2 + h^{2k} + h^{2k_c}\big) + h^{2k_c}\phi_*^{-1}\|g_p^m\|^2_{0,\infty,\Omega^d}\|c^m\|^2_{k_c,\Omega^d},
$$

$$
\begin{aligned}
B = C\Big(h^{2k_c}\big(\phi_*\|c_0\|^2_{k_c,\Omega} + \frac{\phi^*}{\phi_*}\|\partial_t c\|^2_{L^2(0,T;H^{k_c}(\Omega))} + \|c\|^2_{\ell^2(0,T;H^{k_c+1}(\Omega))}\big) \\
+ \frac{\phi^*}{\phi_*}(\Delta t)^2\|\partial_{tt}c\|^2_{L^2(0,T;L^2(\Omega))}\Big),
\end{aligned}
$$

we obtain

$$
\phi_*\|\zeta_c^n\|_\Omega^2 + C_{tr}\Delta t\sum_{m=1}^{n}\|\!|\zeta_c^m|\!\|_c^2 \leq C\left((\Delta t)^2 + h^{2k_f} + h^{2k_c}\right), \tag{4.140}
$$

where C depends on the regularity of the exact solution and the parameters of the problem. $\square$

4.2.2 Numerical experiments

In this section, we use the NGSolve finite element library [49] to implement the numerical examples for the convergence analysis. Let $\Omega = [0,1]^2$ be the unit square with two subregions $\Omega^d = [0,1] \times [0, 0.5]$ and $\Omega^s = [0,1] \times [0.5, 1]$ and $[0,1]$ be the time interval.

4.2.2.1 Example 1: one-way coupling with constant μ, κ, D

We consider $k_f = 3$, $k_c = k_f - 1 = 2$. Let $\alpha = \sqrt{\kappa}(1 + 4\pi^2)/2$, and $D = \begin{bmatrix} 0.01 & 0.005 \\ 0.005 & 0.02 \end{bmatrix}$. We solve the Stokes-Darcy-transport problem with the exact solution given by

$$u^s = (-\frac{1}{2\pi^2} \sin(\pi x_1 + t) e^{(x_2+t)/2}, \frac{1}{\pi} \cos(\pi x_1 + t) e^{(x_2+t)/2})^T, \tag{4.141a}$$

$$u^d = (-2 \sin(\pi x_1 + t) e^{(x_2+t)/2}, \frac{1}{\pi} \cos(\pi x_1 + t) e^{(x_2+t)/2})^T, \tag{4.141b}$$

$$p^s = \frac{\kappa\mu - 2}{\kappa\pi} \cos(\pi x_1 + t) e^{(x_2+t)/2}, \tag{4.141c}$$

$$p^d = -\frac{2}{\kappa\pi} \cos(\pi x_1 + t) e^{(x_2+t)/2}, \tag{4.141d}$$

$$c = \sin(2\pi(x_1 - t)) \cos(2\pi(x_2 - t)). \tag{4.141e}$$

For penalty parameters, we use $\beta_s = 6k_f^2$ and $\beta_{tr} = 6k_c^2$.

Note that this exact solution is chosen to satisfy all of the interface conditions and $\nabla \cdot u^s = 0$ in Ω^s but it does not satisfy homogeneous boundary conditions. In the computation we add extra terms on the right hand side to make them equal.

We present three cases: $\kappa = 1$, $\mu = 10^{-6}$, $\kappa = 10^3$, $\mu = 10^{-6}$ and $\mu = \kappa = 1$.

Tables 4.1, 4.3 and 4.5 present computed errors in the velocity and the pressure in Ω^s and Ω^d for $\kappa = 10^3$, $\mu = 10^{-6}$, $\kappa = 1$, $\mu = 10^{-6}$ and $\kappa = 1$, $\mu = 1$, respectively. We observe optimal rates of convergence for all unknowns, independent of the value of the viscosity and permeability. Furthermore, the magnitude of the

Table 4.1: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 10^3$, $\mu = 10^{-6}$, using BDF2 method in Example 1, Section 4.2.2.1

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega^s}$	Order	$\|p - p_h\|_{\Omega^s}$	Order	$\|\nabla \cdot u_h\|_{\Omega^s}$
1/2	8	2.4e-04	–	4.2e-06	–	5.7e-10
1/4	28	2.2e-05	3.5	7.3e-07	2.5	2.1e-12
1/8	152	6.0e-07	5.2	5.7e-08	3.7	7.5e-15
1/16	576	3.3e-08	4.2	6.4e-09	3.2	4.5e-15
1/32	2348	2.0e-09	4.1	7.9e-10	3.0	1.4e-13
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-06	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-07	3.0	2.2e-12
1/8	152	3.6e-06	5.1	4.0e-08	3.8	8.2e-13
1/16	576	2.2e-07	4.0	5.1e-09	3.0	3.1e-12
1/32	2348	1.3e-08	4.1	6.1e-10	3.1	1.2e-11

pressure error is affected by the reduction of the permeability κ from 10^3 to 1 assuming a fixed small viscosity $\mu = 10^{-6}$. However, the pressure robustness of the method shields the velocity error from this change in the pressure error. Moreover, the last columns in the tables demonstrate that pointwise mass conservation is satisfied in both regions.

The error in L^2 norm and rates of convergence for concentration at final time $t = 0.1$ are presented in Tables 4.2, 4.4 and 4.6 with $\kappa = 10^3$, $\mu = 10^{-6}$; $\kappa = 1$, $\mu = 10^{-6}$ and $\kappa = 1$, $\mu = 1$, respectively. We obtain optimal rates of convergence for c_h.

4.2.2.2 Example 2: one-way coupling with constant μ, κ velocity dependent D In this example, we add the influence of the velocity on the dispersion-diffusion

Table 4.2: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 10^3$, $\mu = 10^{-6}$, using BDF2 in Example 1, Section 4.2.2.1.

h	Cells	$\lVert c - c_h \rVert_\Omega$	Order
1/2	8	1.2e-01	–
1/4	28	2.1e-02	2.5
1/8	152	1.8e-03	3.6
1/16	576	2.2e-04	3.0
1/32	2348	2.5e-05	3.1

tensor and define

$$D(u) = \begin{bmatrix} 1 + u_1^2 & 0 \\ 0 & 1 + u_2^2 \end{bmatrix}.$$

The source terms and boundary conditions for the Stokes-Darcy-transport problem (4.1) are chosen such that the exact solution is given by eq. (4.141). We set $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, We present our results for either small viscosity or small permeability as they can potentially cause issues.

Tables 4.7, 4.9 and 4.11 present computed errors in the velocity and the pressure in Ω^s and Ω^d for $\kappa = 1$, $\mu = 1$, $\kappa = 1$, $\mu = 10^{-6}$, and $\kappa = 10^3$, $\mu = 10^{-6}$, respectively. We see that for three cases, the rates of convergence for the velocity and the pressure are optimal, independent of the value of the viscosity and permeability. We also obtain optimal rates of convergence for c_h as shown in Tables 4.8, 4.10 and 4.12 for all considered cases. In addition, pointwise satisfaction of machine precision in the mass equation is observed through the last columns in Tables 4.7, 4.9 and 4.11 in both Stokes and Darcy regions.

