NUMERICAL ANALYSIS AND SIMULATIONS OF COUPLED FREE FLOW AND DUAL POROSITY MODELS

by

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Dedicated to the memory of my father, in recognition of all the trust and support he has given me throughout his life.

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Dorisa Tabaku

ABSTRACT

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Adviser: Aycil Cesmelioglu, Ph.D.

Coupled free-flow and porous media flow can be found in many applications in science and engineering, including oil extraction, groundwater contamination, etc. However, these problems are hard to solve exactly and fast, accurate numerical methods are necessary to solve them computationally. The coupled problems we consider in this thesis are the dual-porosity Stokes problem and its nonlinear version dual-porosity Navier-Stokes problem that model flow in regions partially occupied by naturally fractured porous media. These problems combine flow in microfractures and the porous matrix, governed by the dual-porosity equations, with free flow in macrofractures/ conduits, governed by the (Navier-)Stokes equations. The dual porosity and (Navier-)Stokes equations are connected through four interface conditions: mass conservation, balance of forces, the Beavers-Joseph-Saffman condition, and a no-exchange condition between the matrix and the conduits.

This thesis introduces, analyzes, and tests through numerical experiments hybridizable discontinuous Galerkin methods for solving Stokes-dual-porosity and Navier-Stokes-dual-porosity problems.

We prove existence, uniqueness, and stability of weak solutions to both problems under appropriate data restriction in the nonlinear case. Then, we introduce hybridizable discontinuous Galerkin (HDG) methods, prove well-posedness of the numerical schemes,

and present a priori error analyses. Our error estimates show optimal convergence in the energy norm for the Stokes velocity and pressures in Stokes, microfracture, and the matrix. In addition, our HDG method is strongly mass conservative and has the typical advantage of computational efficiency compared to the standard discontinuous Galerkin methods.

Our numerical examples not only support our findings but also show optimal convergence rates in all variables in the L^2 norm and show that our method performs well with highly discontinuous permeability fields in the microfractures and the porous matrix.

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CHAPTER ONE

INTRODUCTION

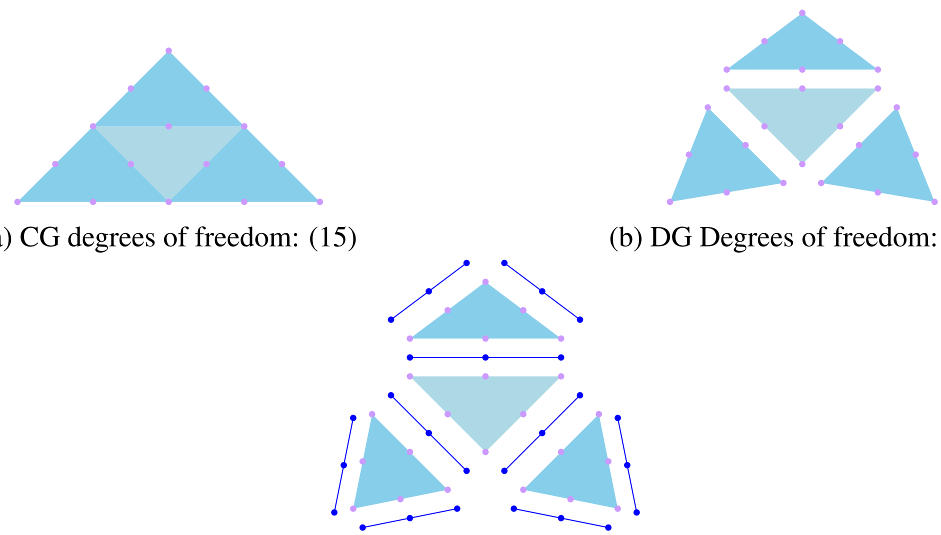
This thesis introduces a hybridizable discontinuous Galerkin (HDG) method for coupled problems arising from the interaction between free flow and flow in porous media. This interaction is relevant in various engineering applications, including bioengineering, industrial filtration, groundwater contamination, and oil and gas extraction [2–6].

A commonly used model for the coupled free flow and porous media flow is the (Navier-)Stokes/Darcy equations and many numerical methods have been developed to solve them. A few examples can be found in [7–15]. However, the (Navier-)Stokes/Darcy model do not consider the heterogeneous nature of a porous medium, which in reality may contain multiple porosities, for example, in a naturally fractured reservoir encountered in hydrology, geothermal systems, and petroleum extraction [16, 17].

The dual porosity model, introduced in [18], allows the presence of two different systems inside a porous media: the matrix system and the microfracture system with different conductivity properties. This model provides a better representation of the naturally fractured porous media [19]. On the other hand, the dual-porosity model itself fails to consider large conduits/macrofractures.

A better suited model to allow for both micro and macrofractures is the dual-porosity-(Navier-)Stokes model introduced in [20] coupling free flow with the dual porosity model. In this model, mass conservation, force balance, the Beavers–Joseph–Saffman condition [21, 22], and a no-exchange condition are imposed on the interface between the free flow and dual porosity domains to close the system. The first three interface conditions are the same as in the coupled Stokes-Darcy model, while the last condition is specific to the dual-porosity-(Navier-)Stokes model. This no exchange condition prescribes that fluid in the matrix can flow into the microfractures, but not into the conduits directly. In [20], the authors provide a weak formulation of the time-dependent dual-porosity-Stokes model, show that it is well-posed and propose and analyze a monolithic finite element method for the model. In [23, 24], stabilized mixed finite element methods are used for both the time-dependent and stationary cases. In [25], the authors propose a coupled finite element method for solving the stationary dual-porosity-Navier-Stokes system where they introduce a re-scaling factor to transfer the original problem into a new space. More recently, [26] presents a monolithic and strongly conservative scheme for the stationary dual-porosity-Navier-Stokes problem based on a symmetric interior penalty discontinuous Galerkin method and a mixed finite element method. Furthermore, for the time-dependent model dual- porosity-(Navier-)Stokes model, various decoupled schemes have been studied, see for example [23, 24, 27–33].

In this thesis, we solve the steady-state dual porosity-(Navier-)Stokes problem monolithically using a strongly mass conservative hybridizable discontinuous Galerkin (HDG) method. The HDG method is one of the numerical methods for solving partial differential equations (PDEs). It combines features from both discontinuous Galerkin (DG) and continuous Galerkin (CG) methods, providing a balance between accuracy and computational efficiency. Although maintaining the advantages of standard DG methods, the HDG method [34] has significantly reduced degrees of freedom which allows for substantial reduction in computational cost and memory storage. This is done by introducing hybrid variables on the mesh faces as traces of unknowns and appropriately defining cross-element numerical fluxes and then using static condensation to obtain a reduced global system in terms of only the degrees of freedom of the trace variables [35]. Once the global system is solved for the trace variables, the unknowns in each element can be solved locally using these trace variables as boundary conditions. This local computation is cheap and can be done in parallel. Therefore, it can significantly reduce computational time for large-scale problems. In Figure 1.1, we illustrate the degrees of freedom of the CG, DG, and HDG methods for a mesh with four triangles using quadratic polynomials for a scalar variable. The blue nodes in fig. 1.1c represent the degrees of freedom of the solution trace, while the rest of the nodes represent the degrees of freedom in elements.



(a) CG degrees of freedom: (15)

(b) DG Degrees of freedom: (24)

(c) HDG Degrees of freedom: (24) and (27) solution trace

Figure 1.1: Example of the degrees of freedom for the CG, DG, and HDG methods on a triangular mesh in 2D by using quadratic polynomial approximation for a scalar variable.

We are unaware of any previous research that uses the HDG method for this problem despite its computational efficiency, high order accuracy, and strong mass conservation property [36, 37].

1.1 Contributions of the thesis

In the first part of this thesis, which focuses on the stationary dual-porosity-Stokes problem, I demonstrate that the weak formulation of the problem is well posed. This is different than the proof of the well-posedness proof in [20] since they used the primal form of the dual-porosity equations and employed a Gårding-type inequality. Whereas, we used the mixed form of the dual-porosity equations which resulted in a weak formulation that is in the form of a regularly perturbed saddle point problem. Therefore, our well-posedness proof relies on the saddle point theory.

To solve this problem numerically, we present a monolithic HDG method that couples the pressure-robust symmetric interior penalty-HDG method [38] for the Stokes problem with a hybridized BDM discretization [39–41] for the dual-porosity problem. This monolithic HDG method can be considered as an extension of the method introduced in [37] for the Stokes/Darcy problem to the dual-porosity-Stokes problem. Our method results in discrete velocities that are divergence-conforming and in the absence of source terms, mass conservation equations are satisfied pointwise on the elements of the mesh. This means that the method is strongly mass conservative. As in the continuous case, the discrete problem also leads to a saddle-point problem with a regular perturbation term. The well-posedness proof requires an inf-sup condition for the velocity-pressure coupling term. Our proof improves the proof of the inf-sup condition in [37] by relaxing the quasi-uniformity assumption.

Next, we provide an a prior error analysis which shows that the Stokes velocity converges optimally in the energy norm, the velocities in the microfractures and the matrix converge with at least order k in the L^2-norm and the pressures converge with the optimal order of k in the L^2-norm when degree k polynomials are used for the velocities and the trace variables and degree $k-1$ polynomials for the pressures. The numerical examples in this thesis support these theoretical results and demonstrate that the method performs well with highly discontinuous permeability in the microfractures and the porous matrix.

In the second part of this thesis, we extend our results to the dual-porosity-Navier-Stokes problem by including a nonlinear convection term in the free flow equations. This adds difficulty to the analysis of both the weak solution and the discrete solution. As typically done with nonlinear problems, we employ the fixed point theory to prove existence and uniqueness of the solutions. This consists of two steps; 1. proving well-posedness of the weak formulation of a linear dual-porosity-Oseen type problem, where the convective velocity is considered as given with certain properties, and whose fixed point solution coincides with the solution of the dual-porosity-Navier-Stokes problem, 2. constructing a bounded operator T based on the weak formulation of the dual-porosity-Oseen problem and showing that this operator has a fixed point by using Schauder and Banach fixed point theorems.

Note that in the weak formulation of the dual-porosity-Stokes problem, the bilinear form $a(\cdot,\cdot)$ in the weak formulation is symmetric and it is possible to use the standard saddle point theory. In the weak formulation of the dual-porosity-Oseen problem in Step 1, this symmetry is broken and an updated saddle point result is needed. Such a result is stated as a remark in [40, Remark 4.3.2] without proof. We provide the proof of this updated saddle point result in the Appendix. Regarding Step 2, showing that the operator T requires an appropriate data assumption which is expected from the analysis of the Navier-Stokes equations.

Next, we extend our HDG method to include a trilinear discretization form for the nonlinear advective term. This nonlinear term from [15] provides upwinding of the numerical flux. Well-posedness of this finite-dimensional problem is shown by a two-step process similar to the one used in the continuous case. The a priori error analysis, based on certain data conditions, shows similar convergence results as in the dual-porosity-Stokes problem. Numerical experiments corroborate our results.

1.2 Thesis outline

As outlined in the table of contents, the remainder of the thesis is organized as follows. In the second chapter, we summarize the basic concepts and essential theorems necessary for this thesis. In the third chapter, we study the dual-porosity-Stokes problem, and in the fourth chapter, we focus on the dual-porosity-Navier-Stokes problem. In the last chapter, we discuss our findings and propose future research ideas to extend the work in the thesis.

CHAPTER TWO

FUNDAMENTAL CONCEPTS: DEFINITIONS, INEQUALITIES, AND THEOREMS

In this chapter, we recall well-known definitions, inequalities, and theorems that are relevant to our thesis.

2.1 Basic definitions

Definition 1. *A norm on a vector space X is a real-valued function on X denoted by $\|\cdot\|$ satisfying the following conditions:*

(i) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$,

(ii) $\|cx\| = |c|\|x\|$ *for every* $x \in X$ *and* $c \in \mathbb{C}$,

(iii) $\|x+y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$.

Definition 2. *A Banach space is a vector space X equipped with a norm $\|\cdot\|$, which is complete, i.e., where all Cauchy sequences in X converge to a limit in X.*

$$x_n \in X : \lim_{m,n\to\infty} \|x_n - x_m\| = 0 \implies \exists x \in X : \lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 3. *A Hilbert space H is a vector space H equipped with an inner product $\langle\cdot,\cdot\rangle$ that satisfies the following properties for all $x, y, z \in H$ and all scalars α:*

1. ***Linearity**:* $\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$

2. ***Symmetry**:* $\langle x, y\rangle = \overline{\langle y, x\rangle}$

3. ***Positivity**:* $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$

The inner product induces a norm $\|\cdot\|$ on H defined by $\|x\| = \sqrt{\langle x, x\rangle}$, and the Hilbert spaces are Banach spaces.

Definition 4. *[42, Definition A.14] Let X be a Banach space over $\mathbb{R}$, equipped with the norm $\|\cdot\|_X$. The dual space X' of X is the set of all linear continous (bounded) functionals from X to $\mathbb{R}$.*

$$X' = \{f : X \to \mathbb{R} \mid f \text{ is linear}\}.$$

For any continous (bounded) linear functional $f \in X'$, the norm of f, denoted by $\|f\|$, is defined as

$$\|f\|_{X'} = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |f(x)|.$$

The problems in our thesis correspond to a system of a partial differential equations posed over a domain $\Omega \subset \mathbb{R}^d$ with space dimension $d = 2,3$. The domain Ω is a bounded, connected, open subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$.

We will use two classes of function spaces in our weak form, and discrete scheme for our problems, so we state only the basic properties and refer the reader to [43, Chap. 5] and [44].

Definition 5. *Let $f : \Omega \to \mathbb{R}$ be a Lebesgue measurable function, and $1 \leq p \leq \infty$ be a real number. We define the norm*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, \mathrm{d}x \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_{L^\infty(\Omega)} := \textit{ess sup}\{|f(x)| \textit{ for a.e. } x \in \Omega\} = \inf\{M > 0 \mid |f(x)| \leq M \textit{ for a.e. } x \in \Omega\}.$$

Thus, the Lebesgue space is the space defined by:

$$L^p(\Omega) := \{f \textit{ Lebesgue measurable} \mid \|f\|_{L^p(\Omega)} < \infty\}.$$

$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$, and $L^2(\Omega)$ is a Hilbert spaces where the scalar product is $\langle f, g \rangle = \int_\Omega f \cdot g \, \mathrm{d}x$.

Definition 6. *[43] Let u be an integrable function on an open set $\Omega \subset \mathbb{R}^n$. v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

if it satisfies the following condition:

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U v \varphi \, dx \tag{3}$$

for all test functions $\varphi \in C_c^\infty(U)$. for all test functions $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω, and α is a multi-index, i.e $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $|\alpha| = \sum_{k=1}^n \alpha_i$.

Definition 7. *[44] A Sobolev space $W^{k,p}(\Omega)$ is a Banach space defined as the space:*

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \, \textit{for all} \, |\alpha| \leq k \right\}$$

equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$$

for $1 \leq p < \infty$, and

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

for $p = \infty$. Ω is an open subset of $\mathbb{R}^n$, k is a non-negative integer, and $D^\alpha u$ denotes α^{th}-weak partial derivative of u.

For $p = 2$, the Sobolov space is a Hilbert space and is denoted by $H^k(\Omega)$.

2.2 Key inequalities and theorems

Proposition 8. *(Hölder inequality). Let $f \in L^p(\Omega), g \in L^q(\Omega)$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1(\Omega)$, and*

$$\int_\Omega f \cdot g \,\mathrm{d}x \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{2.1}$$

The particular case $p = q = 2$ yields the Cauchy–Schwarz inequality:

Proposition 9. *(Cauchy–Schwarz inequality). If $f, g \in L^2(\Omega)$, then $fg \in L^1(\Omega)$, and*

$$\int_\Omega f \cdot g \,\mathrm{d}x \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \tag{2.2}$$

Proposition 10. *(Poincaré inequality). Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. Then, there exists a constant $C > 0$ such that for all $f \in W^{1,p}(\Omega)$ where $f = 0$ on $\Gamma \subset \partial\Omega$ with positive Lebesgue measure, the following inequality holds:*

$$\|f\|_{L^p(\Omega)} \leq C_P \|\nabla f\|_{L^p(\Omega)}. \tag{2.3}$$

Proposition 11. *(The continuous trace inequality). For all $f \in W^{1,p}(\Omega)$ $1 \leq p \leq \infty$, there is a constant $C_{\mathrm{tr,p}}$ such that:*

$$\|f\|_{L^p(\partial\Omega)} \leq C_{\mathrm{tr,p}} \|f\|_{L^p(\Omega)}^{1-1/p} \|f\|_{W^{1,p}(\Omega)}^{1/p} \tag{2.4}$$

In particular one can easily deduce the following two inequalities [45, Equation 1.24]:

$$\|f\|_{L^2(\partial\Omega)} \leq C_{\mathrm{tr},2} \|\nabla f\|_{L^2(\Omega)} \tag{2.5a}$$

$$\|f\|_{L^4(\partial\Omega)} \leq C_{\mathrm{tr},4} \|\nabla f\|_{L^2(\Omega)} \tag{2.5b}$$

Proposition 12 (Sobolov embedding theorem)**.** *[46, Theorem 6.6] Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary.*

Let $1 \leq p < \infty$ and $k \in \mathbb{N}$ be such that $kp < n$. Then, there exists a continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \textit{where } q = \frac{np}{n - kp}.$$

Additionally, if $kp = n$, then there is a continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall 1 \le q < \infty.$$

If $kp > n$, then there is a continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \quad \text{where } \alpha = k - \frac{n}{p}.$$

In particular, we are interested in spaces with $n = 2,3$, and $p = 2$, we highlight the following inequality: There exists a constant $C_{(2,4)} > 0$ such that:

$$\|f\|_{L^4(\Omega)} \le C_{(2,4)} \|\nabla f\|_{L^2(\Omega)}, \tag{2.6}$$

Proposition 13. *(Korn's inequality)[46, Theorems: 6.3-3,6.3-4]. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, and Lipschitz domain. Then, there exists a constant $C_K > 0$ dependent only on Ω such that for all vector fields $u \in [H^1(\Omega)]^n$,*

$$\|u\|_{H^1(\Omega)} \le C_K \left(\|u\|^2_{L^2(\Omega)} + \|\varepsilon(u)\|^2_{L^2(\Omega)} \right)^{1/2},$$

where $\varepsilon(u)$ denotes the symmetric gradient given by:

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

For $u \in [H^1(\Omega)]^n$ with $u = 0$ in $\Gamma \subset \partial\Omega$, such that Γ has a nonzero measure, there exists a constant C_K

$$\|u\|_{H^1(\Omega)} \le C_K \|\varepsilon(u)\|_{L^2(\Omega)}. \tag{2.7}$$

The following two theorems from [46] play a crucial role in demonstrating the existence and uniqueness of the solution for the weak formulation of the dual-porosity-Navier-Stokes problem.