Table 4.3: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 10^{-6}$, using BDF2 method in Example 1, Section 4.2.2.1

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega^s}$	Order	$\|p - p_h\|_{\Omega^s}$	Order	$\|\nabla \cdot u_h\|_{\Omega^s}$
1/2	8	2.5e-04	–	3.5e-03	–	5.7e-10
1/4	28	2.2e-05	3.5	6.9e-04	2.3	2.0e-12
1/8	152	6.1e-07	5.2	5.6e-05	3.6	3.8e-15
1/16	576	3.3e-08	4.2	6.4e-06	3.1	8.8e-14
1/32	2348	2.0e-09	4.1	7.7e-07	3.1	9.4e-14
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-03	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-04	3.0	2.2e-12
1/8	152	3.5e-06	5.1	4.0e-05	3.8	7.2e-13
1/16	576	2.2e-07	4.0	5.1e-06	3.0	3.0e-12
1/32	2348	1.3e-08	4.1	6.1e-07	3.1	1.2e-11

4.2.2.3 Example 3: two-way coupling with constant κ, concentration dependent μ, velocity dependent D

Changing μ from a constant to a concentration dependent function will have serious consequences on the rate of convergence for the velocity. Indeed, we see this effect in our a priori estimate in the energy norm. The convergence rate of the velocity approximation cannot exceed the convergence rate dictated by the concentration convergence rate. Since we prefer compatible discretizations ($k_c = k_f - 1$), for the velocity we can only expect an order of $k_f - 1$ in the energy norm and k_f in the L^2-norm. In this example we assume that the viscosity is defined by the quarter-power mixing rule:

$$\mu(c) = \mu_0\big((\mu_0/\mu_1)^{1/4}c + (1-c)\big)^{-4}, \tag{4.142}$$

Table 4.4: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 10^{-6}$, using BDF2 in Example 1, Section 4.2.2.1.

h	Cells	$\|\|c - c_h\|\|_\Omega$	Order
1/2	8	1.2e-01	–
1/4	28	2.1e-02	2.5
1/8	152	1.8e-03	3.6
1/16	576	2.2e-04	3.0
1/32	2348	2.5e-05	3.1

where $\mu_0 = 0.9$, $\mu_1 = 1.3$. The figure on the left in Figure 4.1 shows a plot of $\mu(c)$ as a function of c. The dispersion/diffusion tensor $D(u)$ is defined as in Section 4.2.2.2.

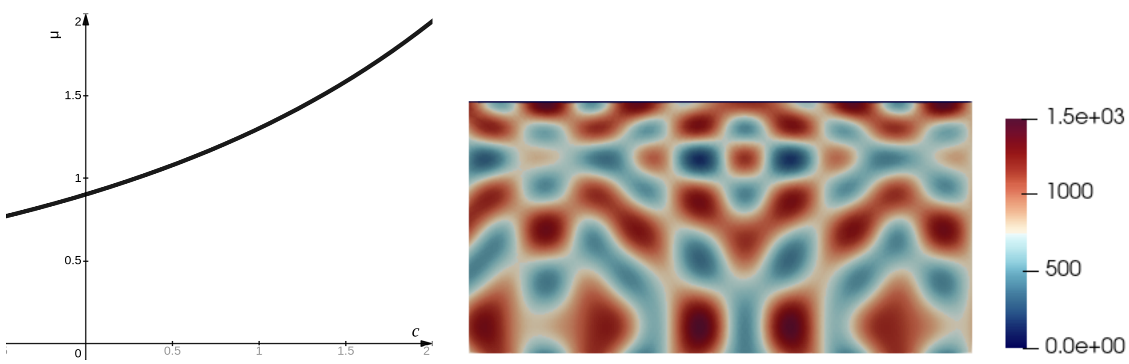


Figure 4.1: The viscosity function $\mu(c)$ (left) defined by eq. (4.142) for Sections 4.2.2.3 and 4.2.2.4 and the permeability field (right) described by eq. (4.144) in the Darcy region $[0, 1] \times [0, 0.5]$ for Section 4.2.2.4.

We set $k_f = 2$, $k_c = 1$, $\Delta t = 10^{-5}$, $T = 0.1$, $\kappa = 1$. In this example, we use BDF3 instead of BDF2. BDF3 is not A-stable but it is close. In fact, its absolute stability region contains the entire negative real line. So it still works as well as the A-stable

Table 4.5: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 1$, using BDF2 method in Example 1, Section 4.2.2.1

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega^s}$	Order	$\|p - p_h\|_{\Omega^s}$	Order	$\|\nabla \cdot u_h\|_{\Omega^s}$
1/2	8	4.3e-04	–	3.1e-02	–	5.7e-10
1/4	28	5.6e-05	3.0	4.6e-03	2.8	2.0e-12
1/8	152	1.4e-06	5.3	2.1e-04	4.4	1.7e-14
1/16	576	7.1e-08	4.3	2.1e-05	3.4	9.3e-14
1/32	2348	3.6e-09	4.3	2.3e-06	3.2	1.0e-13
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	2.0e-03	–	4.2e-03	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-04	3.0	2.2e-12
1/8	152	3.6e-06	5.1	4.0e-05	3.8	7.2e-13
1/16	576	2.2e-07	4.0	5.1e-06	3.0	3.0e-12
1/32	2348	1.3e-08	4.1	6.1e-07	3.1	1.2e-11

methods BE and BDF2. We initialize BDF3 by solving a step of BE, then a step of BDF2.

We introduce the BDF3 operator $d_t^{(3)}$ such that, for all $n \in \{3, \dots, N\}$,

$$d_t^{(3)} u^n = \frac{11u^n - 18u^{n-1} + 9u^{n-2} - 2u^{n-3}}{6\Delta t}. \tag{4.143}$$

In Tables 4.13 and 4.14 and Tables 4.15 and 4.16, we present the results for computed errors in the velocity, the pressure and concentration using BDF3 with $k_f = 2$ and $k_f = 3$, respectively. The approximate concentration converges with an order of $k_c + 1 = k_f$ and the approximate velocity converges with an order of at least k_f as expected in both cases. The velocity field u_h is pointwise divergence-free on the cells.