Theorem 14. *(Schauder Theorem)[46, Theorem 9.12]. Let Y be a closed and convex subset of a Banach space, and let $T : Y \mapsto Y$ be a continuous mapping such that $\overline{T(Y)}$ is compact. Then the operator T has at least one fixed point.*

Theorem 15. *(Brouwer Fixed Point Theorem)[46]. Let V be a convex and compact nonempty subset of a finite dimensional Banach space X, let $T : V \mapsto V$ be a continuous mapping. Then the operator T has at least one fixed point.*

Theorem 16. *(Banach Theorem)[46, Theorem 3.7]. Let (X,d) be a non-empty complete metric space with a contraction mapping $T : X \to X$,*

$$d(Tx,Ty) \le r\,d(x,y), \quad 0 < r < 1. \tag{2.8}$$

Then T admits a unique fixed point $x^* : T(x^*) = x^*$.

2.3 An abstract theory of saddle point problems with perturbed regular terms

Let V and Q be two Hilbert spaces equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively. Let $a(\cdot,\cdot)$ on $V \times V$, $b(\cdot,\cdot)$ on $V \times Q$, and $c(\cdot,\cdot)$ on $Q \times Q$ three continuous bilinear forms, i.e.

$$a(u,v) \le C_a \|u\|_V \|v\|_V \quad \forall u,v \in V, \tag{2.9a}$$

$$b(u,q) \le C_b \|u\|_V \|q\|_Q \quad \forall u \in V, q \in Q, \tag{2.9b}$$

$$c(p,q) \le C_c \|u\|_V \|q\|_Q \quad \forall p,q \in Q. \tag{2.9c}$$

Let $f \in V'$ and $g \in Q'$, where V', Q' denote the dual space of V, Q respectively, and $\|\cdot\|_{V'}, \|\cdot\|_{Q'}$ denote the norms in V', Q' respectively. The abstract saddle point problem with a perturbed regular term reads: Find $u \in V$, and $p \in Q$ such that the following equations hold:

$$\begin{cases} a(u,v)+b(v,p) & =\langle f,v\rangle_{V'\times V}, \quad \forall v\in V, \\ \\ b(u,q)-c(p,q) & =\langle g,q\rangle_{Q'\times Q}, \quad \forall q\in Q. \end{cases} \tag{2.10}$$

As an application of the Riesz representation theorem [46, Theorem 4.6] there exist $A: V\mapsto V'$ a bounded linear form and A^t its adjoint such that:

$$\langle Au,v\rangle_{V'\times V} = \left\langle u, A^t v\right\rangle_{V\times V'} = a(u,v), \quad \forall u,v\in V,$$

similarly, there exist $B: V\mapsto Q'$, $B^t: Q\mapsto V'$ continuous linear form:

$$\langle Bu,p\rangle_{Q'\times Q} = \left\langle u, B^t p\right\rangle_{V\times V'} = b(u,p), \quad \forall u\in V, p\in Q,$$

we denote

$$K := \mathrm{Ker}B \quad \text{ and } \quad H := \mathrm{Ker}B^t.$$

Also, there exist $C: Q\mapsto Q'$ continuous linear operator, and C^t its adjoint such that:

$$\langle Cp,q\rangle_{Q'\times Q} = \left\langle p, C^t q\right\rangle_{Q\times Q'} = c(p,q), \quad \forall p,q\in Q.$$

And, the problem defined by eq. (2.10) can be stated as: Find $u\in V, q\in P$ that satisfy the following system:

$$\begin{cases} Au+B^T p = f \\ \\ Bu - Cp = g \end{cases} \tag{2.11}$$

As a result of the following theorem, we can demonstrate the well-posedness of the dual-porosity-Stokes, and dual-porosity-Navier-Stokes problems.

Theorem 17. *[40, Theorem 4.3.1, Remark 4.3.3] Let $a: V\times V\to\mathbb{R}$, $b: V\times Q\to\mathbb{R}$, and $c: Q\times Q\to\mathbb{R}$ be continuous bilinear forms such that $a(\cdot,\cdot)$ is positive definite and coercive on K and $c(\cdot,\cdot)$ is symmetric positive definite and coercive on H, i.e., they satisfy*

Equation (2.9) *and there exist positive constants* α *and* γ *such that:*

$$a(u,u) \geq \alpha \|u\|_V^2, \quad \forall u \in K, \tag{2.12a}$$

$$c(p,p) \geq \gamma \|p\|_Q^2, \quad \forall p \in H. \tag{2.12b}$$

Let $f \in V'$ *and* $g \in Q'$.

Case 1 *If* $a(\cdot,\cdot)$ *is also symmetric and* $b(\cdot,\cdot)$ *satisfies the following inf-sup condition:*

$$\inf_{q \in H^\perp} \sup_{v \in V} \frac{b(v,q)}{\|q\|_Q \|v\|_V} = \inf_{v \in K^\perp} \sup_{q \in Q} \frac{b(v,q)}{\|q\|_Q \|v\|_V} = \beta > 0, \tag{2.13}$$

then eq. (2.10) *is well-posed and the solution satisfies*

$$\|u\|_V \leq C_1 \|f\|_{V'} + C_2 \|g\|_{Q'}, \tag{2.14a}$$

$$\|p\|_Q \leq C_3 \|f\|_{V'} + C_4 \|g\|_{Q'}, \tag{2.14b}$$

where C_1, C_2, C_3, C_4 *are constants that depend on* α, β, γ, C_a, *and* C_c *and are given as below:*

$$\begin{aligned}
C_1 =& \frac{1}{\alpha\beta^2} \max\left\{\alpha C_c, (\alpha C_c)^{1/2}\sqrt{2\beta^2 + C_a C_c}, (\alpha C_a)^{1/2} C_c, 2(\beta^2 + C_a C_c)\right\} \\
C_2 =& \frac{1}{(\alpha\gamma)^{1/2}\beta^2} \max\Big\{(\alpha\gamma)^{1/2}(\beta + C_a C_c), \\
& 3(\alpha C_c)^{1/2}\sqrt{\beta^2 + C_a C_c}, (C_a\gamma)^{1/2}(\beta + C_a C_c), 3C_a C_c\sqrt{\beta^2 + C_a C_c}\Big\} \\
C_3 =& \frac{1}{(\alpha\gamma)^{1/2}\beta^2} \max\Big\{(\alpha\gamma)^{1/2}(\beta + C_a C_c), 3(\gamma C_a)^{1/2}\sqrt{\beta^2 + C_a C_c}, \\
& (C_c\alpha)^{1/2}(\beta^2 + C_a C_c), 3C_a C_c\sqrt{\beta^2 + C_a C_c}\Big\}, \\
C_4 =& \frac{1}{\gamma\beta^2} \max\left\{\gamma C_a, (\gamma C_a)^{1/2}\sqrt{2\beta^2 + C_a C_c}, (\gamma C_c)^{1/2} C_a, 2(\beta^2 + C_a C_c)\right\}.
\end{aligned}$$

Case 2 *If* $ImB = Q'$ *and* $b(\cdot,\cdot)$ *satisfies the following inf-sup condition:*

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|q\|_Q \|v\|_V} \geq \beta > 0, \tag{2.15}$$

then eq. (2.10) *is well-posed and the solution satisfies Equation* (2.14) *with the following constants:*

$$C_1 = \left(\frac{C_c}{\beta^2} \left(1 + \frac{C_a}{\alpha} \right)^2 + \frac{\sqrt{C_c}}{\beta\sqrt{\alpha}} \left(1 + \frac{C_a}{\alpha} \right) + \frac{1}{\alpha} \right) \tag{2.16a}$$

$$C_2 = \frac{1}{\beta^3}(1 + \frac{C_a}{\alpha}) \left(2C_a C_c \left(1 + \frac{C_a}{\alpha} \right) + \sqrt{2}\beta^2 \right), \tag{2.16b}$$

and $C_3 = \frac{C_a C_1 + 1}{\beta}, C_4 = \frac{C_a C_2}{\beta}$.

In the next chapter, we study the dual-porosity-Stokes problem where the bilinear forms $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ in the weak formulation are both symmetric so we can use first part in Theorem 17 to show the well-posedness. However, in the dual-porosity-Navier-Stokes problem in Section 4.1, the bilinear form $A(w;\cdot,\cdot)$ in the weak formulation of the linearized problem is no longer symmetric. As a result, we cannot apply the first part of Theorem 17 but instead we use the second part of Theorem 17. The proof of the first case of the Theorem 17 can be found in [40, Theorem 4.3.1]. However, the second case of Theorem 17 is stated as a Remark in [40, Remark 4.3.3] without a proof.

Also, we note that in [40], the authors assume finite dimensional spaces by showing only the surjection of the map. We extend proof of the well-posedness on the infinite dimensional spaces for the case when $\mathrm{Im} B = Q'$. Since the proof is very technical, we provide it in the Appendix.

CHAPTER THREE

DUAL-POROSITY-STOKES PROBLEM

Our focus in this chapter is the dual-porosity-Stokes problem, which describes how free fluid flow in conduits is coupled with fluid flow in porous media with two different types of porosities: microfracture and matrix. The work in this chapter is already published in the article [1].

3.1 The model

Consider a domain Ω that is a subset of $\mathbb{R}^{\text{dim}}$, where $\text{dim} = 2, 3$, and has a Lipschitz boundary $\partial\Omega$. It is assumed that Ω is a polygonal or polyhedral region in $\mathbb{R}^2/\mathbb{R}^3$. Let Ω^s be a subdomain of Ω that has free flow, and let Ω^d be a subdomain of Ω that has a porous medium such that $\Omega^s \cap \Omega^d =$, and $\overline{\Omega} = \overline{\Omega}^s \cup \overline{\Omega}^d$. Let Γ^I be the intersection of the closures of Ω^s and Ω^d, Γ^s be the intersection of the boundaries of Ω^s and $\partial\Omega$, and Γ^d be the intersection of the boundaries of Ω^d and $\partial\Omega$. The outward pointing unit normal vector of Ω^j for $j = s, d$ is denoted by n^j. The unit normal vector on Γ^I is denoted by n and is equal to $n^s = -n^d$. See Figure 3.1 for an illustration of a two-dimensional domain Ω.

The Stokes equations define the fluid velocity u and the fluid pressure p, given the kinematic viscosity μ and the source term f:

$$-2\mu\nabla\cdot\varepsilon(u) + \nabla p = f \quad \text{in} \quad \Omega^s, \tag{3.1a}$$

$$\nabla\cdot u = 0 \quad \text{in} \quad \Omega^s, \tag{3.1b}$$

$$u = 0 \quad \text{in} \quad \Gamma^s, \tag{3.1c}$$

where $\varepsilon(u) = \frac{\nabla u + (\nabla u)^T}{2}$ is the strain rate tensor.

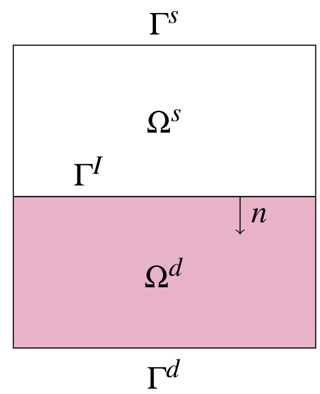


Figure 3.1: Illustration of a free flow/porous medium domain Ω in two dimensions.

In Ω^d, the pressure p and velocity u in the microfracture and, the pressure p^m and velocity u^m in the matrix satisfy the dual-porosity model:

$$
\begin{aligned}
\kappa_f^{-1} u + \nabla p &= 0 && \text{in} \quad \Omega^d, && (3.2a)\\
\nabla \cdot u + \sigma \kappa_m (p - p^m) &= g && \text{in} \quad \Omega^d, && (3.2b)\\
\kappa_m^{-1} u^m + \nabla p^m &= 0 && \text{in} \quad \Omega^d, && (3.2c)\\
\nabla \cdot u^m + \sigma \kappa_m (p^m - p) &= 0 && \text{in} \quad \Omega^d, && (3.2d)\\
u^m \cdot n &= 0 && \text{on} \quad \Gamma^d, && (3.2e)\\
u \cdot n &= 0 && \text{on} \quad \Gamma^d. && (3.2f)
\end{aligned}
$$

The parameters κ_m and κ_f represent the intrinsic permeabilities in the matrix and microfracture, respectively. Additionally, the shape factor σ is a geometric parameter related to the morphology and dimension of microfractures, which controls the fluid flow from the matrix to the microfractures [18]. We assume that $0 < \sigma < \sigma^*$, for some arbitrary constant $\sigma^* > 0$, for more details in the choice of σ^* we refer the reader to [47]. In eq. (3.2b), g represents the source term. The Stokes model and the dual-porosity model are coupled through interface conditions given by the following equations:

$$u^m \cdot n = 0 \qquad \text{on} \quad \Gamma^I, \tag{3.3a}$$

$$u^s \cdot n - u^d \cdot n = 0 \qquad \text{on} \quad \Gamma^I, \tag{3.3b}$$

$$-2\mu\varepsilon(u^s)n \cdot n + p^s = p^d \qquad \text{on} \quad \Gamma^I, \tag{3.3c}$$

$$-2\mu(\varepsilon(u^s)n)^t = \alpha\mu\kappa_f^{-1/2}(u^s)^t \qquad \text{on} \quad \Gamma^I. \tag{3.3d}$$

Note: When we need to differentiate the values of a function on Ω^s and Ω^d, we use notations u^j, and p^j where $u^j := u|_{\Omega^j}$, $p^j := p|_{\Omega^j}$ for $j = s, d$. In the interface condition Equation (3.3d) α is a constant determined by numerical experiments, $w^t = w - (w \cdot n)n$ is the tangential component of a vector w.

The first interface condition is a non-exchange condition between the matrix and the conduits. This means that no flux can cross the interface directly from the matrix system to the conduits. The other three interface conditions are similar to those for the Stokes-Darcy problem [37]. Specifically, the second and third interface conditions describe the conservation of mass between the microfractures and the conduits and the balance of forces. The last interface condition is the Beavers-Joseph-Saffmann interface condition [21, 22] on the tangential component of the normal stress on Γ^I.

3.2 Weak formulation

Besides the Sobolov spaces in Definition 7, we define the following Hilbert spaces relevant to our problems in this thesis:

$$
\begin{aligned}
H^1_{\Gamma^s}(\Omega^s) &:= \{v \in H^1(\Omega^s) : v = 0 \text{ on } \Gamma^s\},\\
H(\mathrm{div},\Omega) &:= \{v \in [L^2(\Omega)]^{\mathrm{dim}} : \nabla \cdot v \in L^2(\Omega)\},\\
H_0(\mathrm{div},\Omega^d) &:= \{v \in H(\mathrm{div},\Omega^d) : v \cdot n = 0 \text{ on } \partial\Omega^d\},\\
L^2_0(\Omega) &:= \{q \in L^2(\Omega) : (q,1)_\Omega \,\mathrm{d}x = 0\},\\
L^2_0(\Omega^d) &:= \{q \in L^2(\Omega^d) : (q,1)_{\Omega^d} \,\mathrm{d}x = 0\}.
\end{aligned}
$$

The space $H(\mathrm{div},\Omega)$ is equipped with the norm

$$
\|v\|_{H(\mathrm{div},\Omega)} = (\|v\|^2_\Omega + \|\nabla \cdot v\|^2_\Omega)^{1/2}.
$$

Remark. *In the rest of the thesis we use the notation $\|\cdot\|_.$ notation for the norm $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_{k,D}$ for the norm $\|\cdot\|_{H^k(D)}$. Also, we denote the L^2-inner product on a domain $E \subset \mathbb{R}^{dim}$ by $(\cdot,\cdot)_E$, and on a $S \subset \mathbb{R}^{dim-1}$ by $\langle\cdot,\cdot\rangle_S$.*

To derive the weak formulation of equations eqs. (3.1) to (3.3), we need the following function spaces:

$$
\begin{aligned}
&V := \{v \in H(\mathrm{div},\Omega) : v|_{\Omega^s} \in H^1_{\Gamma^s}(\Omega^s),\, v \cdot n = 0 \text{ on } \Gamma^d\}, \qquad V^m := H_0(\mathrm{div},\Omega^d),\\
&Q := L^2_0(\Omega), \qquad\qquad Q^m := L^2_0(\Omega^d).
\end{aligned} \tag{3.4}
$$

Remark. *We define $\boldsymbol{u} := (u,u^m)$ as an element of $Z := V \times V^m$, and $\boldsymbol{p} := (p,p^m)$ as an element of $P := Q \times Q^m$, equipped with the following norms:*

$$
\begin{aligned}
\|\boldsymbol{u}\|_Z &:= \left(\|\varepsilon(u^s)\|^2_{\Omega^s} + \|u^d\|^2_{H(div,\Omega^d)} + \|u^m\|^2_{H(div,\Omega^d)} + \|(u^s)^t\|^2_{\Gamma^I}\right)^{1/2},\\
\|\boldsymbol{p}\|_P &:= (\|p\|^2_\Omega + \|p^m\|^2_{\Omega^d})^{1/2}.
\end{aligned} \tag{3.5}
$$

Let $u \in V$, $u^m \in V^m$, $p \in Q$, and $p^m \in Q^m$. First, we test eqs. (3.1a) and (3.2a) with $v \in V$ and eqs. (3.1b) and (3.2b) with $q \in Q$, and we get:

$$-\left(\nabla \cdot 2\mu\varepsilon(u) + \nabla p, v\right)_{\Omega^s} = (f, v)_{\Omega^s}, \tag{3.6a}$$

$$\left(\nabla \cdot u^s, q\right)_{\Omega^s} = 0, \tag{3.6b}$$

$$\left(\kappa_f^{-1} u + \nabla p, v\right)_{\Omega^d} = 0, \tag{3.6c}$$

$$\left(\nabla \cdot u + \sigma\kappa_m(p - p^m), q\right)_{\Omega^d} = (g, q)_{\Omega^d}. \tag{3.6d}$$

By taking the scalar product of eqs. (3.2c) and (3.2d) respectively with $v^m \in V^m$ and $q^m \in Q^m$, and integrating over Ω^d, we get:

$$\left(\kappa_m^{-1} u^m + \nabla p^m, v^m\right)_{\Omega^d} = 0, \tag{3.7a}$$

$$\left(\nabla \cdot u^m + \sigma\kappa_m(p^m - p), q^m\right)_{\Omega^d} = 0. \tag{3.7b}$$

Since $\varepsilon(u)$ is symmetric, $(\varepsilon(u), \varepsilon(v))_{\Omega^s} = (\varepsilon(u), \nabla v)_{\Omega^s}$. Therefore, integrating by parts in eqs. (3.6a) and (3.6c), noting that $v = 0$ on Γ^s and $v \cdot n = 0$ on Γ^d, and adding the resulting equations, we get:

$$\begin{aligned}\left(2\mu\varepsilon(u), \varepsilon(v)\right)_{\Omega^s} + \left(\kappa_f^{-1} u, v\right)_{\Omega^d} - (p, \nabla \cdot v)_{\Omega} \\ - \langle 2\mu\varepsilon(u^s)n, v\rangle_{\Gamma^I} + \langle p^s, v^s \cdot n^s\rangle_{\Gamma^I} + \langle p^d, v^d \cdot n^d\rangle_{\Gamma^I} \\ = (f, v)_{\Omega^s}.\end{aligned}$$

Multiplying eq. (3.3c) by n and adding to the fourth interface condition we get

$$p^s n - 2\mu\varepsilon(u^s)n = p^d n + \alpha\mu\kappa_f^{-1/2}(u^s)^t. \tag{3.8}$$

Therefore, since $n = n^s = -n^d$ on Γ^I, we can simplify the above equations and obtain

$$\left(2\mu\varepsilon(u),\varepsilon(v)\right)_{\Omega^s} + \left(\kappa_f^{-1}u,v\right)_{\Omega^d} - (p,\nabla\cdot v)_{\Omega} + \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,(v^s)^t\rangle_{\Gamma^I} = (f,v)_{\Omega^s}. \quad (3.9)$$

Next, we integrate by parts the second term on equation eq. (3.7a) and apply $v^m \cdot n = 0$ on $\partial\Omega^d$ to get

$$\left(\kappa_m^{-1}u^m,v^m\right)_{\Omega^d} - \left(p^m,\nabla\cdot v^m\right)_{\Omega^d} = 0. \quad (3.10)$$

To formulate the problem, we combine equations eqs. (3.9) and (3.10) with equations eqs. (3.6b), (3.6d) and (3.7b) and arrive at the weak formulation: Find $u \in V, u^m \in V^m, p \in Q, p^m \in Q^m$ such that:

$$\left(2\mu\varepsilon(u),\varepsilon(v)\right)_{\Omega^s} + \left(\kappa_f^{-1}u,v\right)_{\Omega^d} - (p,\nabla\cdot v)_{\Omega} + \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,(v^s)^t\rangle_{\Gamma^I} = (f,v)_{\Omega^s}, \quad (3.11a)$$

$$(\nabla\cdot u,q)_{\Omega} + \left(\sigma\kappa_m(p-p^m),q\right)_{\Omega^d} = (g,q)_{\Omega^d}, \quad (3.11b)$$

$$\left(\kappa_m^{-1}u^m,v^m\right)_{\Omega^d} - \left(p^m,\nabla\cdot v^m\right)_{\Omega^d} = 0, \quad (3.11c)$$

$$-\left(\nabla\cdot u^m,q^m\right)_{\Omega^d} + \left(\sigma\kappa_m(p^m-p),q^m\right)_{\Omega^d} = 0. \quad (3.11d)$$

for any $v \in V, v^m \in V^m, q \in Q, q^m \in Q^m$. Next, let us define the following bilinear forms for $u,v \in V, u^m,v^m \in V^m, p,q \in Q, p^m,q^m \in Q^m$:

$$\begin{aligned} a((u,u^m),(v,v^m)) &:= \left(2\mu\varepsilon(u),\varepsilon(v)\right)_{\Omega^s} + \left(\kappa_f^{-1}u,v\right)_{\Omega^d} - (p,\nabla\cdot v)_{\Omega} && (3.12a)\\ &\quad + \left(\kappa_m^{-1}u^m,v^m\right)_{\Omega^d} + \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,(v^s)^t\rangle_{\Gamma^I}, \\ b((u,u^m),(q,q^m)) &:= -(\nabla\cdot u,q)_{\Omega} - \left(\nabla\cdot u^m,q^m\right)_{\Omega}, && (3.12b)\\ c((p,p^m),(q,q^m)) &:= \left(\sigma\kappa_m(p^m-p),(q-q^m)\right)_{\Omega^d}. && (3.12c) \end{aligned}$$

Equation (3.11) can be written in the form:

Find $(u,u^m) \in Z, (p,p^m) \in P$ such that:

$$a((u,u^m),(v,v^m)) + b((v,v^m),(p,p^m)) = (f,v)_{\Omega^s} \qquad \forall (v,v^m) \in Z, \tag{3.13a}$$

$$b((u,u^m),(q,q^m)) - c((p,p^m),(q,q^m)) = -(g,q)_{\Omega^d} \qquad \forall (q,q^m) \in P. \tag{3.13b}$$

We use saddle point theory to prove that Equation (3.13) is well-posed. Therefore, we must show that the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ satisfy the conditions in Case 1, Theorem 17.

Lemma 18. *The following inf sup condition for the bilinear form $b(\cdot,\cdot)$ holds for some constant constant $C > 0$, depending only on Ω:*

$$\forall \boldsymbol{q} \in P, \qquad C\|\boldsymbol{q}\|_P \leq \sup_{\boldsymbol{v}\in Z\setminus\{(0,0)\}} \frac{b(\boldsymbol{v},\boldsymbol{q})}{\|\boldsymbol{v}\|_Z}.$$

Proof. Let $\boldsymbol{q} = (q,q^m) \in P$. $q \in L_0^2(\Omega)$, the standard inf-sup condition [40, p.463], implies the existence of $v \in H_0^1(\Omega)$ such that $\nabla \cdot v = -q$ in Ω and $\|v\|_{1,\Omega} \leq c\|q\|_\Omega$. Furthermore, the inf-sup condition for $H_0(\mathrm{div},\Omega^d)$ and $L_0^2(\Omega^d)$ [42, p.117-118], implies the existence of $v^m \in H_0(\mathrm{div},\Omega^d)$ such that $\nabla \cdot v^m = -q^m$ in Ω^d and $\|v^m\|_{H(\mathrm{div},\Omega^d)} \leq C\|q^m\|_{\Omega^d}$. Using the trace inequality [45, (1.24)], we obtain:

$$\begin{aligned}\|\boldsymbol{v}\|_Z =& \left(\|\varepsilon(v^s)\|^2_{\Omega^s} + \|v^d\|^2_{H(\mathrm{div},\Omega^d)} + \|v^m\|^2_{H(\mathrm{div},\Omega^d)} + \|(v^s)^t\|^2_{\Gamma^I}\right)^{\frac{1}{2}} \\ \leq& C\left(\|q\|^2_\Omega + \|q^m\|^2_{\Omega^d}\right)^{\frac{1}{2}} = C\|\boldsymbol{q}\|_P.\end{aligned}$$

Our results from the above, combined with the definition of the bilinear form $b(\cdot,\cdot)$, lead us to the following inequality:

$$\frac{b(\boldsymbol{v},\boldsymbol{q})}{\|\boldsymbol{v}\|_Z} = \frac{\|\boldsymbol{q}\|_P^2}{\|\boldsymbol{v}\|_Z} \geq C\|\boldsymbol{q}\|_P,$$

the result follows by applying the definition of the supremum. ∎

Lemma 19. *The bilinear form $b(\cdot,\cdot)$, defined in eq.* (3.12b)*, is continuous on $Z\times P$, that is,*

$$|b(\boldsymbol{u},\boldsymbol{q})| \leq \|\boldsymbol{u}\|_Z \|\boldsymbol{q}\|_P \quad \forall(\boldsymbol{u},\boldsymbol{q}) \in Z\times P.$$

Proof. Since, the trace $\mathrm{tr}(\varepsilon(u))$ of $\varepsilon(u^s)$ and the trace-free part of $\varepsilon(u^s)$ are orthogonal in the Frobenius inner product for almost every point on Ω^s, we have $\|\varepsilon(u^s)\|_{\Omega^s} \geq \|\mathrm{tr}(\varepsilon(u^s))\|_{\Omega^s}$. By applying the Cauchy–Schwarz inequality and that $\nabla\cdot u^s$ is equivalent to the trace of $\varepsilon(u^s)$:

$$\begin{aligned}
&|b(\boldsymbol{u},\boldsymbol{q})| \leq \|q^s\|_{\Omega^s}\|\nabla\cdot u^s\|_{\Omega^s} + \|q^d\|_{\Omega^d}\|\nabla\cdot u^d\|_{\Omega^d} + \|q^m\|_{\Omega^d}\|\nabla\cdot u^m\|_{\Omega^d}\\
&\leq \|q^s\|_{\Omega^s}\|\varepsilon(u^s)\|_{\Omega^s} + \|q^d\|_{\Omega^d}\|u^d\|_{H(\mathrm{div},\Omega^d)} + \|q^m\|_{\Omega^d}\|u^m\|_{H(\mathrm{div},\Omega^d)}\\
&\leq (\|q^s\|^2_{\Omega^s} + \|q^d\|^2_{\Omega^d} + \|q^m\|^2_{\Omega^d})^{\frac{1}{2}}(\|\varepsilon(u^s)\|^2_{\Omega^s} + \|u^d\|^2_{H(\mathrm{div},\Omega^d)} + \|u^m\|^2_{H(\mathrm{div},\Omega^d)})^{\frac{1}{2}}\\
&\leq \|\boldsymbol{q}\|_P\|\boldsymbol{u}\|_Z.
\end{aligned}$$

■

We now introduce the operator $B : Z \mapsto P$ such that

$$(B\boldsymbol{u},\boldsymbol{p}) = b(\boldsymbol{u},\boldsymbol{p}) \quad \forall(\boldsymbol{u},\boldsymbol{p}) \in Z\times P. \tag{3.14}$$

Lemma 20. *The bilinear form $a(\cdot,\cdot)$ given in eq.* (3.12a) *is symmetric and continuous on $Z\times Z$, furthermore $a(\cdot,\cdot)$ is coercive on* $\mathrm{Ker}B$*, i.e.,*

$$|a(\boldsymbol{u},\boldsymbol{v})| \leq \tilde{C}_b\|\boldsymbol{u}\|_Z\|\boldsymbol{v}\|_Z \qquad \forall\boldsymbol{u},\boldsymbol{v}\in Z, \tag{3.15a}$$

$$a(\boldsymbol{u},\boldsymbol{u}) \geq \tilde{C}_c\|\boldsymbol{u}\|^2_Z \qquad \forall\boldsymbol{u}\in \mathrm{Ker}B, \tag{3.15b}$$

where $\tilde{C}_b = \max\{2\mu, \kappa_f^{-1}, \alpha\mu\kappa_f^{-1/2}, \kappa_m^{-1}\}$ *and* $\tilde{C}_c = \min\{2\mu, \kappa_f^{-1}, \alpha\mu\kappa_f^{-1/2}, \kappa_m^{-1}\}$.

Proof. On the basis of its definition, the bilinear form $a(\cdot,\cdot)$ is symmetric. To establish continuity, we can apply the Cauchy-Schwarz inequality twice in the definition of $a(\cdot,\cdot)$

(see eq. (3.12a)).

$$
\begin{aligned}
|a(\boldsymbol{u},\boldsymbol{v})| \leq & 2\mu \left\|\varepsilon(u)\right\|_{\Omega^s} \left\|\varepsilon(v)\right\|_{\Omega^s} + \kappa_f^{-1} \left\|u\right\|_{\Omega^d} \left\|v\right\|_{\Omega^d} + \kappa_m^{-1} \left\|u^m\right\|_{\Omega^d} \left\|v^m\right\|_{\Omega^d} \\
& + \alpha\mu\kappa_f^{-1/2} \left\|(u^s)^t\right\|_{\Gamma^I} \left\|(v^s)^t\right\|_{\Gamma^I} \\
\leq & (2\mu \left\|\varepsilon(u)\right\|_{\Omega^s}^2 + \kappa_f^{-1} \left\|u\right\|_{\Omega^d}^2 + \alpha\mu\kappa_f^{-1/2} \left\|(u^s)^t\right\|_{\Gamma^I}^2 + \kappa_m^{-1} \left\|u^m\right\|_{\Omega^d}^2)^{1/2} \\
& \cdot (2\mu \left\|\varepsilon(v)\right\|_{\Omega^s}^2 + \kappa_f^{-1} \left\|v\right\|_{\Omega^d}^2 + \alpha\mu\kappa_f^{-1/2} \left\|(v^s)^t\right\|_{\Gamma^I}^2 + \kappa_m^{-1} \left\|v^m\right\|_{\Omega^d}^2)^{1/2} \\
\leq & \tilde{C}_b \left\|\boldsymbol{u}\right\|_Z \left\|\boldsymbol{v}\right\|_Z .
\end{aligned}
\tag{3.16}
$$

To prove that $a(\cdot,\cdot)$ is coercive on $\operatorname{Ker} B$, $\boldsymbol{u} = (u,u^m) \in \operatorname{Ker} B$, then we observe that $\nabla \cdot u = 0$ in Ω and $\nabla \cdot u^m = 0$ in Ω^d. Thus,

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{u}) &= 2\mu \left\|\varepsilon(u)\right\|_{\Omega^s}^2 + \kappa_f^{-1} \left\|u\right\|_{\Omega^d}^2 + \kappa_m^{-1} \left\|u^m\right\|_{\Omega^d}^2 + \alpha\mu\kappa_f^{-1/2} \left\|(u^s)^t\right\|_{\Gamma^I}^2 \\
&\geq \tilde{C}_c (\left\|\varepsilon(u)\right\|_{\Omega^s}^2 + \left\|u\right\|_{\Omega^d}^2 + \left\|u^m\right\|_{\Omega^d}^2 + \left\|(u^s)^t\right\|_{\Gamma^I}^2) = \tilde{C}_c \left\|\boldsymbol{u}\right\|_Z^2 .
\end{aligned}
\tag{3.17}
$$

■

Lemma 21. *The bilinear form $c(\cdot,\cdot)$, given in eq.* (3.12c)*, is symmetric, continuous, and positive semidefinite on $P \times P$.*