Table 4.6: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 1$, using BDF2 in Example 1, Section 4.2.2.1.

h	Cells	$\|\|c - c_h\|\|_\Omega$	Order
$1/2$	8	1.2e-01	–
$1/4$	28	2.1e-02	2.5
$1/8$	152	1.8e-03	3.6
$1/16$	576	2.2e-04	3.0
$1/32$	2348	2.5e-05	3.1

4.2.2.4 Example 4: Realistic example

In this section, we now test our method on a more realistic example where the solution is not known and the permeability field in the Darcy region is highly heterogeneous. The domain and the boundary conditions for the flow problem are as defined in Section 2.3.2.2 and we set the permeability to

$$\kappa(x, y) = 700(1+0.5(\sin(10\pi x)\cos(20\pi y^2)+\cos(6.4\pi x)\cos(6.4\pi x)\sin(9.2\pi y)))+100, \tag{4.144}$$

and $\mu(c)$ as in eq. (4.142) as seen in Figure 4.1. The porosity takes different values in the Stokes and Darcy regions, that is, $\phi = 1$ in Ω^s and $\phi = 0.4$ in Ω^d. The dispersion/diffusion tensor is

$$D(u) = \begin{cases} \delta I, & \text{in } \Omega^s, \\ \phi d_m \mathbb{I} + d_l |u| \mathbb{T} + d_t |u| (\mathbb{I} - \mathbb{T}), & \text{in } \Omega^d, \end{cases}$$

where d_l, d_t, and d_m represent longitudinal and transverse dispersivities and the molecular diffusivity, respectively, $\mathbb{I}$ denotes the identity matrix, and $\mathbb{T} = uu^T/|u|^2$ and u^T is the transpose of the vector u. Under the assumption of $d_l \geq d_t$, $D(u)$ satisfies the assumptions that we made in our analysis [25, 50].

Table 4.7: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 1$, using BDF2 method in Example 2, Section 4.2.2.2

			Stokes solution (Ω^s)			
h	Cells	$\Vert u - u_h\Vert_{\Omega^s}$	Order	$\Vert p - p_h\Vert_{\Omega^s}$	Order	$\Vert\nabla\cdot u_h\Vert_{\Omega^s}$
1/2	8	4.3e-04	–	3.1e-02	–	5.7e-10
1/4	28	5.6e-05	3.0	4.6e-03	2.8	2.0e-12
1/8	152	1.4e-06	5.3	2.1e-04	4.4	1.7e-14
1/16	576	7.1e-08	4.3	2.1e-05	3.4	9.3e-14
1/32	2348	3.6e-09	4.3	2.3e-06	3.2	1.0e-13
			Darcy solution (Ω^d)			
h	Cells	$\Vert u - u_h\Vert_{\Omega_d}$	Order	$\Vert p - p_h\Vert_{\Omega_d}$	Order	$\Vert\nabla\cdot u_h + \Pi_Q g^d\Vert_{\Omega_d}$
1/2	8	2.0e-03	–	4.2e-03	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-04	3.0	2.2e-12
1/8	152	3.6e-06	5.1	4.0e-05	3.8	7.2e-13
1/16	576	2.2e-07	4.0	5.1e-06	3.0	3.0e-12
1/32	2348	1.3e-08	4.1	6.1e-07	3.1	1.2e-11

In this numerical experiment, we choose $\delta = 10^{-6}$, $d_m = 1.e-5$, $d_l = 1.e-5$, and $d_t = 1.e-5$ and assume that the initial concentration of the plume of contaminant is

$$c_0(x) = \begin{cases} 0.95 & \text{if } \sqrt{(x_1-0.2)^2 + (x_2-0.7)^2} < 0.1, \\ 0.05 & \text{otherwise.} \end{cases}$$

Finally, we set $k_f = 3$, $k_c = 2$, $\alpha = 0.5$, $T = 20$, $\Delta t = 10^{-3}$, $h = 1/80$ and all of the source/sink terms to zero. The results are obtained by using BDF2 time stepping.

Figure 4.2 shows the computed velocity field after one time step and at the final time. In the Darcy region Ω^d, the flow field avoids the areas with low permeability.

Figure 4.3 presents the plume of contaminant spreading through the surface water region and infiltrating the porous medium with snapshots at 6 different

Table 4.8: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 1$, using BDF2 in Example 2, Section 4.2.2.2.

h	Cells	$\|\|c - c_h\|\|_\Omega$	Order
1/2	8	1.2e-01	–
1/4	28	2.8e-02	2.9
1/8	152	2.3e-03	3.6
1/16	576	2.5e-04	3.2
1/32	2348	2.6e-05	3.2

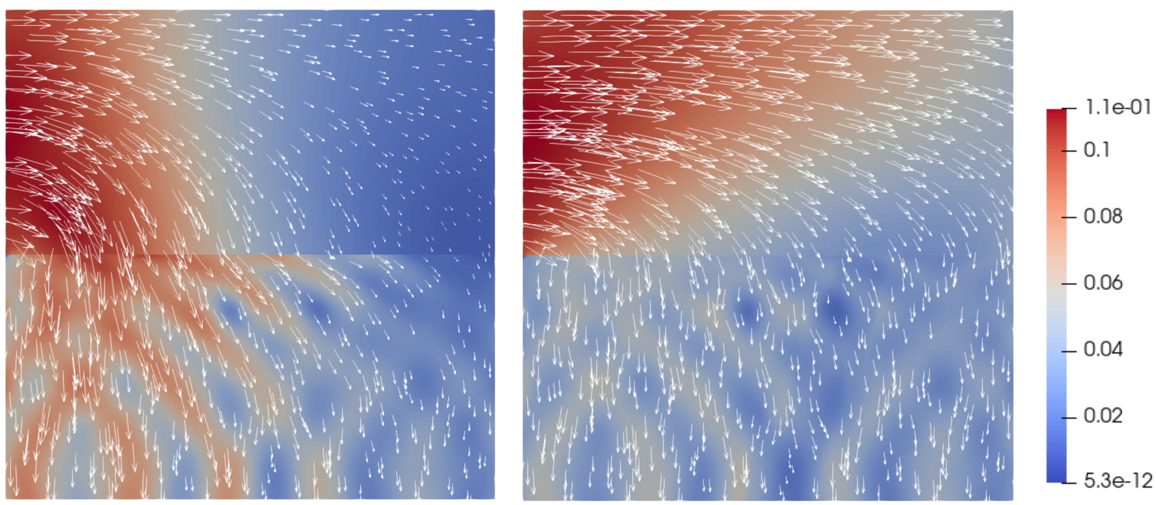


Figure 4.2: The velocity field after one time step (left) and at the final time (right). The color represents the magnitude of the velocity.

instances in time. The contaminant plume stays compact while in the surface water region. Once it reaches the subsurface region it spreads out following a path dictated by the heterogeneous permeability structure of the porous medium as we expected.