Proof. It is clear from the definition of $c(\cdot,\cdot)$ that the form is symmetric. Using the Cauchy–Schwarz inequality, we show the continuity of $c(\cdot,\cdot)$. :

$$
\begin{aligned}
c(\boldsymbol{p},\boldsymbol{q}) = \sigma\kappa_m(p-p^m,q-q^m)_{\Omega^d} &\leq \sigma\kappa_m \left\|p-p^m\right\|_{\Omega^d} \left\|q-q^m\right\|_{\Omega^d} \\
&\leq \sigma\kappa_m(\left\|p\right\|_{\Omega^d} + \left\|p^m\right\|_{\Omega^d})(\left\|q\right\|_{\Omega^d} + \left\|q^m\right\|_{\Omega^d}) \\
&\leq 2\sigma\kappa_m(\left\|p\right\|_{\Omega^d}^2 + \left\|p^m\right\|_{\Omega^d}^2)^{1/2}(\left\|q\right\|_{\Omega^d}^2 + \left\|q^m\right\|_{\Omega^d}^2)^{1/2} \\
&\leq 2\sigma^*\kappa_m \left\|\boldsymbol{p}\right\|_P \left\|\boldsymbol{q}\right\|_P .
\end{aligned}
$$

Furthermore,

$$
c(\boldsymbol{p},\boldsymbol{p}) = \sigma\kappa_m \left\|p-p^m\right\|_{\Omega^d}^2 \geq 0 \quad \forall \boldsymbol{p} \in P,
$$

which implies the bilinear form is positive semidefinite ■

We now reach the well-possessedness of Equation (3.13):

Theorem 22. *Given* $f \in [L^2(\Omega^s)]^{dim}$ *and* $g \in L^2(\Omega^d)$*, the weak formulation eq.* (3.13) *has a unique solution. Moreover,*

$$\|\boldsymbol{u}\|_Z + \|\boldsymbol{p}\|_P \leq C\left(\|f\|_{\Omega^s} + \|g\|_{\Omega^d}\right). \tag{3.18}$$

Proof. We can observe that $a(\cdot,\cdot)$ is symmetric and continuous on $Z \times Z$, and coercive on $\mathrm{Ker}\, B$, based on Lemma 20. Similarly, Lemma 19 establishes that $b(\cdot,\cdot)$ is continuous on $Z \times P$, and Lemma 21 shows that $c(\cdot,\cdot)$ is symmetric, continuous, and positive semidefinite on $P \times P$. Lemma 18 confirms B as bounded and surjective, which implies that $\mathrm{Im}\, B$ is closed and $\mathrm{Ker}\, B^T = (\mathrm{Im}\, B)^\perp = \{\mathbf{0}\}$. By virtue of the corollary of $\mathrm{Ker}\, B^T = \{\mathbf{0}\}$, $c(\cdot,\cdot)$ is coercive on $\mathrm{Ker}\, B^T$.

Finally, the result follows from Theorem 17. ■

3.3 The HDG method

3.3.1 Notation

We introduce an HDG method to solve the equations eqs. (3.1) to (3.3). This method is an extension of the idea proposed in [37]. Let us denote by $\mathscr{T}^j$ a triangulation of Ω^j, $j = s,d$, such that $\mathscr{T}^s$ and $\mathscr{T}^d$ match at the interface Γ^I, and by $\mathscr{T} := \mathscr{T}^s \cup \mathscr{T}^d$. We denote by h_K the diameter of K, for $K \in \mathscr{T}$, and

$$h = \max_{K \in \mathscr{T}} h_K.$$

For $j = s,d$, let $\mathscr{F}_i^j$ denote the set of all interior facets in Ω^j, $\mathscr{F}_b^j$ denote the set of all facets on the boundary Γ^j for $j = s,d$, and $\mathscr{F}^I$ denote the set of all facets that lie on Γ^I.

We define $\mathscr{F}^j := \mathscr{F}_i^j \cup \mathscr{F}_b^j \cup \mathscr{F}^I$, and the set of all facets in $\overline{\Omega}$ is denoted by $\mathscr{F}$. Let Γ_0^j be the union of all facets in $\overline{\Omega}^j$, for $j = s,d$ and Γ_0 be the union of all facets in $\overline{\Omega}$.

We begin by defining the discrete velocity and pressure spaces on Ω and Ω^j, where $j = s,d$:

$$
\begin{aligned}
V_h &:= \Big\{ v_h \in [L^2(\Omega)]^{\dim} : v_h \in [P_k(K)]^{\dim} \quad \forall K \in \mathscr{T} \Big\}, \\
V_h^j &:= \Big\{ v_h \in [L^2(\Omega^j)]^{\dim} : v_h \in [P_k(K)]^{\dim} \quad \forall K \in \mathscr{T}^j \Big\} \quad j = s,d, \\
Q_h &:= \Big\{ q_h \in L^2(\Omega) : q_h \in P_{k-1}(K) \quad \forall K \in \mathscr{T} \Big\}, \\
Q_h^j &:= \Big\{ q_h \in L^2(\Omega^j) : q_h \in P_{k-1}(K) \quad \forall K \in \mathscr{T}^j \Big\}, \quad j = s,d, \\
Q_h^m &:= \Big\{ r_h \in L_0^2(\Omega^d) : r_h \in P_{k-1}(K) \quad \forall K \in \mathscr{T}^d \Big\},
\end{aligned}
$$

where $P_k(K)$ denotes the polynomial space of total degree at most k in K.

Also, we define the discrete trace space for the Stokes velocity and pressure in the Stokes region on Γ_0^s, as well as the pressures in microfractures and the matrix on Γ_0^d:

$$
\begin{aligned}
\bar{V}_h &:= \Big\{ \bar{v}_h \in [L^2(\Gamma_0^s)]^{\dim} : \bar{v}_h \in [P_k(F)]^{\dim} \quad \forall F \in \mathscr{F}^s, \quad \bar{v}_h = 0 \text{ on } \Gamma^s \Big\}, \\
\bar{Q}_h^j &:= \Big\{ \bar{q}_h^j \in L^2(\Gamma_0^j) : \bar{q}_h^j \in P_k(F) \quad \forall F \in \mathscr{F}^j \Big\}, \quad j = s,d.
\end{aligned}
$$

To simplify the notation, we define some additional spaces:

$$
\boldsymbol{V}_h := V_h \times \bar{V}_h, \quad \boldsymbol{Q}_h := Q_h \times \bar{Q}_h^s \times \bar{Q}_h^d, \quad \boldsymbol{Q}_h^m := Q_h^m \times \bar{Q}_h^d,
$$

$$
\boldsymbol{Z}_h := \boldsymbol{V}_h \times V_h^d, \quad \boldsymbol{P}_h := \boldsymbol{Q}_h \times \boldsymbol{Q}_h^m.
$$

The following norms apply to the last two spaces above:

$$\|\boldsymbol{v}\|_{Z_h}^2 := \|\varepsilon(v^s)\|_{\Omega^s}^2 + \sum_{K\in\mathscr{T}^s} h_K^{-1}\|v-\bar{v}\|_{\partial K}^2 + \|v^d\|_{\Omega^d}^2 + \|v^m\|_{\Omega^d}^2 + \|\bar{v}_h^t\|_{\Gamma^I}^2,$$

$$\|\boldsymbol{v}\|_{Z_h^*}^2 := \|\boldsymbol{v}\|_{Z_h} + \sum_{K\in\mathscr{T}^s} h_K^2 |\boldsymbol{v}|_{2,K}^2,$$

$$\|\boldsymbol{q}\|_{P_h}^2 := \|q\|_{\Omega}^2 + \|q^m\|_{\Omega^d}^2 + \sum_{j=s,d}\sum_{K\in\mathscr{T}^j} h_K\|\bar{q}^j\|_K^2 + \sum_{K\in\mathscr{T}^d} h_K\|\bar{q}^m\|_K^2,$$

for all $\boldsymbol{v} = (v,\bar{v},v^m) \in \boldsymbol{Z}_h, \boldsymbol{q} = (q,\bar{q}^s,\bar{q}^d,q^m,\bar{q}^m) \in \boldsymbol{P}_h$. In the discrete scheme, we use the following notations $(\cdot,\cdot)_{\Omega^j} := \sum_{K\in\mathscr{T}^j}(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_{\partial\mathscr{T}^j} := \sum_{K\in\mathscr{T}^s}\langle\cdot,\cdot\rangle_{\partial K}$ for $j = s,d$.

3.3.2 Discrete scheme

In this section, we develop the HDG scheme for the equations eqs. (3.1) to (3.3).

Let $\boldsymbol{\sigma}^s = 2\mu\varepsilon(u^s) - p^s\mathbb{I}$ be the Stokes stress tensor. By multiplying eq. (3.1a) with a test function $v_h \in V_h$, integrating on an element $K \in \mathscr{T}$, applying integration by parts, and summing over all elements, we obtain the following:

$$\left(\boldsymbol{\sigma}^s, \nabla v_h\right)_{\Omega^s} - \langle\hat{\boldsymbol{\sigma}}^s n^s, v_h\rangle_{\mathscr{T}^s} = (f, v_h)_{\Omega^s}, \tag{3.21}$$

where $\hat{\boldsymbol{\sigma}}^s$ is the numerical flux that we choose as

$$\hat{\boldsymbol{\sigma}}^s = 2\mu\varepsilon(u^s) - \bar{p}^s\mathbb{I} - \frac{2\mu\beta}{h}(u^s - \bar{u}) \otimes n^s.$$

To guarantee the continuity of the numerical flux, we impose:

$$\left(\bar{v}_h, \hat{\boldsymbol{\sigma}}^s n^s\right)_{\Omega^s} = \langle\bar{v}_h, \boldsymbol{\sigma}^s n\rangle_{\Gamma^I}, \quad \forall\bar{v}_h \in \bar{V}_h.$$

From eq. (3.8), we have

$$\boldsymbol{\sigma}^s n = -p^d n - \alpha\mu\kappa^{-1/2}(u^s)^t$$

on Γ^I. Replacing p^d and u^s by $\bar{p}^d$ and $\bar{u}$ on $\mathscr{F}^I$ and noting that $\bar{v}_h \cdot \bar{u}^t = \bar{v}_h^t \cdot \bar{u}^t$, we get the following:

$$\langle \bar{v}_h, \hat{\sigma}^s n^s \rangle_{\mathscr{T}^s} = -\langle \bar{v}_h \cdot n, \bar{p}^d \rangle_{\Gamma^I} - \langle \alpha\mu\kappa^{-1/2} \bar{v}_h^t, \bar{u}^t \rangle_{\Gamma^I}.$$

Plugging $\hat{\sigma}^s$ into the above equation and recalling that $\bar{v}_h \cdot n$ and $\bar{p}^s$ are single valued and $\bar{v}_h = 0$ in Γ^s, we obtain:

$$\begin{aligned} -\langle \bar{v}_h \cdot n, \bar{p}^s \rangle_{\Gamma^I} + \langle 2\mu\varepsilon(u)n, \bar{v}_h \rangle_{\partial\mathscr{T}^s} - \langle \frac{2\beta\mu}{h}(u - \bar{u}), \bar{v}_h \rangle_{\partial\mathscr{T}^s} \\ = -\langle \bar{v}_h \cdot n, \bar{p}^d \rangle_{\Gamma^I} - \langle \alpha\mu\kappa^{-1/2} \bar{v}_h^t, \bar{u}^t \rangle_{\Gamma^I}. \end{aligned} \tag{3.22}$$

We group the terms after adding eq. (3.21) to eq. (3.22), to obtain:

$$\begin{aligned} &2\mu(\varepsilon(u), \varepsilon(v_h))_{\Omega^s} - (p, \nabla \cdot v_h)_{\Omega^s} - 2\mu\langle \varepsilon(u)n, v_h - \bar{v}_h \rangle_{\partial\mathscr{T}^s} \\ &+ \frac{2\beta\mu}{h}\langle u - \bar{u}, v_h - \bar{v}_h \rangle_{\partial\mathscr{T}^s} + \langle \bar{p}^s, v_h \cdot n \rangle_{\partial\mathscr{T}^s} \\ &- \langle \bar{p}^s, \bar{v}_h \cdot n \rangle_{\Gamma^I} + \langle \bar{p}^d, \bar{v}_h \cdot n \rangle_{\Gamma^I} + \langle \alpha\kappa^{-1/2} \bar{v}_h^t, \bar{u}^t \rangle_{\Gamma^I} = (f, v_h)_{\Omega^s}. \end{aligned}$$

Afterward, by adding a consistent and symmetrical term, we arrive at the following:

$$\begin{aligned} &2\mu(\varepsilon(u), \varepsilon(v_h))_{\Omega^s} - (p, \nabla \cdot v_h)_{\Omega^s} - 2\mu\langle \varepsilon(u)n, v_h - \bar{v}_h \rangle_{\partial\mathscr{T}^s} \\ &- 2\mu\langle \varepsilon(v)n, u_h - \bar{u}_h \rangle_{\partial\mathscr{T}^s} + \frac{2\beta\mu}{h}\langle u - \bar{u}, v_h - \bar{v}_h \rangle_{\partial\mathscr{T}^s} + \langle \bar{p}^s, v_h \cdot n \rangle_{\partial\mathscr{T}^s} \\ &- \langle \bar{p}^s, \bar{v}_h \cdot n \rangle_{\Gamma^I} + \langle \bar{p}^d, \bar{v}_h \cdot n \rangle_{\Gamma^I} + \langle \alpha\kappa^{-1/2} \bar{v}_h^t, \bar{u}^t \rangle_{\Gamma^I} = (f, v_h)_{\Omega^s}. \end{aligned}$$

Similarly, we multiply *eq*. (3.1b) by q_h, where q_h belongs to the set Q_h. Afterward, we integrate the resulting expression into each element K belonging to the set $\mathscr{T}^s$ and arrive:

$$(q_h, \nabla \cdot u)_{\Omega^s} = 0.$$

To impose normal continuity of the Stokes velocity we set

$$\langle \bar{q}_h^s, (u - \bar{u}) \cdot n^s \rangle_{\partial\mathscr{T}^s} = 0 \quad \forall \bar{q}_h^s \in \bar{Q}_h^s.$$

Subtracting the former equation from the latter, since $\bar{u}, \bar{q}_h^s$ are single-valued at the interfaces and $\bar{u} = 0$ on Γ^s yields:

$$-(q_h, \nabla \cdot u)_{\Omega^s} + \left\langle \bar{q}_h^s, u \cdot n^s \right\rangle_{\partial \mathscr{T}^s} - \left\langle \bar{q}_h^s, \bar{u} \cdot n \right\rangle_{\Gamma^I} = 0. \tag{3.24}$$

Next, eq. (3.2a) is multiplied by a test function $v_h \in V_h$, using integration by parts, replacing p^d with $\bar{p}^d$, and summing over all elements in $\mathscr{T}^d$ to get:

$$(\kappa_f^{-1} u, v_h)_{\Omega^d} - (p, \nabla \cdot v_h)_{\Omega^d} + \langle \bar{p}^d, v_h \cdot n^d \rangle_{\partial \mathscr{T}^d} = 0 \tag{3.25}$$

Multiplying eq. (3.2b) by a test function $q_h \in Q_h$, integrating over an element K, then summing over all elements in $\mathscr{T}^d$ we get:

$$(\nabla \cdot u, q_h)_{\Omega^d} + \big(\sigma \kappa_m (p - p^m), q_h\big)_{\Omega^d} = (g, q_h)_{\Omega^d}.$$

To impose the normal continuity of u we set:

$$\langle \bar{q}_h^d, u \cdot n^d \rangle_{\partial \mathscr{T}^d} = -\langle \bar{q}_h^d, u \cdot n \rangle_{\Gamma^I} \quad \forall \bar{q}_h^d \in \bar{Q}_h^d.$$

We subtract the preceding equation from the current one, and obtain:

$$-(q_h, \nabla \cdot u)_{\Omega^d} + \langle \bar{q}_h^d, u \cdot n^d \rangle_{\partial \mathscr{T}^d} + \langle \bar{q}_h^d, u \cdot n \rangle_{\Gamma^I} - \big(\sigma \kappa_m (p - p^m), q_h\big)_{\Omega^d} = -(g, q_h)_{\Omega^d}. \tag{3.26}$$

Similarly, we multiply eq. (3.2c) by a test function $v_h^m \in V_h^d$, integrate by parts over an element K, sum over all elements $K \in \mathscr{T}^d$, and replace p^m in the facets with $\bar{p}^m$ we get:

$$(\kappa_m^{-1} u^m, v_h^m)_{\Omega^d} - (p^m, \nabla \cdot v_h^m)_{\Omega^d} + \langle \bar{p}^m, v_h^m \cdot n^d \rangle_{\partial \mathscr{T}^d} = 0. \tag{3.27}$$

Last, multiplying eq. (3.2d) by a test function $q_h^m \in Q_h^m$, integrating over an element K, then summing over all elements $K \in \mathscr{T}^d$ we obtain:

$$\big(q_h^m, \nabla \cdot u^m\big)_{\Omega^d} + \big(\sigma \kappa_m (p^m - p), q_h^m\big)_{\Omega^d} = 0.$$

To ensure the normal continuity of u^m, we set:

$$\left\langle \bar{q}_h^m, u^m \cdot n^d \right\rangle_{\partial \mathscr{T}^d} = -\left\langle \bar{q}_h^m, u^m \cdot n \right\rangle_{\Gamma^I} = 0 \quad \forall \bar{q}_h^m \in \bar{Q}_h^d.$$

Combining the last two equations as before, we get:

$$\left(q_h^m, \nabla \cdot u^m\right)_{\Omega^d} - \left(\sigma \kappa_m (p^m - p), q_h^m\right)_{\Omega^d} + \langle \bar{q}_h^m, u^m \cdot n^d \rangle_{\partial \mathscr{T}^d} = 0. \tag{3.28}$$

Collecting eqs. (3.23) to (3.28) results in the following HDG method for the dual-porosity-Stokes problem defined by eqs. (3.1) to (3.3):

Find $\boldsymbol{u}_h := (u_h, \bar{u}_h, u_h^m) \in \boldsymbol{Z}_h$, $\boldsymbol{p}_h := (p_h, \bar{p}_h^s, \bar{p}_h^d, p_h^m, \bar{p}_h^m) \in \boldsymbol{P}_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) = (f, v_h)_{\Omega^s} \qquad \forall \boldsymbol{v}_h := (v_h, \bar{v}_h, v_h^m) \in \boldsymbol{Z}_h, \tag{3.29a}$$

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) - c_h(\boldsymbol{p}_h, \boldsymbol{q}_h) = -(g, q_h)_{\Omega^d} \qquad \forall \boldsymbol{q}_h := (q_h, \bar{q}_h^s, \bar{q}_h^d, q_h^m, \bar{q}_h^m) \in \boldsymbol{P}_h, \tag{3.29b}$$

where

$$a_h(\boldsymbol{u}, \boldsymbol{v}) := a_h^s(\boldsymbol{u}, \boldsymbol{v}) + a_h^d(u, v) + a_h^m(u^m, v^m) + a_h^I(\bar{u}, \bar{v}), \tag{3.30}$$

$$b_h(\boldsymbol{p}, \boldsymbol{v}) := \sum_{j=s,d} [b_h^j(\boldsymbol{p}, v) + b_h^{I,j}(\bar{p}^j, \bar{v})] + b_h^d(\boldsymbol{p}^m, v^m), \tag{3.31}$$

$$c_h(\boldsymbol{p}, \boldsymbol{q}) := \sigma \kappa_m (p^m - p, q^m - q)_{\Omega^d}, \tag{3.32}$$

and

$$
\begin{aligned}
a_h^s(\boldsymbol{u},\boldsymbol{v}) :=& \big(2\mu\varepsilon(u),\varepsilon(v)\big)_{\Omega^s} - \langle 2\mu\varepsilon(u)n^s, v-\bar{v}\rangle_{\partial\mathscr{T}^s} - \langle 2\mu\varepsilon(v)n^s, u-\bar{u}\rangle_{\partial\mathscr{T}^s} \\
&+ \langle 2\beta\mu h_K^{-1}(u-\bar{u}), v-\bar{v}\rangle_{\partial\mathscr{T}^s}, \\
a_h^d(u,v) :=& (\kappa_f^{-1}u,v)_{\Omega^d}, \\
a_h^m(u^m,v^m) :=& (\kappa_m^{-1}u^m,v^m)_{\Omega^d}, \\
a_h^I(\bar{u},\bar{v}) :=& \langle \alpha\mu\kappa_f^{-1/2}\bar{u}^t,\bar{v}^t\rangle_{\Gamma^I}, \\
b_h^j(\boldsymbol{p},v) :=& -(p,\nabla\cdot v)_{\Omega^j} + \langle \bar{p}^j, v\cdot n\rangle_{\partial\mathscr{T}^j}, \quad j=s,d, \\
b_h^{I,j}(\bar{p}^j,\bar{v}) :=& -\langle \bar{p}^j,\bar{v}\cdot n^j\rangle_{\Gamma^I}, \quad j=s,d.
\end{aligned}
$$

3.3.3 Properties of the numerical scheme

The lemma below shows that the velocity solution u_h is $H(\text{div};\Omega)-$conforming , u_h^m is $H(\text{div};\Omega^d)-$conforming, and u_h^s is pointwise divergence-free in the elements in $\mathscr{T}^s$. Additionally, eq. (3.2b) is satisfied point-wise on the elements in $\mathscr{T}^d$ up to the error of the L^2-projection of the source term g into Q_h^d, and eq. (3.2d) is satisfied point-wise on the elements in $\mathscr{T}^d$.

Lemma 23. *The solution to eq.* (3.29) *satisfies:*

$$
\begin{aligned}
[\![u_h^j\cdot n]\!] &= 0 && \forall x\in F, \quad \forall F\in\mathscr{F}^j\backslash\mathscr{F}^I, \quad j=s,d, && \text{(3.33a)}\\
u_h^j\cdot n &= \bar{u}_h\cdot n && \forall x\in F, \quad \forall F\in\mathscr{F}^I, \quad j=s,d, && \text{(3.33b)}\\
[\![u_h^m\cdot n]\!] &= 0 && \forall x\in F, \quad \forall F\in\mathscr{F}^d, && \text{(3.33c)}\\
\nabla\cdot u_h^s &= 0 && \forall x\in K, \quad \forall K\in\mathscr{T}^s, && \text{(3.33d)}\\
\sigma\kappa_m(p_h^d-p_h^m)+\nabla\cdot u_h^d &= \Pi_Q^d g && \forall x\in K, \quad \forall K\in\mathscr{T}^d, && \text{(3.33e)}\\
\sigma\kappa_m(p_h^m-p_h^d)+\nabla\cdot u_h^m &= 0 && \forall x\in K, \quad \forall K\in\mathscr{T}^d, && \text{(3.33f)}
\end{aligned}
$$

where $[\![\cdot]\!]$ *is the standard jump operator and* Π_Q^d *denotes the* L^2*-projection onto* Q_h^d.

Proof. Let $\boldsymbol{v}_h = 0$, $q_h = 0$, and $q_h^m = 0$ in eq. (3.29) then the following holds:

$$\sum_{j=s,d} \langle \bar{q}_h^j, u_h \cdot n \rangle_{\partial \mathscr{T}^j} + \langle \bar{q}_h^m, u_h^m \cdot n \rangle_{\partial \mathscr{T}^d} - \sum_{j=s,d} \langle \bar{q}_h^j, \bar{u}_h \cdot n^j \rangle_{\Gamma^I} = 0.$$

for all $(\bar{q}_h^s, \bar{q}_h^d, \bar{q}_h^m) \in \bar{Q}_h^s \times \bar{Q}_h^d \times \bar{Q}_h^d$. Therefore,

$$\sum_{j=s,d} \left[\left\langle \bar{q}_h^j, [\![u_h \cdot n]\!] \right\rangle_{\mathscr{F}_i^j \cup \mathscr{F}_b^j} + \left\langle \bar{q}_h^j, (u_h^j - \bar{u}_h) \cdot n^j \right\rangle_{\mathscr{F}^I} \right] + \left\langle \bar{q}_h^m, [\![u_h^m \cdot n]\!] \right\rangle_{\mathscr{F}^d} = 0. \quad (3.34)$$

Then, by using

$$\bar{q}_h^j = \begin{cases} [\![u_h \cdot n]\!] & \text{on } F \in \mathscr{F}_i^j \cup \mathscr{F}_b^j, \\ (u_h^j - \bar{u}_h) \cdot n^j & \text{on } F \in \mathscr{F}^I, \end{cases}$$

and $\bar{q}_h^m = [\![u_h^m \cdot n]\!]$ on $F \in \mathscr{F}^d$ in eq. (3.34), we obtain:

$$0 = \sum_{j=s,d} \left[\sum_{F \in \mathscr{F}_i^j \cup \mathscr{F}_b^j} \| [\![u_h \cdot n]\!] \|_F^2 + \sum_{F \in \mathscr{F}^I} \| (u_h^j - \bar{u}_h) \cdot n^j \|_F^2 \right] + \sum_{F \in \mathscr{F}^d} \| [\![u_h^m \cdot n]\!] \|_F^2, \tag{3.35}$$

resulting in Equations (3.33a) to (3.33c). To prove eqs. (3.33d) to (3.33f), set $\boldsymbol{v}_h = \mathbf{0}$, $q_h^m = 0$, and $\bar{q}_h^j = 0$, $j = s, d, m$ in eq. (3.29).

Next, we choose

$$q_h = \begin{cases} \nabla \cdot u_h^s & \text{in } \Omega^s, \\ \sigma \kappa_m (p_h^d - p_h^m) + \nabla \cdot u_h^d - \Pi_Q^d g & \text{in } \Omega^d, \end{cases}$$

to get:

$$
\begin{aligned}
(\nabla\cdot u_h^s, \nabla\cdot u_h^s)_{\Omega^s} + \left(\sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g, \nabla\cdot u_h^d\right)_{\Omega^s} \\
+ \left(\sigma\kappa_m(p_h^d - p_h^m), \sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g\right)_{\Omega^d} \\
- \left(g, \sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g\right)_{\Omega^d} \\
= \left(\sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g, \sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g\right)_{\Omega^d} \\
= \left\|\sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g\right\|_{\Omega^d}^2 = 0,
\end{aligned}
$$

thus we arrive at eqs. (3.33d) and (3.33e). Finally, setting $\boldsymbol{v}_h = \boldsymbol{0}$, $q_h = 0$, $\bar{q}_h^j = 0$, $j = s,d,m$, and $q_h^m = \sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m$ in eq. (3.29), we obtain

$$
\begin{aligned}
&\left(\sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m, \nabla\cdot u_h^m\right)_{\Omega^d} + \left(\sigma\kappa_m(p_h^m - p_h^d), \sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m\right)_{\Omega^d} \\
&= \left(\sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m, \sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m\right)_{\Omega^d} = \|\sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m\|_{\Omega^d}^2 = 0,
\end{aligned}
$$

which yields eq. (3.33f). ■

The next lemma shows that the HDG method eq. (3.29) is a consistent discretization of the dual-porosity-Stokes problem eqs. (3.1) to (3.3).

Lemma 24 (Consistency)**.** *Let* (u, u^m, p, p^m) *be the solution of the dual-porosity-Stokes problem eqs.* (3.1) *to* (3.3)*, where trace of* u*, and trace of* p^j *(*$j = s,d,m$*) on the mesh skeleton is denoted by* $\bar{u}$ *and* $\bar{p}^j$*, respectively , then* $(u, \bar{u}, u^m)$ *and* $(p, \bar{p}^s, \bar{p}^d, p^m, \bar{p}^m)$ *satisfy eq.* (3.29).

Proof. The proof is similar to [37, Lemma 1]. Using smoothness of u, single-valuedness of $\bar{v}_h$, that $\bar{v}_h = 0$ on Γ^s, and eq. (3.3d) we get:

$$
\begin{aligned}
a_h((u,\bar{u},u^m),\boldsymbol{v}_h) &= (2\mu\varepsilon(u),\varepsilon(v_h))_{\Omega^s} - \langle 2\mu\varepsilon(u)n^s,(v_h-\bar{v}_h)\rangle_{\partial\mathscr{T}^s} \\
&\quad + \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,\bar{v}_h^t\rangle_{\Gamma^I} + (\kappa_f^{-1}u,v_h)_{\Omega^d} + (\kappa_m^{-1}u^m,v_h^m)_{\Omega^d} \\
&= -(\nabla\cdot(2\mu\varepsilon(u)),v_h)_{\Omega^s} - \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,\bar{v}_h^t\rangle_{\Gamma^I} + \langle 2\mu(n^s\cdot\varepsilon(u)n^s)n^s,\bar{v}_h\rangle_{\Gamma^I} \\
&\quad + \langle \alpha\mu\kappa_f^{-1/2}(u^s)^t,\bar{v}_h^t\rangle_{\Gamma^I} + (\kappa_f^{-1}u,v_h)_{\Omega^d} + (\kappa_m^{-1}u^m,v_h^m)_{\Omega^d} \\
&= -(\nabla\cdot 2\mu\varepsilon(u),v_h)_{\Omega^s} + \langle 2\mu(n^s\cdot\varepsilon(u)n^s)n^s,\bar{v}_h\rangle_{\Gamma^I} \\
&\quad + (\kappa_f^{-1}u,v_h)_{\Omega^d} + (\kappa_m^{-1}u^m,v_h^m)_{\Omega^d}.
\end{aligned}
$$

Furthermore, after integration-by-parts,

$$
\begin{aligned}
&\sum_{j=s,d}[b_h^j((p,\bar{p}^s,\bar{p}^d,p^m,\bar{p}^m),v_h) + b_h^{I,j}(\bar{p}^j,\bar{v}_h)] + b_h^d((p^m,\bar{p}^m),v_h^m) \\
&= -(p,\nabla\cdot v_h)_{\Omega^s} + \langle p,v_h\cdot n\rangle_{\partial\mathscr{T}^s} - \langle p^s,\bar{v}_h\cdot n^s\rangle_{\Gamma^I} - (p,\nabla\cdot v_h)_{\Omega^d} + \langle p,v_h\cdot n\rangle_{\partial\mathscr{T}^d} \\
&\quad - \langle p^d,\bar{v}_h\cdot n^d\rangle_{\Gamma^I} - (p^m,\nabla\cdot v_h^m)_{\Omega^d} + \langle p^m,v_h^m\cdot n^d\rangle_{\partial\mathscr{T}^d} \\
&= (\nabla p,v_h)_{\Omega} + \langle p^d-p^s,\bar{v}_h\cdot n\rangle_{\Gamma^I} + (\nabla p^m,v_h^m)_{\Omega^d}.
\end{aligned}
$$

Combining the above two results, and using smoothness of u and u^m, single-valuedness of $\bar{q}_h^j$, $j = s, d, m$, and eqs. (3.1) to (3.3).

$$
\begin{aligned}
&a_h((u,\bar{u},u^m),\boldsymbol{v}_h) + \sum_{j=s,d}[b_h^j((p,\bar{p}^s,\bar{p}^d,p^m,\bar{p}^m),v_h) + b_h^{I,j}(p^j,\bar{v}_h)] + b_h^d(\boldsymbol{p}^m,v_h^m) \\
&\quad + \sum_{j=s,d}\Big(b_h^j(\boldsymbol{q}_h,u) + b_h^{I,j}(\bar{q}_h^j,u)\Big) + b_h^d(\boldsymbol{q}_h^m,u^m) - c_h((p,\bar{p}^s,\bar{p}^d,p^m,\bar{p}^m),\boldsymbol{q}_h) \\
&= -(2\mu\varepsilon(u),\nabla v_h)_{\Omega^s} + \langle 2\mu(n^s\cdot\varepsilon(u)n^s),n^s\cdot\bar{v}_h\rangle_{\Gamma^I} + (\kappa_f^{-1}u,v_h)_{\Omega^d} \\
&\quad + (\kappa_m^{-1}u^m,v_h^m)_{\Omega^d} + (\nabla p,v_h)_\Omega + \langle (p^d-p^s)n,\bar{v}_h\rangle_{\Gamma^I} + (\nabla p^m,v_h^m)_{\Omega^d} \\
&\quad - (q_h,\nabla\cdot u)_\Omega + \langle \bar{q}_h^s,u\cdot n^s\rangle_{\partial\mathscr{T}^s} + \langle \bar{q}_h^d,u\cdot n^d\rangle_{\partial\mathscr{T}^d} \\
&\quad - \langle \bar{q}_h^s,u\cdot n^s\rangle_{\Gamma^I} - \langle \bar{q}_h^d,u\cdot n^d\rangle_{\Gamma^I} \\
&\quad - (q_h^m,\nabla\cdot u^m)_{\Omega^d} + \langle \bar{q}_h^m,u^m\cdot n^d\rangle_{\partial\mathscr{T}^d} - (\sigma\kappa_m(p^m-p),q_h^m-q_h)_{\Omega^d} \\
&= -(\nabla\cdot(2\mu\varepsilon(u)) - \nabla p,v_h)_{\Omega^s} + (\kappa_f^{-1}u + \nabla p,v_h)_{\Omega^d} + (\kappa_m^{-1}u^m + \nabla p^m,v_h^m)_{\Omega^d} \\
&\quad + \langle 2\mu(n^s\cdot\varepsilon(u)n^s) + (p^d-p^s)n^s,\bar{v}_h\rangle_{\Gamma^I} - (q_h,\nabla\cdot u + \sigma\kappa_m(p-p^m))_{\Omega^d} \\
&\quad - (q_h^m,\nabla\cdot u^m + \sigma\kappa_m(p^m-p))_{\Omega^d} \\
&= (f,v_h)_{\Omega^s} - (g,q_h)_{\Omega^d}.
\end{aligned}
$$

Thus, we have proven the lemma. ■

3.3.4 Stability of the discrete scheme

The next lemmas will help us establish the stability of the discrete problem given in Equation (3.29).