Table 4.9: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 10^{-6}$, using BDF2 method in Example 2, Section 4.2.2.2

			Stokes solution (Ω^s)			
h	Cells	$\lVert u - u_h \rVert_{\Omega^s}$	Order	$\lVert p - p_h \rVert_{\Omega^s}$	Order	$\lVert \nabla \cdot u_h \rVert_{\Omega^s}$
1/2	8	2.5e-04	–	3.5e-03	–	5.7e-10
1/4	28	2.2e-05	3.5	6.9e-04	2.3	2.0e-12
1/8	152	6.1e-07	5.2	5.6e-05	3.6	3.8e-15
1/16	576	3.3e-08	4.2	6.4e-06	3.1	8.8e-14
1/32	2348	2.0e-09	4.1	7.7e-07	3.1	9.4e-14
			Darcy solution (Ω^d)			
h	Cells	$\lVert u - u_h \rVert_{\Omega_d}$	Order	$\lVert p - p_h \rVert_{\Omega_d}$	Order	$\lVert \nabla \cdot u_h + \Pi_Q g^d \rVert_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-03	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-04	3.0	2.2e-12
1/8	152	3.5e-06	5.1	4.0e-05	3.8	7.2e-13
1/16	576	2.2e-07	4.0	5.1e-06	3.0	3.0e-12
1/32	2348	1.3e-08	4.1	6.1e-07	3.1	1.2e-11

Table 4.10: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 1$, $\mu = 10^{-6}$, using BDF2 in Example 2, Section 4.2.2.2.

h	Cells	$\lvert\lvert c - c_h \rvert\rvert_{\Omega}$	Order
1/2	8	1.2e-01	–
1/4	28	2.8e-02	2.9
1/8	152	2.3e-03	3.6
1/16	576	2.5e-04	3.2
1/32	2348	2.6e-05	3.2

Table 4.11: Table of convergence for the HDG method with $k_f = 3$, $k_c = k_f - 1$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 10^3$, $\mu = 10^{-6}$, using BDF2 method in Example 2, Section 4.2.2.2

				Stokes solution (Ω^s)		
h	Cells	$\\|u - u_h\\|_{\Omega^s}$	Order	$\\|p - p_h\\|_{\Omega^s}$	Order	$\\|\nabla \cdot u_h\\|_{\Omega^s}$
1/2	8	2.4e-04	–	4.2e-06	–	5.7e-10
1/4	28	2.2e-05	3.5	7.3e-07	2.5	2.1e-12
1/8	152	6.0e-07	5.2	5.7e-08	3.7	7.5e-15
1/16	576	3.3e-08	4.2	6.4e-09	3.2	4.5e-15
1/32	2348	2.0e-09	4.1	7.9e-10	3.0	1.4e-13
				Darcy solution (Ω^d)		
h	Cells	$\\|u - u_h\\|_{\Omega_d}$	Order	$\\|p - p_h\\|_{\Omega_d}$	Order	$\\|\nabla \cdot u_h + \Pi_Q g^d\\|_{\Omega_d}$
1/2	8	1.9e-03	–	4.2e-06	–	5.7e-10
1/4	28	1.2e-04	4.0	5.4e-07	3.0	2.2e-12
1/8	152	3.6e-06	5.1	4.0e-08	3.8	8.2e-13
1/16	576	2.2e-07	4.0	5.1e-09	3.0	3.1e-12
1/32	2348	1.3e-08	4.1	6.1e-10	3.1	1.2e-11

Table 4.12: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-4}$, $T = 0.1$, $\kappa = 10^3$, $\mu = 10^{-6}$, using BDF2 in Example 2, Section 4.2.2.2.

h	Cells	$\|\|c - c_h\|\|_\Omega$	Order
1/2	8	1.2e-01	–
1/4	28	2.8e-02	2.9
1/8	152	2.3e-03	3.6
1/16	576	2.5e-04	3.2
1/32	2348	2.6e-05	3.2

Table 4.13: Table of convergence for the HDG method with $k_f = 2$, $k_c = 1$, $\Delta t = 10^{-5}$, $T = 0.1$, $\kappa = 1$, $\mu = \mu(c)$, using BDF3 method in Example 3, Section 4.2.2.3

			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega^s}$	Order	$\|p - p_h\|_{\Omega^s}$	Order	$\|\nabla \cdot u_h\|_{\Omega^s}$
1/2	8	4.1e-03	–	9.7e-02	–	2.0e-07
1/4	28	9.4e-04	2.1	3.3e-02	1.6	2.9e-09
1/8	152	9.3e-05	3.3	5.6e-03	2.5	4.5e-11
1/16	576	1.9e-05	2.3	1.3e-03	2.1	7.0e-13
1/32	2348	4.7e-06	2.0	3.3e-04	2.0	8.0e-14
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	7.0e-02	–	2.3e-02	–	1.3e-06
1/4	28	3.1e-02	1.2	9.4e-03	1.3	5.4e-08
1/8	152	6.6e-03	2.2	1.4e-03	2.7	2.7e-10
1/16	576	1.4e-03	2.3	3.8e-04	1.9	5.4e-12
1/32	2348	3.5e-04	2.0	8.6e-05	2.1	4.8e-12

Table 4.14: Table of convergence by the HDG method for concentration with $k_f = 2$, $k_c = 1$, $\Delta t = 10^{-5}$, $T = 0.1$, $\kappa = 1$, $\mu = \mu(c)$, using BDF3 in Example 3, Section 4.2.2.3.

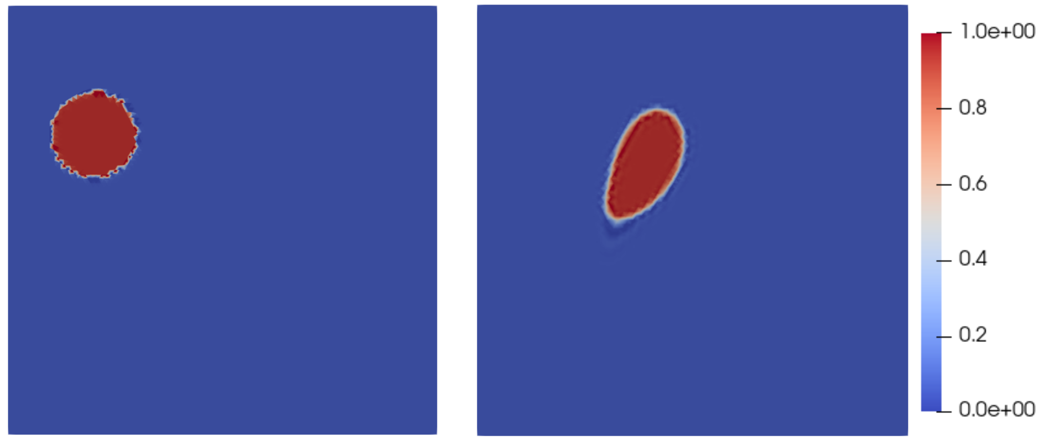
h	Cells	$\|\|c - c_h\|\|_\Omega$	Order
1/2	8	2.6e-01	–
1/4	28	1.0e-01	1.4
1/8	152	1.9e-02	2.4
1/16	576	4.3e-03	2.1
1/32	2348	1.0e-03	2.1

Table 4.15: Table of convergence for the HDG method with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-6}$, $T = 0.1$, $\kappa = 1$, $\mu = \mu(c)$, using BDF3 method in Example 3, Section 4.2.2.3