Lemma 25 (Coercivity)**.** *Let $C_e = \min\{c_{\text{ae}}^s\mu, \alpha\kappa_f^{-1/2}, \kappa_f^{-1}, \kappa_m^{-1}\}$ where $C_{tr} > 0$ is the constant of a discrete trace inequality independent of h, μ, κ_f, κ_m, and σ, and $c_{\text{ae}}^s = (1 - C_{tr}^2/\beta)\min(1,\beta)$. Then for sufficiently large penalty parameter $\beta > 0$,*

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq C_e\|\boldsymbol{v}_h\|_{Z_h}^2. \tag{3.36}$$

Proof. The result follows the same steps as the proof of [38, Lemma 4.2]. First note that

$$\begin{aligned} a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) =& 2\mu \Big(\|\varepsilon(v_h)\|^2_{\Omega^s} - \langle 2\varepsilon(v_h)n^s, v_h - \bar{v}_h \rangle_{\partial \mathscr{T}^s} + \beta \sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Big) \\ & + \alpha \kappa_f^{-1/2} \|\bar{v}_h^t\|^2_{\Gamma^I} + \kappa_f^{-1} \|v_h\|^2_{\Omega^d} + \kappa_m^{-1} \|v_h^m\|^2_{\Omega^d}. \end{aligned} \tag{3.37}$$

Applying the Cauchy–Schwarz inequality and a discrete trace inequality [48, Lemma 1.46] on the second term on the right hand side, we get:

$$\begin{aligned} \langle 2\varepsilon(v_h)n^s, v_h - \bar{v}_h \rangle_{\partial \mathscr{T}^s} \le& 2\Big(\sum_{K\in\mathscr{T}^s} h_K \|\varepsilon(v_h)\|^2_{\partial K} \Big)^{\frac{1}{2}} \Big(\sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Big)^{\frac{1}{2}} \\ \le& 2C_{\mathrm{tr}} \|\varepsilon(v_h)\|_{\Omega^s} \Big(\sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Big)^{\frac{1}{2}}, \end{aligned}$$

where C_{tr} is the discrete trace inequality constant depending on the shape regularity of meshes, k, and dim. We refer to [49] for explicit dependence of C_{tr} on k and dim. Combine this with eq. (3.37) and recall the inequality
$x^2 - 2\psi xy + y^2 \ge (1-\psi^2)(x^2+y^2)/2$, which holds for all $x, y \in \mathbb{R}$ and $0 < \psi < 1$.
Choose $x = \|\varepsilon(v_h)\|_{\Omega^s}$, $y^2 = \beta \sum_{K\in\mathscr{T}^s} h_K^{-1} \|v - \bar{v}\|^2_{\partial K}$, and $\psi = C_{\mathrm{tr}} \beta^{-\frac{1}{2}}$ to find:

$$\begin{aligned} & a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \\ & \ge 2\mu \Bigg[\|\varepsilon(v_h)\|^2_{\Omega^s} - 2C_{\mathrm{tr}} \|\varepsilon(v_h)\|_{\Omega^s} \Big(\sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Big)^{\frac{1}{2}} \\ & \qquad + \beta \sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Bigg] + \alpha \kappa_f^{-1/2} \|\bar{v}_h^t\|^2_{\Gamma^I} + \kappa_f^{-1} \|v_h\|^2_{\Omega^d} + \kappa_m^{-1} \|v_h^m\|^2_{\Omega^d} \\ & \ge (\mu(1 - C_{\mathrm{tr}}^2/\beta) \Big(\|\varepsilon(v_h)\|^2_{\Omega^s} + \beta \sum_{K\in\mathscr{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|^2_{\partial K} \Big) + \alpha \kappa_f^{-1/2} \|\bar{v}_h^t\|^2_{\Gamma^I} \\ & \quad + \kappa_f^{-1} \|v_h\|^2_{\Omega^d} + \kappa_m^{-1} \|v_h^m\|^2_{\Omega^d} \\ & \ge C_e (\|\boldsymbol{v}_h\|^2_{V_h^s} + \|\bar{v}_h^t\|^2_{\Gamma^I} + \|v_h\|^2_{\Omega^d} + \|v_h^m\|^2_{\Omega^d}), \end{aligned}$$

proving eq. (3.36). ■

Lemma 26 (Boundedness)**.** *The bilinear forms $a_h^s(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ satisfy*

$$a_h^s(\boldsymbol{u}_h,\boldsymbol{v}_h) \le C_c^s \|\boldsymbol{u}_h\|_{V_h^s}\|\boldsymbol{v}_h\|_{V_h^s} \qquad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \boldsymbol{V}_h^s, \tag{3.38a}$$

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \le C_c \|\boldsymbol{u}_h\|_{Z_h}\|\boldsymbol{v}_h\|_{Z_h} \qquad \forall \boldsymbol{u}_h,\boldsymbol{v}_h \in \boldsymbol{Z}_h, \tag{3.38b}$$

where $C_c^s := 2\mu\max(1+C_{\rm tr},\beta+C_{\rm tr})$ and $C_c := \max(C_c^s,\kappa_f^{-1},\kappa_m^{-1},\alpha\mu\kappa_f^{-1/2})$ and with $C>0$ a constant independent of μ, κ_f, κ_m, and σ.

Proof. The proof is similar to the proof of [37, Lemma 3]. We start by proving eq. (3.38a). Let $\boldsymbol{u}_h,\boldsymbol{v}_h\in\boldsymbol{Z}_h$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} a_h^s(\boldsymbol{u}_h,\boldsymbol{v}_h) \le 2\mu\Big(&\|\varepsilon(u_h)\|_{\Omega^s}\|\varepsilon(v_h)\|_{\Omega^s} + \sum_{K\in\mathscr{T}^s}\|\varepsilon(u_h)\|_{\partial K}\|v_h-\bar v_h\|_{\partial K} \\ &+ \sum_{K\in\mathscr{T}^s}\|\varepsilon(v_h)\|_{\partial K}\|u_h-\bar u_h\|_{\partial K} + \sum_{K\in\mathscr{T}^s}\beta h_K^{-1}\|u_h-\bar u_h\|_{\partial K}\|v_h-\bar v_h\|_{\partial K}\Big). \end{aligned} \tag{3.39}$$

Note that by the discrete trace inequality [48, Lemma 1.46] and the Cauchy–Schwarz inequality,

$$\begin{aligned} &\sum_{K\in\mathscr{T}^s}\|\varepsilon(u_h)\|_{\partial K}\|v_h-\bar v_h\|_{\partial K} + \sum_{K\in\mathscr{T}^s}\|\varepsilon(v_h)\|_{\partial K}\|u_h-\bar u_h\|_{\partial K} \\ &\quad+ \sum_{K\in\mathscr{T}^s}\beta h_K^{-1}\|u_h-\bar u_h\|_{\partial K}\|v_h-\bar v_h\|_{\partial K} \\ &\le \sum_{K\in\mathscr{T}^s} C_{\rm tr}\|\varepsilon(u_h)\|_K h_K^{-1/2}\|v_h-\bar v_h\|_{\partial K} + \sum_{K\in\mathscr{T}^s} C_{\rm tr}\|\varepsilon(v_h)\|_K h_K^{-1/2}\|u_h-\bar u_h\|_{\partial K} \\ &\quad+ \sum_{K\in\mathscr{T}^s}\beta h_K^{-1/2}\|u_h-\bar u_h\|_{\partial K} h_K^{-1/2}\|v_h-\bar v_h\|_{\partial K} \\ &\le C_{\rm tr}\|\varepsilon(u_h)\|_{\Omega^s}\Big(\sum_{K\in\mathscr{T}^s} h_K^{-1}\|v_h-\bar v_h\|_{\partial K}^2\Big)^{\frac12} + C_{\rm tr}\|\varepsilon(v_h)\|_{\Omega^s}\Big(\sum_{K\in\mathscr{T}^s} h_K^{-1}\|u_h-\bar u_h\|_{\partial K}^2\Big)^{\frac12} \\ &\quad+ \beta\Big(\sum_{K\in\mathscr{T}^s} h_K^{-1}\|u_h-\bar u_h\|_{\partial K}^2\Big)^{1/2}\Big(\sum_{K\in\mathscr{T}^s} h_K^{-1}\|v_h-\bar v_h\|_{\partial K}^2\Big)^{1/2}. \end{aligned} \tag{3.40}$$

Combining eqs. (3.39) and (3.40) and using the Cauchy–Schwarz inequality once more, we get

$$
\begin{aligned}
&a_h^s(\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\leq 2\mu\Big((1+C_{\mathrm{tr}})\|\varepsilon(u_h)\|_{\Omega^s}^2 + (\beta + C_{\mathrm{tr}})\sum_{K\in\mathscr{T}^s} h_K^{-1}\|u_h - \bar{u}_h\|_{\partial K}^2\Big)^{\frac{1}{2}} \\
&\quad \times \Big((1+C_{\mathrm{tr}})\|\varepsilon(v_h)\|_{\Omega^s}^2 + (\beta + C_{\mathrm{tr}})\sum_{K\in\mathscr{T}^s} h_K^{-1}\|v_h - \bar{v}_h\|_{\partial K}^2\Big)^{\frac{1}{2}} \leq C_c^s\|\boldsymbol{u}_h\|_{V_h^s}\|\boldsymbol{v}_h\|_{V_h^s}.
\end{aligned}
$$

To prove eq. (3.38b) we use eq. (3.38a) and the Cauchy–Schwarz inequality to find:

$$
\begin{aligned}
a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) &= a_h^s(\boldsymbol{u}_h,\boldsymbol{v}_h) + a_h^d(u_h,v_h) + a_h^m(u_h^m,v_h^m) + a_h^I(\bar{u}_h,\bar{v}_h) \\
&\leq C_c^s\|\boldsymbol{u}_h\|_{V_h^s}\|\boldsymbol{v}_h\|_{V_h^s} + \kappa_f^{-1}\|u_h\|_{\Omega^d}\|v_h\|_{\Omega^d} + \kappa_m^{-1}\|u_h^m\|_{\Omega^d}\|v_h^m\|_{\Omega^d} \\
&\quad + \alpha\mu\kappa_f^{-1/2}\|\bar{u}_h^t\|_{\Gamma^I}\|\bar{v}_h^t\|_{\Gamma^I}. \\
&\leq \max(C_c^s, \kappa_f^{-1}, \kappa_m^{-1}, \alpha\mu\kappa_f^{-1/2})\Big(\|\boldsymbol{u}_h\|_{V_h^s}^2 + \|u_h\|_{\Omega^d}^2 + \|u_h^m\|_{\Omega^d}^2 + \|\bar{u}_h^t\|_{\Gamma^I}^2\Big)^{1/2} \\
&\qquad \times \Big(\|\boldsymbol{v}_h\|_{V_h^s}^2 + \|v_h\|_{\Omega^d}^2 + \|v_h^m\|_{\Omega^d}^2 + \|\bar{v}_h^t\|_{\Gamma^I}^2\Big)^{1/2}. \\
&\leq C_c\|\boldsymbol{u}_h\|_{Z_h}\|\boldsymbol{v}_h\|_{Z_h}.
\end{aligned}
$$

■

Corollary 27. *If the first component of a_h^s belongs to $\boldsymbol{V}^s(h) := \boldsymbol{V}_h^s + (\widetilde{V}^s \times \bar{\widetilde{V}}^s)$, where*

$$\widetilde{V}^s := \{v \in [H^2(\Omega^s)]^{dim} : v = 0 \text{ on } \Gamma^s\}$$

and $\bar{\widetilde{V}}^s$ is the trace space of $\widetilde{V}^s$ on Γ_0^s, eq. (3.38a) *becomes*

$$a_h^s(\boldsymbol{u},\boldsymbol{v}_h) \leq C_c^{s,*}\|\boldsymbol{u}\|_{V_h^{s,*}}\|\boldsymbol{v}_h\|_{V_h^s}, \quad \forall \boldsymbol{u} \in \boldsymbol{V}^s(h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s,$$

where

$$\|\boldsymbol{u}\|_{V_h^{s,*}}^2 := \|\boldsymbol{u}\|_{V_h^s}^2 + \sum_{K\in\mathscr{T}^s} h_K^2|u|_{2,K}^2,$$

with a slightly different constant $C_c^{s,*}$ due to the use of a continuous trace inequality instead of a discrete one. The dependence of the constants on the problem parameters, however, stays the same.

To prove an inf-sup condition, we introduce the following interpolation operators (see, e.g., [50, Lemma 7] or [40, Proposition 2.5.3]):

Lemma 28. *There exist interpolation operators $\Pi_V : H_0(\mathrm{div},\Omega) \mapsto V_h \cap H_0(\mathrm{div},\Omega)$, $\Pi_V^m : H_0(\mathrm{div},\Omega^d) \mapsto V_h^m \cap H_0(\mathrm{div},\Omega^d)$ such that for all $u \in [H^{k+1}(K)]^{\mathrm{dim}}$, $K \in \mathscr{T}^d$, and $u^m \in [H^{k+1}(K)]^{\mathrm{dim}}$, $K \in \mathscr{T}^d$ the following hold:*

1. $(q, \nabla\cdot(u - \Pi_V u))_K = 0$ *for all* $q \in P_{k-1}(K)$, $K \in \mathscr{T}$.
2. $(q^m, \nabla\cdot(u^m - \Pi_V^m u^m))_K = 0$ *for all* $q^m \in P_{k-1}(K)$, $K \in \mathscr{T}^d$.
3. $\langle \bar{q}, (u - \Pi_V u)\cdot n\rangle_F = 0$ *for all* $\bar{q} \in P_k(F)$, $F \in \mathscr{F}$.
4. $\langle \bar{q}^m, (u^m - \Pi_V u^m)\cdot n\rangle_F = 0$ *for all* $\bar{q}^m \in P_k(F)$, $F \in \mathscr{F}^d$.
5. $\|u - \Pi_V u\|_{p,K} \le C h_K^{l-p} |u|_{l,K}$ *with* $p = 0,1,2$ *and* $\max(1,p) \le l \le k+1$, $K \in \mathscr{T}$.
6. $\|u^m - \Pi_V^m u^m\|_{p,K} \le C h_K^{l-p} |u^m|_{l,K}$ *with* $p = 0,1,2$ *and* $\max(1,p) \le l \le k+1$, $K \in \mathscr{T}^d$.
7. $\|\nabla\cdot(u - \Pi_V u)\|_K \le C h_K^l |\nabla\cdot u|_{l,K}$ *with* $1 \le l \le k$, $K \in \mathscr{T}$.
8. $\|\nabla\cdot(u^m - \Pi_V^m u^m)\|_K \le C h_K^l |\nabla\cdot u^m|_{l,K}$ *with* $1 \le l \le k$, $K \in \mathscr{T}^d$.

Furthermore, we denote the L^2-projection onto $\bar{V}_h$ by $\bar{\Pi}_V$. For $v \in [H^l(K)]^{\mathrm{dim}}$, $1 \le l \le k+1$, we have:

$$\|v - \bar{\Pi}_V v\|_{\partial K} \le C h_K^{l-1/2} \|v\|_{H^l(K)}, \tag{3.41a}$$

$$\|\Pi_V v - \bar{\Pi}_V v\|_{\partial K} \le C h_K^{l-1/2} \|v\|_{H^l(K)}. \tag{3.41b}$$

Next we prove an inf-sup condition. We start by splitting $b_h(\cdot,\cdot)$ as follows:

$$b_h(\boldsymbol{v}_h,\boldsymbol{q}_h) = b_h^1(\boldsymbol{v}_h,(q_h,q_h^m)) + b_h^2(\boldsymbol{v}_h,(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m)),$$

where

$$\begin{aligned} b_h^1(\boldsymbol{v}_h,(q_h,q_h^m)) &= -\sum_{j=s,d}(q_h,\nabla\cdot v_h)_{\Omega^j} - (q_h^m,\nabla\cdot v_h^m)_{\Omega^d},\\ b_h^2(\boldsymbol{v}_h,(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m)) &= \sum_{j=s,d}(\langle\bar{q}_h^j, v_h\cdot n\rangle_{\partial\mathscr{T}^j} + b_h^{I,j}(\bar{q}_h^j,\bar{v}_h)) + \langle\bar{q}_h^m, v_h^m\cdot n\rangle_{\partial\mathscr{T}^d}. \end{aligned}$$

According to [51, Theorem 31.1], we only need to prove that $b_h^1(\cdot,\cdot)$ satisfies the inf-sup condition in

$$\boldsymbol{Z}_h^0 := Ker(b_h^2) = \{\boldsymbol{v}_h\in\boldsymbol{Z}_h: \quad b_h^2(\boldsymbol{v}_h,(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m)) = 0 \quad \forall(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m)\in\bar{Q}_h^s\times\bar{Q}_h^d\times\bar{Q}_h^d\}$$

and $b_h^2(\cdot,\cdot)$ satisfies the inf-sup condition in $\boldsymbol{Z}_h\times\boldsymbol{P}_h$. The following two lemmas serve this purpose.

Lemma 29. *There exists a constant $C>0$, independent of h, such that for any $(q_h,q_h^m)\in Q_h\times Q_h^m$,*

$$C(\|q_h\|_\Omega^2 + \|q_h^m\|_{\Omega^d}^2)^{1/2} \le \sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h^0,\boldsymbol{v}_h\neq\boldsymbol{0}} \frac{b_h^1(\boldsymbol{v}_h,(q_h,q_h^m))}{\|\boldsymbol{v}_h\|_{Z_h}}. \tag{3.42}$$

Proof. Let $(q_h,q_h^m)\in Q_h\times Q_h^m$. Since $q_h\in L_0^2(\Omega)$ and $q_h^m\in L_0^2(\Omega^d)$, by the standard inf-sup condition, there exist $v\in[H_0^1(\Omega)]^{\dim}$ and $v^m\in[H_0^1(\Omega^d)]^{\dim}$ such that:

$$\begin{aligned} -\nabla\cdot v &= q_h \text{ in } \Omega, & C\|v\|_{1,\Omega} &\le \|q_h\|_\Omega,\\ -\nabla\cdot v^m &= q_h^m \text{ in } \Omega^d, & C\|v^m\|_{1,\Omega^d} &\le \|q_h^m\|_{\Omega^d}. \end{aligned}$$

By Lemma 28, eq. (3.41), and [45, (1.24)],

$$\begin{aligned}\|(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\|_{Z_h}^2 =&\|\varepsilon(\Pi_V v)\|_{\Omega^s}^2+\sum_{K\in\mathscr{T}^s} h_K^{-1}\|\Pi_V v-\bar{\Pi}_V v\|_{\partial K}^2\\ &+\|\Pi_V v\|_{\Omega^d}^2+\|\Pi_V^m v^m\|_{\Omega^d}^2+\|\bar{\Pi}_V (v^s)^t\|_{\Gamma^I}^2\\ \le& C(\|v\|_{1,\Omega^s}^2+\|v^m\|_{1,\Omega^d}^2+\|v\|_{1,\Omega^d}^2).\end{aligned} \tag{3.43}$$

Observe also that $(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\in \boldsymbol{Z}_h^0$ by the fact that $\bar{q}_h^j$, $j=s,d,m$ is single-valued and the properties of Π_V and $\bar{\Pi}_V$. Therefore,

$$\begin{aligned}\sup_{\boldsymbol{v}_h\in \boldsymbol{Z}_h^0,\boldsymbol{v}_h\neq \boldsymbol{0}}\frac{b_h^1(\boldsymbol{v}_h,(q_h,q_h^m))}{\|\boldsymbol{v}_h\|_{Z_h}} =& \sup_{\boldsymbol{v}_h\in \boldsymbol{Z}_h^0,\boldsymbol{v}_h\neq \boldsymbol{0}} -\frac{\sum_{j=s,d}(q_h,\nabla\cdot v_h)_{\Omega^j}+(q_h^m,\nabla\cdot v_h^m)_{\Omega^d}}{\|\boldsymbol{v}_h\|_{Z_h}}\\ \ge& \frac{\sum_{j=s,d}(q_h,\nabla\cdot \Pi_V v)_{\Omega^j}+(q_h^m,\nabla\cdot \Pi_V^m v^m)_{\Omega^d}}{\|(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\|_{Z_h}}\\ \ge& C\frac{\|q_h\|_{\Omega}^2+\|q_h^m\|_{\Omega^d}^2}{(\|v\|_{1,\Omega^s}^2+\|v\|_{1,\Omega^d}^2+\|v^m\|_{1,\Omega^d}^2)^{1/2}}\\ \ge& C(\|q_h\|_{\Omega}^2+\|q_h^m\|_{\Omega^d}^2)^{1/2}.\end{aligned} \tag{3.44}$$

■

Lemma 30. *There exists a constant $C>0$, independent of h, such that for any $(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m)\in \bar{Q}_h^s\times\bar{Q}_h^d\times\bar{Q}_h^d$,*

$$\begin{aligned}C\Bigg(\sum_{j=s,d}\sum_{K\in\mathscr{T}^j} h_K\|\bar{q}_h^j\|_{\partial K}^2+\sum_{K\in\mathscr{T}^d} h_K\|\bar{q}_h^m\|_{\partial K}^2\Bigg)^{1/2}&\\ \le \sup_{\boldsymbol{v}_h\in \boldsymbol{Z}_h,\boldsymbol{v}_h\neq\boldsymbol{0}}\frac{b_h^2(\boldsymbol{v}_h,(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m))}{\|\boldsymbol{v}_h\|_{Z_h}}&.\end{aligned} \tag{3.45}$$

Proof. The proof is similar to that of [52, Lemma 3]. We start by introducing an operator [41, Proposition 2.10] to lift $\bar{q}_h^m\in\bar{Q}_h^m$ to Ω^d and $\bar{q}_h^j\in\bar{Q}_h^j$ to Ω^j, $j=s,d$. Let $R_k(\partial K):=\{\bar{q}\in L^2(\partial K):\bar{q}\in P_k(F),\ \forall F\subset\partial K\}$ and let $L:R_k(\partial K)\mapsto[P_k(K)]^{\dim}$ be the

BDM local lifting operator that satisfies for all $\bar{q}_h \in R_k(\partial K)$:

$$(L\bar{q}_h)\cdot n = h_K \bar{q}_h \text{ on } \partial K, \qquad \|L\bar{q}_h\|_K \le C_0 h_K^{3/2} \|\bar{q}_h\|_{\partial K}, \tag{3.46a}$$

$$\|\nabla(L\bar{q}_h)\|_K \le C_0 h_K^{1/2} \|\bar{q}_h\|_{\partial K}, \qquad \|L\bar{q}_h\|_{\partial K} \le C_0 h_K \|\bar{q}_h\|_{\partial K}. \tag{3.46b}$$

Here the constant $C_0 \ge 1$ only depends on the shape regularity of the mesh and the polynomial degree k. Using the same argument as in [37, Lemma 6], we define

$$L\bar{q}_h = \begin{cases} L\bar{q}_h^s & \forall K \in \mathscr{T}^s, \\ L\bar{q}_h^d & \forall K \in \mathscr{T}^d. \end{cases}$$

Then, $(L\bar{q}_h, 0, L\bar{q}_h^m) \in \boldsymbol{Z}_h$ and

$$\begin{aligned} &\sum_{j=s,d} \left(\langle \bar{q}_h^j, L\bar{q}_h \cdot n \rangle_{\partial \mathscr{T}^j} + b_h^{I,j}(\bar{q}_h^j, 0) \right) + \langle \bar{q}_h^m, L\bar{q}_h^m \cdot n \rangle_{\partial \mathscr{T}^d} \\ &\qquad = \sum_{j=s,d} \sum_{K \in \mathscr{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K \in \mathscr{T}^d} h_K \|\bar{q}_h^m\|_{\partial K}^2. \end{aligned}$$

Furthermore, by eq. (3.46),

$$\begin{aligned} \|(L\bar{q}_h, 0, L\bar{q}_h^m)\|_{Z_h}^2 =& \|\varepsilon(L\bar{q}_h^s)\|_{\Omega^s}^2 + \sum_{K \in \mathscr{T}^s} h_K^{-1} \|L\bar{q}_h^s\|_{\partial K}^2 \\ &+ \|L\bar{q}_h^d\|_{\Omega^d}^2 + \|L\bar{q}_h^m\|_{\Omega^d}^2 \\ \le& 2C_0^2 \left(\sum_{j=s,d} \sum_{K \in \mathscr{T}^j} h_K \|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K \in \mathscr{T}^d} h_K \|\bar{q}_h^m\|_{\partial K}^2 \right). \end{aligned} \tag{3.47}$$

Therefore,

$$
\begin{aligned}
&\sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h,\boldsymbol{v}_h\neq\boldsymbol{0}} \frac{b_h^2(\boldsymbol{v}_h,(\bar{q}_h^s,\bar{q}_h^d,\bar{q}_h^m))}{\|\boldsymbol{v}_h\|_{Z_h}} \\
&= \sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h,\boldsymbol{v}_h\neq\boldsymbol{0}} \frac{\sum_{j=s,d}(\langle\bar{q}_h^j, v_h\cdot n\rangle_{\partial\mathscr{T}^j} + b_h^{I,j}(\bar{q}_h^j,\bar{v}_h)) + \langle\bar{q}_h^m, v_h^m\cdot n\rangle_{\partial\mathscr{T}^d}}{\|\boldsymbol{v}_h\|_{Z_h}} \\
&\qquad\qquad\geq \frac{\sum_{j=s,d}\langle\bar{q}_h^j, L\bar{q}_h^j\cdot n\rangle_{\partial\mathscr{T}^j} + \langle\bar{q}_h^m, L\bar{q}_h^m\cdot n\rangle_{\partial\mathscr{T}^d}}{\|(L\bar{q}_h,0,L\bar{q}_h^m)\|_{Z_h}} \\
&\qquad\qquad\geq \frac{\sum_{j=s,d}\sum_{K\in\mathscr{T}^j} h_K\|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K\in\mathscr{T}^d} h_K\|\bar{q}_h^m\|_{\partial K}^2}{\sqrt{2}C_0\left(\sum_{j=s,d}\sum_{K\in\mathscr{T}^j} h_K\|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K\in\mathscr{T}^d} h_K\|\bar{q}_h^m\|_{\partial K}^2\right)^{1/2}} \\
&\qquad\qquad= (\sqrt{2}C_0)^{-1}\left(\sum_{j=s,d}\sum_{K\in\mathscr{T}^j} h_K\|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K\in\mathscr{T}^d} h_K\|\bar{q}_h^m\|_{\partial K}^2\right)^{1/2}.
\end{aligned}
$$

The result follows with $C = (\sqrt{2}C_0)^{-1}$. ■

The following inf-sup condition is a direct consequence of Equations (3.42) and (3.45) and [51, Theorem 3.1].

Theorem 31. *There exists a constant $\beta_p > 0$, independent of h, such that for all $\boldsymbol{p}_h \in \boldsymbol{P}_h$,*

$$\beta_p\|\boldsymbol{p}_h\|_{P_h} \leq \sup_{\boldsymbol{v}_h\in\boldsymbol{Z}_h,\boldsymbol{v}_h\neq\boldsymbol{0}} \frac{\boldsymbol{b}_h(\boldsymbol{p}_h,\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{Z_h}}. \tag{3.48}$$

We end this section by proving well-posedness of the HDG method.

Theorem 32. *If $\beta > \beta_0$, then the discrete problem eq.* (3.29) *is well-posed.*

Proof. Since the problem is finite dimensional and linear, it is sufficient to show uniqueness of the solution. Choosing $f = 0$, $g = 0$, $\boldsymbol{v}_h = \boldsymbol{u}_h$ and $\boldsymbol{q}_h = -\boldsymbol{p}_h$ in eq. (3.29), we arrive:

$$a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) + c(\boldsymbol{p}_h,\boldsymbol{p}_h) = 0.$$

By Lemma 25,

$$C_e(\|\boldsymbol{u}_h\|_{V_h^s}^2 + \|\bar{u}_h^t\|_{\Gamma^I}^2 + \|u_h\|_{\Omega^d}^2 + \|u_h^m\|_{\Omega^d}^2) + \sigma\kappa_m\|p_h^m - p_h\|_{\Omega^d}^2 \leq 0.$$

Thus $p_h^m = p_h$ in Ω^d and $\boldsymbol{u}_h = \mathbf{0}$. Plugging these values in eq. (3.29), we obtain:

$$\boldsymbol{b}_h(\boldsymbol{p}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h.$$

Therefore, $\boldsymbol{p}_h = \mathbf{0}$ by Theorem 31, concluding the proof. ■

3.4 A priori error analysis

In the next part, we will analyze the errors in the HDG method given in Equation (3.29). We will be using the BDM interpolation operator Π_V as defined in Lemma 28 , and the L^2-projection operators $\bar{\Pi}_V$ for $\bar{V}_h$, Π_Q for Q_h, Π_Q^m for Q_h^m, and $\bar{\Pi}_Q^j$ for $\bar{Q}_h^j$, where $j = s, d$, as well as for $k \geq 0$ and $0 \leq l \leq k$. The following holds:

$$\|q - \Pi_Q q\|_K \leq C h_K^l \|q\|_{l,K} \qquad \forall q \in H^l(K), \tag{3.49a}$$

$$\|q - \bar{\Pi}_Q^d q\|_{\partial K} \leq C h_K^{l+1/2} \|q\|_{l+1,K} \qquad \forall q \in H^{l+1}(K). \tag{3.49b}$$

3.4.1 Some useful notations

In this section and section 4.4 we will use the following notation for convenience. $\boldsymbol{\Pi} u := (\Pi_V u, \bar{\Pi}_V u^s, \Pi_V^m u^m)$, $\boldsymbol{\Pi} p := (\Pi_Q p, \bar{\Pi}_Q^s p^s, \bar{\Pi}_Q^d p^d, \Pi_Q^m p^m, \bar{\Pi}_Q^d p^m)$,

$$e_u^I = u - \Pi_V u, \qquad e_u^h = u_h - \Pi_V u, \qquad e_{u^m}^I = u^m - \Pi_V^m u^m, \qquad e_{u^m}^h = u_h^m - \Pi_V^m u^m,$$
$$e_p^I = p - \Pi_Q p, \qquad e_p^h = p_h - \Pi_Q p, \qquad e_{p^m}^I = p^m - \Pi_Q^m p^m, \qquad e_{p^m}^h = p_h^m - \Pi_Q^m p^m,$$

and

$$\bar{e}_{u^s}^I = u^s|_{\Gamma_0^s} - \bar{\Pi}_V u^s, \qquad \bar{e}_{u^s}^h = \bar{u}_h^s - \bar{\Pi}_V u^s,$$
$$\bar{e}_{p^s}^I = p^s|_{\Gamma_0^s} - \bar{\Pi}_Q^s p^s, \qquad \bar{e}_{p^s}^h = p_h^s - \bar{\Pi}_Q^s p^s,$$
$$\bar{e}_{p^j}^I = p^j|_{\Gamma_0^j} - \bar{\Pi}_Q^d p^j, \qquad \bar{e}_{p^j}^h = p_h^j - \bar{\Pi}_Q^d p^j, \quad j = d, m.$$

$$
\begin{aligned}
\boldsymbol{e}_u^I &= (e_u^I, \bar{e}_u^I, e_{u^m}^I), & \boldsymbol{e}_u^h &= (e_u^h, \bar{e}_u^h, e_{u^m}^h),\\
\boldsymbol{e}_p^I &= (e_p^I, \bar{e}_{p^s}^I, \bar{e}_{p^d}^I, e_{p^m}^I, \bar{e}_{p^m}^I), & \boldsymbol{e}_p^h &= (e_p^h, \bar{e}_{p^s}^h, \bar{e}_{p^d}^h, e_{p^m}^h, \bar{e}_{p^m}^h),\\
\boldsymbol{e}_{p^j}^I &= (e_{p^j}^I, \bar{e}_{p^j}^I), & \boldsymbol{e}_{p^j}^h &= (e_{p^j}^h, \bar{e}_{p^j}^h),
\end{aligned}
$$

where $j = s, d, m$. By [37, Lemmas 7 and 8],

$$
\|\boldsymbol{e}_u^I\|_{V_h^{s,*}} \le Ch^{l-1}\|u\|_{l,\Omega^s}, \quad 2 \le l \le k+1, \tag{3.50}
$$

and

$$
\|\boldsymbol{e}_{p^s}^I\|_{Q_h^s} \le Ch^l\|p^s\|_{l,\Omega^s}, \quad \|\boldsymbol{e}_{p^j}^I\|_{Q_h^j} \le Ch^l\|p^j\|_{l,\Omega^d}, \quad 0 \le l \le k, \quad j = d, m. \tag{3.51}
$$

3.4.1.1 <u>Error equation</u> The following lemma presents the error equations used to derive our error estimates in Theorem 34 and Corollary 35.