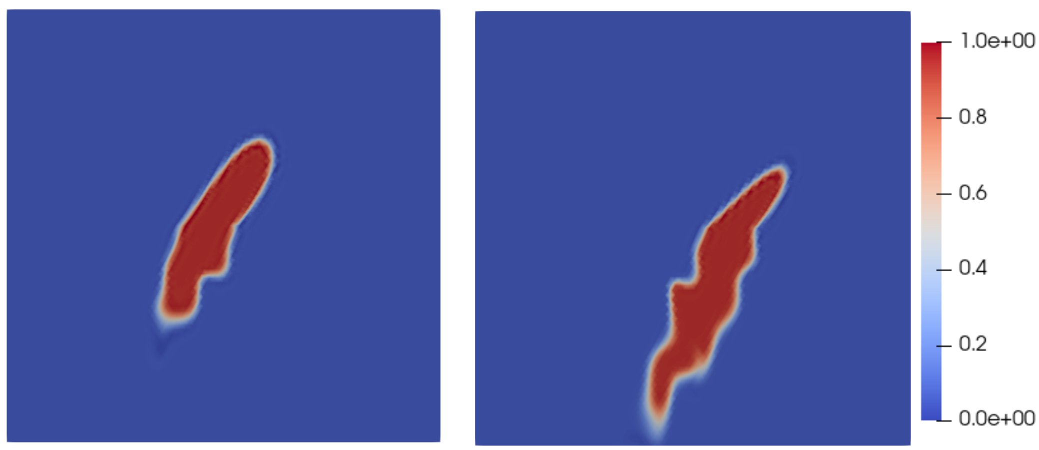
			Stokes solution (Ω^s)			
h	Cells	$\|u - u_h\|_{\Omega^s}$	Order	$\|p - p_h\|_{\Omega^s}$	Order	$\|\nabla \cdot u_h\|_{\Omega^s}$
1/2	8	7.7e-04	–	5.2e-02	–	5.7e-10
1/4	28	9.7e-05	3.0	8.2e-03	2.7	2.1e-12
1/8	152	4.1e-06	4.6	3.8e-04	4.4	3.2e-14
1/16	576	2.1e-07	4.3	4.0e-05	3.3	2.2e-14
1/32	2348	2.2e-08	3.3	4.7e-06	3.1	9.1e-14
			Darcy solution (Ω^d)			
h	Cells	$\|u - u_h\|_{\Omega_d}$	Order	$\|p - p_h\|_{\Omega_d}$	Order	$\|\nabla \cdot u_h + \Pi_Q g^d\|_{\Omega_d}$
1/2	8	2.3e-02	–	6.0e-03	–	5.7e-10
1/4	28	3.6e-03	2.7	6.4e-04	3.2	2.2e-12
1/8	152	3.3e-04	3.4	4.1e-05	4.0	8.1e-13
1/16	576	3.8e-05	3.1	5.1e-06	3.0	2.9e-12
1/32	2348	4.5e-06	3.1	6.4e-07	3.0	1.2e-11

Table 4.16: Table of convergence by the HDG method for concentration with $k_f = 3$, $k_c = 2$, $\Delta t = 10^{-6}$, $T = 0.1$, $\kappa = 1$, $\mu = \mu(c)$, using BDF3 in Example 3, Section 4.2.2.3.

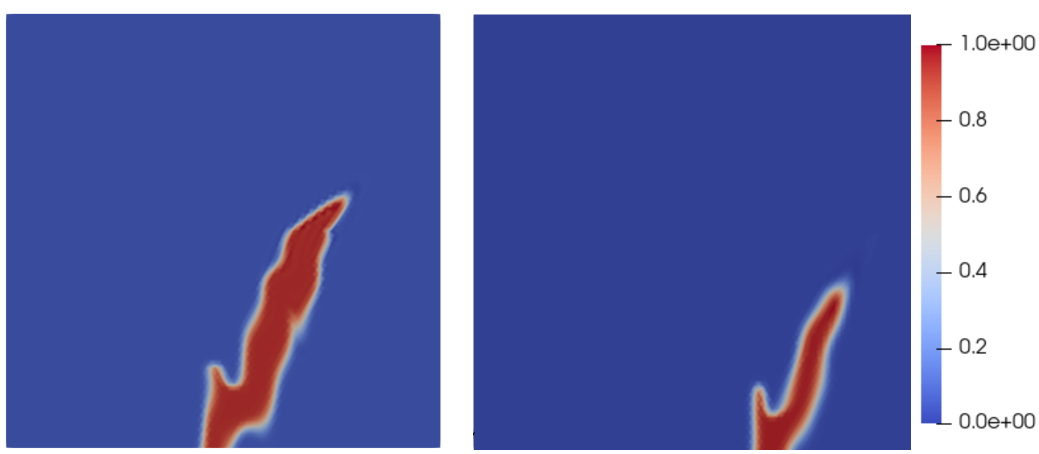
h	Cells	$\|\|c - c_h\|\|_{\Omega}$	Order
1/2	8	1.0e-01	–
1/4	28	1.4e-02	2.9
1/8	152	1.2e-03	3.6
1/16	576	1.2e-04	3.2
1/32	2348	1.3e-05	3.2



(a) The plume at $t = 0$. (b) The plume at $t = 2.5$.



(d) The plume at $t = 5.0$. (e) The plume at $t = 7.5$.



(g) The plume at $t = 10$. (h) The plume at $t = 15$.

Figure 4.3: The plume of contaminant at times $t = 0, 2.5, 5, 7.5, 10, 15$. The color represents the concentration values at time t.

CHAPTER FIVE

CONCLUSIONS AND FUTURE WORK

In this thesis, we studied a partial differential equations system, namely the fully coupled time-dependent Stokes-Darcy-transport problem, describing the transport of a chemical species via the coupled free flow and porous media flow. This system consists of two main ingredients: the time-dependent Stokes-Darcy problem, and the time-dependent advection-diffusion problem.

In Chapter 2, we analyzed an HDG discretization for the time-dependent Stokes-Darcy problem which models the interaction between free flow governed by the Stokes equations and porous media flow governed by Darcy's equations. After deriving the fully-discrete strongly mass conservative HDG discretization for this problem, we established its well-posedness and carried out an a priori error analysis in the energy norm. We provided a series of numerical examples demonstrating strong mass conservation, robustness with respect to the problem parameters for the viscosity and permeability, and optimal convergence rates in the L^2-norm.

In Chapter 3, we analyzed an HDG discretization for the time-dependent advection-diffusion equation that models the diffusion and advection of a chemical species under the influence of a given velocity field. After introducing the semi-discretization, we proved the existence and uniqueness of the solution as well as a priori error estimates in the energy norm. Numerical experiments exhibiting optimal convergence rates for different boundary conditions were presented.

In Chapter 4, we analyzed a numerical discretization for the fully coupled time-dependent Stokes–Darcy-transport problem where the fluid viscosity and source/sink terms depend on the concentration and the dispersion/diffusion tensor depends on the fluid velocity. We have employed a sequential strategy in which we used HDG methods to solve the Stokes/Darcy and transport subproblems at each time step. Strong mass conservation and pressure robustness properties of the flow discretization are crucial to provide a velocity input to approximate the concentration accurately without artificial mass. We proved well-posedness and a priori error estimates for the velocity and the concentration in the energy norm. We presented numerical experiments that demonstrate optimal convergence.

We close this chapter by listing some extensions of this work that may be interesting to pursue. A natural extension is to derive and analyze HDG methods for the time-dependent coupled Navier-Stokes/Darcy problem. The nonlinearity of the Navier-Stokes equations need to be treated correctly and more assumptions on the data would be required for the analysis. The strong mass conservation property of our method will prove to be quite useful in each iteration. Then including the full coupling of this to the transport equation would be a highly nontrivial extension. Especially its analysis will require a technical proof in proving existence and uniqueness.

Our numerical algorithm preserves high order convergence rate for smooth solutions and the analysis for low regularity solutions is still open. An extra weight in the penalty term can be used to relax the requirement on the polynomial degree due to unbounded dispersion-diffusion tensor. For realistic numerical experiments, it may be worthwhile to investigate the effect of additional stabilization techniques involving cut-off functions [50], diffusion upwinding [42, 45], streamline diffusion [11], or slope limiters. Different decoupling techniques and time stepping schemes will also be investigated.

APPENDIX A

ABSTRACT THEORY FOR SADDLE POINT PROBLEMS

In this section, we list some results on saddle point problems. Let X and M be two reflexive Banach spaces, $f \in X'$, $g \in M'$, and consider two bilinear forms $a \in \mathcal{L}(X \times X; \mathbb{R})$ and $b \in \mathcal{L}(X \times M; \mathbb{R})$. We consider the following problem: Find $u \in X$ and $p \in M$ such that

$$\begin{cases} a(u,v) + b(v,p) &= f(v), \quad \forall v \in X, \\ b(u,q) &= g(q), \quad \forall q \in M. \end{cases} \tag{A.1}$$

Theorem A.1 ([26, Theorem 2.34]). *Problem eq.* (A.1) *is well-posed if and only if*

$$\begin{cases} \exists \alpha > 0, \quad \inf_{u \in \mathrm{Ker}(B)} \sup_{v \in \mathrm{Ker}(B)} \frac{a(u,v)}{\|u\|_X \|v\|_X} \geq \alpha, \\ (\forall u \in \mathrm{Ker}(B), \quad a(u,v) = 0) \implies (v = 0), \end{cases} \tag{A.2}$$

and

$$\exists \beta > 0, \quad \inf_{q \in M} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_M}, \tag{A.3}$$

where $\mathrm{Ker}(B) = \{v \in X;\, \forall q \in M,\, b(v,q) = 0\}$. *Furthermore, the following a priori estimates hold:*

$$\begin{cases} \|u\|_X \leq c_1 \|f\|_{X'} + c_2 \|g\|_{M'}, \\ \|p\|_M \leq c_3 \|f\|_{X'} + c_4 \|g\|_{M'}, \end{cases}$$

where c_1, c_2, c_3 and c_4 are constants depending on α, β and a.

The next theorem is on twofold saddle point problems and is necessary for the the proof of the inf-sup condition Theorem 2.

Theorem A.2 ([34, Theorem 3.1]). *Let U, P_1, and P_2 be reflexive Banach spaces, and let $b_1 : P_1 \times U \to \mathbb{R}$, and $b_2 : P_2 \times U \to \mathbb{R}$ be bilinear and continuous. Then the following are equivalent:*

1. *There exists* $c > 0$ *such that*

$$\sup_{v \in U} \frac{b_1(p_1, v) + b_2(p_2, v)}{\|v\|_U} \geq c(\|p_1\|_{P_1} + \|p_2\|_{P_2}),\ \forall (p_1, p_2) \in P_1 \times P_2. \quad \text{(A.4)}$$

2. *There exists* $c > 0$ *such that*

$$\sup_{v \in Z_{b_2}} \frac{b_1(p_1, v)}{\|v\|_U} \geq c\|p_1\|_{P_1},\ \forall p_1 \in P_1 \ \textit{and}\ \sup_{v \in U} \frac{b_2(p_2, v)}{\|v\|_U} \geq c\|p_2\|_{P_2},\ \forall p_2 \in P_2, \quad \text{(A.5)}$$

   where $Z_{b_2} = \{v \in U : b_2(p_2, v) = 0 \ \textit{for all}\ p_2 \in P_2\}$.

APPENDIX B

PROOF OF THE INF-SUP CONDITION

This section includes the proof of Theorem 2 [16, Theorem 1]. Here we present it with a minor modification for completeness. We start by splitting the velocity-pressure coupling term $b_h^j(\boldsymbol{q}_h^j, v_h)$ in eq. (2.25a) into two parts as follows:

$$b_h^j(\boldsymbol{q}_h^j, v_h) = b_h^{j,1}(q_h, v_h) + b_h^{j,2}(\bar{q}_h^j, v_h),$$

where for $j = s, d$,

$$b_h^{j,1}(q_h, v_h) = -\sum_{K\in\mathcal{T}^j} \int_K q_h \nabla\cdot v_h \, \mathrm{d}x, \quad b_h^{j,2}(\bar{q}_h^j, v_h) = \sum_{K\in\mathcal{T}^j} \int_{\partial K} \bar{q}_h^j v_h \cdot n^j \, \mathrm{d}s.$$

We will prove intermediate results for $b_h^{j,1}(q_h, v_h)$ and $b_h^{j,2}(\bar{q}_h^j, v_h) + b_h^{I,j}(\bar{q}_h^j(\bar{q}_h^j, \bar{v}_h)$ and then combine them by using an inf-sup condition for twofold saddle point problems. We introduce the following space for these results.

$$\boldsymbol{V}_h^0 := \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h : \sum_{j=s,d} \big(b_h^{j,2}(\bar{q}_h^j, v_h) + b_h^{I,j}(\bar{q}_h^j, \bar{v}_h)\big) = 0 \quad \forall(\bar{q}_h^s, \bar{q}_h^d) \in \bar{Q}_h^s \times \bar{Q}_h^d\Big\}.$$

Lemma B.1. *There exists a constant $c_{\text{inf}} > 0$, independent of h, such that for any $q_h \in Q_h$,*

$$c_{\text{inf}}\|q_h\|_\Omega \leq \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h^0\\ \boldsymbol{v}_h\neq 0}} \frac{\sum_{j=s,d} b_h^{j,1}(q_h, v_h)}{|||\boldsymbol{v}_h|||_v} = \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h^0\\ \boldsymbol{v}_h\neq 0}} \frac{-\sum_{j=s,d}\sum_{K\in\mathcal{T}^j} \int_K q_h \nabla\cdot v_h \, \mathrm{d}x}{|||\boldsymbol{v}_h|||_v}.$$

Proof. Let $q_h \in Q_h \subset L_0^2(\Omega)$. By the surjectivity of the divergence operator, see [24, Theorem 6.5], there exists a $v \in [H_0^1(\Omega)]^{\dim}$ such that

$$-\nabla\cdot v = q_h \text{ and } \gamma\|v\|_{H^1(\Omega)} \leq \|q_h\|_\Omega. \tag{B.1}$$

Then, noting that $q_h \in P_{k-1}(K)$ for all $K \in \mathcal{T}$, Lemma 4(i.) implies that

$$\|q_h\|_\Omega^2 = \int_\Omega q_h(-\nabla \cdot v)\ \mathrm{d}x = -\int_\Omega q_h \nabla \cdot \Pi_V v\ \mathrm{d}x. \tag{B.2}$$

By the triangle inequality and using Lemma 4(iv.) with $k = l = 1$, we bound the Stokes part of the norm as follows:

$$\|\nabla(\Pi_V v)\|_K \le \|v - \Pi_V v\|_{1,K} + \|v\|_{1,K} \le C\|v\|_{1,K}. \tag{B.3}$$

By eq. (B.3) and eq. (2.49) with $\ell = 1$, we have

$$\begin{aligned} |\!|\!|(\Pi_V v, \overline{\Pi}_V v)|\!|\!|_{v,s}^2 &= \sum_{K\in\mathcal{T}^s} \left(\|\nabla(\Pi_V v)\|_K^2 + h_K^{-1}\|\Pi_V v - \bar{\Pi}_V v\|_{\partial K}^2\right) \\ &\le C \sum_{K\in\mathcal{T}^s} \|v\|_{1,K}^2 = C\|v\|_{1,\Omega^s}^2. \end{aligned} \tag{B.4}$$

For the Darcy part of the norm, using the triangle inequality and Lemma 4(iv.), we get

$$\|\Pi_V v\|_{\Omega^d} \le \|\Pi_V v - v\|_{\Omega^d} + \|v\|_{\Omega^d} \le C\|v\|_{1,\Omega^d}. \tag{B.5}$$

Also, by the trace inequalities eq. (2.31) and eq. (2.35), and by Lemma 4(iv.), the interface term in the norm is bounded as follows:

$$\begin{aligned} \sum_{j=1}^{\dim-1} \gamma^j \|\bar{\Pi}_V v \cdot \tau^j\|_{\Gamma^I}^2 &= \sum_{j=1}^{\dim-1} \gamma^j \|(\Pi_V v)^s \cdot \tau^j\|_{\Gamma^I}^2 \le (\sum_{j=1}^{\dim-1} \gamma^j)\|(\Pi_V v)^s\|_{\Gamma^I}^2 \\ &\le C(\|(\Pi_V v)^s - v^s\|_{\Gamma^I}^2 + \|v^s\|_{\Gamma^I}^2) \\ &\le C(\sum_{K\in\mathcal{T}^s} \|\Pi_V v - v\|_{\partial K}^2) + C\|v\|_{\Gamma^I}^2 \\ &\le C(\sum_{K\in\mathcal{T}^s} h_K\|v\|_{H^1(K)}^2) + C\|v\|_{\Gamma^I}^2 \le C\|v\|_{1,\Omega^s}^2. \end{aligned} \tag{B.6}$$

From eqs. (B.4) to (B.6) and the definition of $\|\!|.|\!\|_v$, we have

$$\|\!|(\Pi_V v, \bar{\Pi}_V v)|\!\|_v \leq C\|v\|_{1,\Omega}. \tag{B.7}$$

Now observe that $(\Pi_V v, \bar{\Pi}_V v) \in \boldsymbol{V}_h^0$. Indeed, by Lemma 4(iii.) and since $\bar{q}_h^j$ is single-valued,

$$\begin{aligned}
&\sum_{j=s,d} \Big(b_h^{j,2}(\bar{q}_h^j, \Pi_V v) + b_h^{I,j}(\bar{q}_h^j, \bar{\Pi}_V v) \Big) \\
&\qquad = \sum_{j=s,d} \Big(\sum_{K\in\mathcal{T}^j} \int_{\partial K} \bar{q}_h^j \Pi_V v \cdot n^j \, \mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^j \Pi_V v \cdot n^j \, \mathrm{d}s \Big) \\
&\qquad = \sum_{j=s,d} \Big(\int_{\Gamma^I} \bar{q}_h^j \Pi_V v \cdot n^j \, \mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^j \Pi_V v \cdot n^j \, \mathrm{d}s \Big) = 0.
\end{aligned}$$

Hence, by eqs. (B.1) and (B.2),

$$\sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^0 \\ \boldsymbol{v}_h \neq \boldsymbol{0}}} \frac{-\int_\Omega q_h \nabla \cdot v_h \, \mathrm{d}x}{\|\!|\boldsymbol{v}_h|\!\|_v} \geq \frac{-\int_\Omega q_h \nabla \cdot \Pi_V v \, \mathrm{d}x}{\|\!|(\Pi_V v, \bar{\Pi}_V v)|\!\|_v} \geq \frac{\|q_h\|_\Omega^2}{C\|v\|_{1,\Omega}} \geq \frac{\gamma}{C}\|q_h\|_\Omega.$$

This completes the proof by letting $c_{\inf} = \frac{\gamma}{C}$. □

For the second intermediate result, we define

$$R_k(\partial K) := \{\bar{q} \in L^2(\partial K),\ \bar{q} \in P_k(F),\ \forall F \in \partial K\}$$

and let $L : R_k(\partial K) \to [P_k(K)]^{\dim}$ be the BDM local lifting operator which satisfies for all $\bar{q}_h \in R_k(\partial K)$ the following [10]:

$$(L\bar{q}_h) \cdot n = \bar{q}_h, \qquad \|L\bar{q}_h\|_K \leq C h_K^{1/2} \|\bar{q}_h\|_{\partial K} \tag{B.8}$$

$$\|\nabla(L\bar{q}_h)\|_K^2 \leq C h_K^{-1} \|\bar{q}_h\|_{\partial K}^2, \qquad \|L\bar{q}_h\|_{\partial K}^2 \leq C \|\bar{q}_h\|_{\partial K}^2. \tag{B.9}$$

Lemma B.2. *There exists a constant $\bar{c}_{\text{inf}} > 0$, independent of h, such that for any $(\bar{q}_h^s, \bar{q}_h^d) \in \bar{Q}_h^s \times \bar{Q}_h^d$,*

$$\Big(\bar{c}_{\text{inf}} \sum_{j=s,d} \sum_{K\in\mathcal{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2\Big)^{1/2} \le \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h \\ \boldsymbol{v}_h\neq\boldsymbol{0}}} \frac{\sum_{j=s,d} \Big(b_h^{j,2}(\bar{q}_h^j, v_h) + b_h^{I,j}(\bar{q}_h^j, \bar{v}_h)\Big)}{\|\boldsymbol{v}_h\|_v}. \tag{B.10}$$

Proof. Let $\bar{q}_h^j \in \bar{Q}_h^j$ and set $w_h^j := L\bar{q}_h^j$ for $j = s, d$. Since $w_h^j \in [P_k(K)]^{\dim}$ for all $K \in \mathcal{T}^j$, $j = s, d$,

$$w_h = \begin{cases} w_h^s & \text{in } \Omega^s, \\ w_h^d & \text{in } \Omega^d \end{cases}$$

belongs to $[P_k(K)]^{\dim}$ for all $K \in \mathcal{T}$. Using eq. (B.9), we get

$$\begin{aligned} |||(w_h, 0)|||_{v,s}^2 &= \sum_{K\in\mathcal{T}^s} (\|\nabla(L\bar{q}_h^s)\|_K^2 + h_K^{-1}\|L\bar{q}_h^s - 0\|_{\partial K}^2) \\ &\le C \sum_{K\in\mathcal{T}^s} h_K^{-1}\|\bar{q}_h^s\|_{\partial K}^2. \end{aligned} \tag{B.11}$$

Using eq. (B.8), we have

$$\begin{aligned} \|w_h\|_{\Omega^d}^2 = \|w_h^d\|_{\Omega^d}^2 &= \sum_{K\in\mathcal{T}^d} \|L\bar{q}_h^d\|_K^2 \\ &\le C \sum_{K\in\mathcal{T}^d} h_K\|\bar{q}_h^d\|_{\partial K}^2 \le C \sum_{K\in\mathcal{T}^d} h_K^{-1}\|\bar{q}_h^d\|_{\partial K}^2, \end{aligned} \tag{B.12}$$

where we used $h_K < Ch_K^{-1}$.

From eqs. (B.11) and (B.12), we obtain

$$|||(w_h, 0)|||_v^2 = |||(w_h, 0)|||_{v,s}^2 + \|w_h\|_{\Omega^d}^2 \le C \sum_{j=s,d} \Big(\sum_{K\in\mathcal{T}^j} h_K^{-1}\|\bar{q}_h^j\|_{\partial K}^2\Big). \tag{B.13}$$

Using eq. (B.8), we have $w_h^j \cdot n = \bar{q}_h^j,\ j = s, d$. Then

$$
\begin{aligned}
&\sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \boldsymbol{v}_h \neq \boldsymbol{0}}} \frac{\sum\limits_{j=s,d} \big(b_h^{j,2}(\bar{q}_h^j, v_h) + b_h^{I,j}(\bar{q}_h^j, \bar{v}_h)\big)}{\|\boldsymbol{v}_h\|_v} \\
&= \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \boldsymbol{v}_h \neq \boldsymbol{0}}} \frac{\sum\limits_{j=s,d} \big(\sum\limits_{K \in \mathcal{T}^j} \int_{\partial K} \bar{q}_h^j v_h \cdot n^j \ \mathrm{d}s - \int_{\Gamma^I} \bar{q}_h^j \bar{v}_h \cdot n^j \ \mathrm{d}s \big)}{\|\boldsymbol{v}_h\|_v} \\
&\geq \frac{\sum\limits_{j=s,d} \big(\sum\limits_{K \in \mathcal{T}^j} \|\bar{q}_h^j\|_{\partial K}^2 \big)}{\|(w_h, 0)\|_v} \quad (\text{choosing } \boldsymbol{v}_h = (w_h, 0)) \\
&\geq \frac{\sum\limits_{j=s,d} \big(\sum\limits_{K \in \mathcal{T}^j} \|\bar{q}_h^j\|_{\partial K}^2 \big)}{C \big(\sum\limits_{j=s,d} \sum\limits_{K \in \mathcal{T}^j} h_K^{-1} \|\bar{q}_h^j\|_{\partial K}^2 \big)^{1/2}} \quad (\text{by eq. (B.13)}) \\
&\geq \frac{h_{\max}^{-1} \sum\limits_{j=s,d} \big(\sum\limits_{K \in \mathcal{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2 \big)}{C \big(h_{\min}^{-2} \sum\limits_{j=s,d} \sum\limits_{K \in \mathcal{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2 \big)^{1/2}} \quad (\text{since } h_{\min} \leq h_K \leq h_{\max}) \\
&\geq C h_{\min} h_{\max}^{-1} \big(\sum_{j=s,d} \sum_{K \in \mathcal{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2 \big)^{1/2}, \qquad \text{(B.14)}
\end{aligned}
$$

where $h_{\min}$ and $h_{\max}$ denote the minimum and maximum diameter over all cells, respectively. Setting $c_{\inf} = C h_{\min} h_{\max}^{-1}$, eq. (B.10) follows. □

Proof of Theorem 2. Let $U = \boldsymbol{V}_h$, $P_1 = Q_h$ and $P_2 = \bar{Q}_h^s \times \bar{Q}_h^d$ and define $b_1(\cdot,\cdot) = \sum\limits_{j=s,d} b_h^{j,1}(q_h, v_h)$ and $b_2(\cdot,\cdot) = \sum\limits_{j=s,d} \Big(b_h^{j,2}(\bar{q}_h^j, v_h) + b_h^{I,j}(\bar{q}_h^j, \bar{v}_h) \Big)$. By Lemmas B.1 and B.2, the conditions in eq. (A.5) are satisfied. Therefore, by the equivalence of eq. (A.5) and eq. (A.4) in Theorem A.2, and the definition of $\|\cdot\|_p$, the result follows. □

APPENDIX C
MORE USEFUL RESULTS

We have the following discrete integration by parts:

$$\sum_{m=0}^{N}(x^{m+1}-x^m)y^m = x^{N+1}y^N - x^0y^0 + \sum_{m=0}^{N-1}(y^m - y^{m+1})x^{m+1}. \tag{C.1}$$

As a result of Taylor's theorem, we have

Theorem C.1 ([12, Lemma 3.2]). *For a function z defined on $D \times [0,T]$, assuming enough regularity, we have the following results*

$$\Delta t \sum_{n=1}^{k} \|\partial_t z^n - d_t z^n\|_D^2 \leq C(\Delta t)^2 \|\partial_{tt} z\|^2_{L^2(0,T;L^2(D))}, \tag{C.2}$$

$$\Delta t \sum_{n=1}^{k} \|d_t z^n\|^2_{m,D} \leq \|\partial_t z\|^2_{L^2(0,T;H^m(D))}, \quad m = 0,1 \tag{C.3}$$

where $\|f\|_{L^2(0,T;X)} = \left(\int_0^T \|f(t)\|_X^2\right)^{1/2}$.

Lemma C.1 (Grönwall's inequality, [4, Corollary 1.2]). *Let $f(t), g(t), h(t)$ be continuous functions defined on $[0,T]$ such that $g(t)$ is non-negative and $h(t)$ is nondecreasing in $[0,T]$. Let k be a nonnegative constant. If*

$$f(t) + \int_0^t g(s)\,\mathrm{d}s \leq h(t) + \int_0^t kf(s)\,\mathrm{d}s \quad \forall t \in [0,T],$$

then

$$f(t) + \int_0^t g(s)\,\mathrm{d}s \leq h(t)e^{kt} \quad \forall t \in [0,T]. \tag{C.4}$$

Lemma C.2 (Discrete Grönwall's inequality, [39, Lemma 27]). *Let $\Delta t, B > 0$ and let $\{a_n\}_n$, $\{b_n\}_n$, $\{c_n\}_n, \{d_n\}_n$ be sequences of nonnegative numbers satisfying*

$$\forall n \geq 1, \quad a_n + \Delta t \sum_{m=1}^{n} b_m \leq B + \Delta t \sum_{m=1}^{n} d_m a_m + \Delta t \sum_{m=1}^{n} c_m.$$

If $d_m\Delta t < 1$ *for all* n*, then*

$$\forall n \geq 1, \quad a_n + \Delta t \sum_{m=1}^{n} b_m \leq e^{\Delta t \sum\limits_{m=1}^{n} \frac{d_m}{1-\Delta t d_m}} \Big(B + \Delta t \sum_{m=1}^{n} c_m \Big).$$

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