Lemma 33 (Error equation)**.** *For any $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$, the following holds:*

$$
a_h(\boldsymbol{e}_u^h, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h) = a_h^s(\boldsymbol{e}_u^I, \boldsymbol{v}_h) + a_h^d(e_u^I, v_h) + a_h^m(e_{u^m}^I, v_h^m), \tag{3.52a}
$$

$$
b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^h) - c_h(\boldsymbol{e}_p^h, \boldsymbol{q}_h) = 0. \tag{3.52b}
$$

Proof. By Lemma 24 and eq. (3.29),

$$
a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, \boldsymbol{v}_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h), \tag{3.53a}
$$

$$
b_h(\boldsymbol{q}_h, \boldsymbol{u}) - c_h(\boldsymbol{p}, \boldsymbol{q}_h) = b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) - c_h(\boldsymbol{p}_h, \boldsymbol{q}_h), \tag{3.53b}
$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\boldsymbol{q}_h \in \boldsymbol{P}_h$. Subtracting $a_h(\boldsymbol{\Pi}u, \boldsymbol{v}_h) + b_h(\boldsymbol{\Pi}p, \boldsymbol{v}_h)$ from both sides of eq. (3.53a) and $b_h(\boldsymbol{q}_h, \boldsymbol{\Pi}u) - c_h(\boldsymbol{\Pi}p, \boldsymbol{q}_h)$ from both sides of eq. (3.53b), we find

$$
\begin{aligned}
a_h(\boldsymbol{e}_u^h, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h) &= a_h(\boldsymbol{e}_u^I, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^I, \boldsymbol{v}_h),\\
b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^h) - c_h(\boldsymbol{e}_p^h, \boldsymbol{q}_h) &= b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^I) - c_h(\boldsymbol{e}_p^I, \boldsymbol{q}_h).
\end{aligned}
$$

By the definition of bilinear forms, we obtain

$$
\begin{aligned}
&a_h(\boldsymbol{e}_u^I, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^I, \boldsymbol{v}_h) = a_h^s(\boldsymbol{e}_u^I, \boldsymbol{v}_h) + a_h^I(\bar{e}_u^I, \bar{v}_h) + a_h^d(e_u^I, v_h) + a_h^m(e_{u^m}^I, v_h^m) \\
&\quad + b_h^s(\boldsymbol{e}_{p^s}^I, v_h) + b_h^d(\boldsymbol{e}_{p^d}^I, v_h) + b_h^d(\boldsymbol{e}_{p^m}^I, v_h^m) + b_h^{I,s}(\bar{e}_{p^s}^I, \bar{v}_h) + b_h^{I,d}(\bar{e}_{p^d}^I, \bar{v}_h), \\
&b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^I) - c_h(\boldsymbol{e}_p^I, \boldsymbol{q}_h) = b_h^s(\boldsymbol{q}_h^s, e_u^I) + b_h^d(\boldsymbol{q}_h^d, e_u^I) + b_h^d(\boldsymbol{q}_h^m, e_{u^m}^I) \\
&\quad + b_h^{I,s}(\bar{q}_h^s, \bar{e}_u^I) + b_h^{I,d}(\bar{q}_h^d, \bar{e}_u^I) - c_h(\boldsymbol{e}_p^I, \boldsymbol{q}_h).
\end{aligned}
$$

Since Π_Q^j, $\bar{\Pi}_Q^j$, $j = d, m$ are L^2-projections, we note that

$$
\begin{aligned}
b_h^s(\boldsymbol{e}_{p^s}^I, v_h) = 0, \quad b_h^d(\boldsymbol{e}_{p^d}^I, v_h) = 0, \quad b_h^d(\boldsymbol{e}_{p^m}^I, v_h^m) = 0, \\
b_h^{I,d}(\bar{e}_{p^d}^I, \bar{v}_h) = 0, \quad b_h^{I,s}(\bar{e}_{p^s}^I, \bar{v}_h) = 0, \quad c_h(\boldsymbol{e}_p^I, \boldsymbol{q}_h) = 0.
\end{aligned}
$$

Furthermore, by Lemma 28,

$$
b_h^s(\boldsymbol{q}_h^s, e_u^I) = 0, \quad b_h^d(\boldsymbol{q}_h^d, e_u^I) = 0, \quad b_h^d(\boldsymbol{q}_h^m, e_{u^m}^I) = 0,
$$

and by the definition of the L^2-projection $\bar{\Pi}_V$, we have

$$
b_h^{I,s}(\bar{q}_h^s, \bar{e}_u^I) = 0, \quad b_h^{I,d}(\bar{q}_h^d, \bar{e}_u^I) = 0, \quad a_h^I(\bar{e}_u^I, \bar{v}_h) = 0.
$$

These identities lead to the conclusion. ■

Theorem 34. *Let $(\boldsymbol{u}, \boldsymbol{p}) \in \boldsymbol{Z} \times \boldsymbol{P}$ be the solution of the dual-porosity-Stokes problem eqs.* (3.1) *to* (3.3) *such that $u^s \in [H^{k+1}(\Omega^s)]^{\dim}$, $u^d \in [H^k(\Omega^d)]^{\dim}$, $u^m \in [H^k(\Omega^d)]^{\dim}$, $p^j \in H^k(\Omega^j)$, $j = s, d$, $p^m \in H^k(\Omega^d)$, $k \geq 1$. Let $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$ be the solution to eq.* (3.29)*. Then,*

$$
\begin{aligned}
&\|\boldsymbol{e}_u^h\|_{V_h^s} + \|e_u^h\|_{\Omega^d} + \|e_{u^m}^h\|_{\Omega^d} && (3.54a) \\
&\qquad \leq C C_e^{-1} C_m h^k (\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}), \\
&\|\boldsymbol{e}_p^h\|_{P_h} \leq C \beta_p^{-1} (C_c C_e^{-1} + 1) C_m h^k (\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}), && (3.54b)
\end{aligned}
$$

where C_e is the ellipticity constant in Lemma 25, β_p is the inf-sup constant in Theorem 31, C_c is the boundedness constant in eq. (3.38b)*, $C_m = \max(C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1})$ in which $C_c^{s,*}$ is the boundedness constant in Corollary 27, and C is a generic constant independent of h, σ, μ, κ_f, and κ_m.*

Proof. Choose $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{e}_u^h, -\boldsymbol{e}_p^h)$ in eq. (3.52), and add the two equations together. By Lemma 25

$$C_e(\|\boldsymbol{e}_u^h\|_{V_h^s}^2 + \|e_u^h\|_{\Omega^d}^2 + \|e_{u^m}^h\|_{\Omega^d}^2) \\ \leq a_h(\boldsymbol{e}_u^h, \boldsymbol{e}_u^h) + c_h(\boldsymbol{e}_p^h, \boldsymbol{e}_p^h) = a_h^s(\boldsymbol{e}_u^I, \boldsymbol{e}_u^h) + a_h^d(e_u^I, e_u^h) + a_h^m(e_{u^m}^I, e_{u^m}^h). \quad (3.55)$$

We bound the right hand side of eq. (3.55) by Corollary 27, the Cauchy–Schwarz inequality, and the definition of $\|\cdot\|_{V_h^{s,*}}$ as follows:

$$\begin{aligned} & a_h^s(\boldsymbol{e}_u^I, \boldsymbol{e}_u^h) + a_h^d(e_u^I, e_u^h) + a_h^m(e_{u^m}^I, e_{u^m}^h) \\ & \leq C_c^{s,*}\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}\|\boldsymbol{e}_u^h\|_{V_h^s} + \kappa_f^{-1}\|e_u^I\|_{\Omega^d}\|e_u^h\|_{\Omega^d} + \kappa_m^{-1}\|e_{u^m}^I\|_{\Omega^d}\|e_{u^m}^h\|_{\Omega^d} \\ & \leq \max(C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1})(\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}^2 + \|e_u^I\|_{\Omega^d}^2 + \|e_{u^m}^I\|_{\Omega^d}^2)^{1/2} \\ & \quad \times (\|\boldsymbol{e}_u^h\|_{V_h^s}^2 + \|e_u^h\|_{\Omega^d}^2 + \|e_{u^m}^h\|_{\Omega^d}^2)^{1/2}. \end{aligned} \quad (3.56)$$

Equation (3.54a) now follows from eqs. (3.55) and (3.56), Lemma 28, and eq. (3.41). We next prove eq. (3.54b). By eq. (3.52a), Lemma 26, and following the same steps as in eq. (3.56), we have :

$$\begin{aligned} |b_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h)| & \leq |a_h(\boldsymbol{e}_u^h, \boldsymbol{v}_h)| + |a_h^s(\boldsymbol{e}_u^I, \boldsymbol{v}_h^s)| + |a_h^d(e_u^I, v_h^d)| + |a_h^m(e_{u^m}^I, v_h^m)|, \\ & \leq (C_c\|\boldsymbol{e}_u^h\|_{Z_h} + \max(C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1})(\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}^2 + \|e_u^I\|_{\Omega^d}^2 + \|e_{u^m}^I\|_{\Omega^d}^2)^{1/2})\|\boldsymbol{v}_h\|_{Z_h}. \end{aligned}$$

Combining this with the inf-sup condition in Theorem 31, Lemma 28, eq. (3.41), and observing that a modification of eq. (3.54a),

$$
\begin{aligned}
\|\boldsymbol{e}_p^h\|_{P_h} &\leq \beta_p^{-1} \sup_{\boldsymbol{v}_h \in \boldsymbol{Z}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{\boldsymbol{b}_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{Z_h}} \\
&\leq \beta_p^{-1}(C_c\|\boldsymbol{e}_u^h\|_{Z_h} + \max(C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1})(\|\boldsymbol{e}_u^I\|^2_{V_h^{s,*}} + \|e_u^I\|^2_{\Omega^d} + \|e_{u^m}^I\|^2_{\Omega^d})^{1/2}) \\
&\leq C\beta_p^{-1}(C_cC_e^{-1} + 1)\max(C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1})h^k(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}),
\end{aligned}
$$

which is the desired result. ■

Corollary 35. *Under the assumptions of Theorem 34,*

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{V_h^s} + &\|u - u_h\|_{\Omega^d} + \|u^m - u_h^m\|_{\Omega^d} \\
&\leq C(C_e^{-1}C_m + 1)h^k\left(\|u\|_{k+1,\Omega^s} + \|u\|_{k+1,\Omega^d} + \|u^m\|_{k+1,\Omega^d}\right),
\end{aligned} \tag{3.57a}
$$

$$
\begin{aligned}
\|\boldsymbol{p} - \boldsymbol{p}_h\|_{P_h} \leq &Ch^k(\|p\|_{k,\Omega^s} + \|p\|_{k,\Omega^d} + \|p^m\|_{k,\Omega^d}) \\
&+ C\beta_p^{-1}(C_cC_e^{-1} + 1)C_m h^k(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}),
\end{aligned} \tag{3.57b}
$$

where $\boldsymbol{u} := (u, u^s|_{\Gamma_0^s}, u^m)$, $\boldsymbol{p} := (p, p^s|_{\Gamma_0^s}, p^d|_{\Gamma_0^d}, p^m, p^m|_{\Gamma_0^m})$, and C is a constant a constant which depends on the shape regularity of meshes and the polynomial degree k but is independent of h, σ, μ, κ_f, and κ_m, and where

$$
\begin{aligned}
C_m &= \max(2\mu\max(1 + C, \beta + C), \kappa_f^{-1}, \kappa_m^{-1}), \\
C_e &= \min(\mu(1 - C^2/\beta)\min(1, \beta), \alpha\kappa_f^{-1/2}, \kappa_f^{-1}, \kappa_m^{-1}), \\
C_c &= \max(2\mu\max(1 + C_{\mathrm{tr}}, \beta + C_{\mathrm{tr}}), \kappa_f^{-1}, \kappa_m^{-1}, \alpha\mu\kappa_f^{-1/2}).
\end{aligned}
$$

Proof. We start by proving eq. (3.57a). By a triangle inequality, eq. (3.54a), eq. (3.50), and Lemma 28, we obtain:

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{V_h^s} + \|u-u_h\|_{\Omega^d} + \|u^m - u_h^m\|_{\Omega^d}
&\le \|\boldsymbol{e}_u^h\|_{V_h^s} + \|e_u^h\|_{\Omega^d} + \|e_{u^m}^h\|_{\Omega^d} + \|\boldsymbol{e}_u^I\|_{V_h^s} + \|e_u^I\|_{\Omega^d} + \|e_{u^m}^I\|_{\Omega^d} \\
&\le CC_e^{-1}C_m h^k(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}) \\
&\quad + Ch^k(\|u\|_{k+1,\Omega^s} + \|u\|_{k+1,\Omega^d} + \|u^m\|_{k+1,\Omega^d}) \\
&\le C(C_e^{-1}C_m + 1)h^k\Big(\|u\|_{k+1,\Omega^s} + \|u\|_{k+1,\Omega^d} + \|u^m\|_{k+1,\Omega^d}\Big).
\end{aligned}
$$

We next prove eq. (3.57b). By the triangle inequality, eq. (3.51), and eq. (3.54b),

$$
\begin{aligned}
\|\boldsymbol{p}-\boldsymbol{p}_h\|_{P_h} &\le \|\boldsymbol{e}_p^I\|_{P_h} + \|\boldsymbol{e}_p^h\|_{P_h} \\
&\le Ch^k(\|p\|_{k,\Omega^s} + \|p\|_{k,\Omega^d} + \|p^m\|_{k,\Omega^d}) \\
&\quad + C\beta_p^{-1}(C_c C_e^{-1} + 1)C_m h^k(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}).
\end{aligned}
$$

■

3.5 Numerical Examples

The numerical examples in this section were implemented in Python using the NGSolve library [53, 54]. The interior penalty parameter in eq. (3.29) is assigned as $\beta = 10k^2$ for all examples, which, according to the explicit upper bound of C_{tr} in [49], is determined to be sufficient for stability if k is less than or equal to 5. All simulations are performed on an unstructured mesh.

3.5.1 Example 1

We validate our theoretical results by using a manufactured solution adapted from [37, Section 6.1]. Let $\Omega = (0,1)^2$, $\Omega^s = (0,1) \times (0.5,1)$, and $\Omega^d = (0,1) \times (0,0.5)$. We choose the following parameters for the scheme $\kappa_m = 1$, $\kappa_f = 1$, $\sigma = 0.9$, $\mu = 1$ and $\alpha = \mu\sqrt{\kappa_f}(1+4\pi^2)/2$, for the convergence rates presented in tables 3.1 to 3.3.

Boundary conditions and source terms are set in such a way that the exact solution to our problem is given by:

$$u^s = \begin{bmatrix} -2\pi^{-2}\sin(\pi x)\exp(y/2) \\ \pi^{-1}\cos(\pi x)\exp(y/2), \end{bmatrix}, \qquad p^s = \frac{\kappa_f \mu - 2}{\kappa_f \pi}\cos(\pi x)\exp(y/2),$$

$$u^d = \begin{bmatrix} -2\sin(\pi x)\exp(y/2), \\ \pi^{-1}\cos(\pi x)\exp(y/2) \end{bmatrix}, \qquad p^d = -\frac{2}{\kappa_f \pi}\cos(\pi x)\exp(y/2),$$

$$u^m = \begin{bmatrix} -\sin(\pi x)\cos(2\pi y) \\ -2\cos(\pi x)\sin(2\pi y) \end{bmatrix}, \qquad p^m = \frac{1}{\kappa_m \pi}\cos(\pi x)\cos(2\pi y).$$

Table 3.1 presents the L^2-errors and convergence rates for u and p in Ω^s and Table 3.2 and Table 3.2 present the L^2-errors and convergence rates for u and p in Ω^d, and u^m and p^m in Ω^d together with a column demonstrating mass conservation. These tables show that the rates of convergence of $\|u_h - u\|_{\Omega^s}$, $\|u_h - u\|_{\Omega^d}$, $\|u_h^m - u^m\|_{\Omega^d}$, $\|p_h - p\|_{\Omega^s}$, $\|p_h - p\|_{\Omega^d}$, $\|p_h^m - p^m\|_{\Omega^d}$, $\|\nabla(u_h - u)\|_{\Omega^s}$, $\|\nabla\cdot(u_h - u)\|_{\Omega^d}$, and $\|\nabla\cdot(u_h^m - u^m)\|_{\Omega^d}$ are at least k or higher, thereby corroborating Corollary 35 and eqs. (3.33d) to (3.33f). Note that the tables actually show that $\|u_h - u\|_{\Omega^s}$, $\|u_h - u\|_{\Omega^d}$, and $\|u_h^m - u^m\|_{\Omega^d}$ converge with an asymptotic rate of convergence of $k+1$, even though this is not shown by our error analysis.

Next, we compare the amount of time it takes to assemble and solve the system with and without static condensation. For this test, we fix the mesh size as $h = 1/64$ and polynomial degrees as $k = 3$ and $k = 4$. With static condensation, it takes 4.5 and 12 seconds, whereas without static condensation, the computation takes more than 9 and 39 minutes, for $k = 3$ and $k = 4$, respectively. This clearly demonstrates the computational advantage of using static condensation to compute the HDG solution.

In the Stokes region in Ω^s

df	Cells	$\|e_u\|_{\Omega^s}$	Rate	$\|e_p\|_{\Omega^s}$	Rate	$\|\nabla e_u\|_{\Omega^s}$	Rate	$\|\nabla \cdot u_h\|_{\Omega^s}$
$k=2$								
3.3e+02	8	4.4e-03	-	1.6e-01	-	7.4e-02	-	4.4e-16
1.1e+03	28	7.4e-04	2.6	3.8e-02	2.1	2.0e-02	1.9	1.1e-15
5.5e+03	152	5.2e-05	3.8	7.7e-03	2.3	3.8e-03	2.4	2.0e-15
2.0e+04	570	4.9e-06	3.4	1.9e-03	2.0	8.2e-04	2.2	3.6e-15
8.3e+04	2346	5.7e-07	3.1	4.9e-04	2.0	2.0e-04	2.0	6.8e-15
3.4e+05	9520	6.8e-08	3.1	1.3e-04	2.0	4.9e-05	2.0	1.3e-14
$k=3$								
5.0e+02	8	4.4e-04	-	4.0e-02	-	9.9e-03	-	5.0e-15
1.7e+03	28	6.1e-05	2.8	6.8e-03	2.5	2.1e-03	2.2	8.1e-15
8.6e+03	152	1.4e-06	5.4	3.1e-04	4.4	1.3e-04	4.0	2.1e-14
3.2e+04	570	7.5e-08	4.3	3.1e-05	3.3	1.5e-05	3.2	4.0e-14
1.3e+05	2346	3.8e-09	4.3	3.4e-06	3.2	1.6e-06	3.2	8.5e-14
5.3e+05	9520	2.2e-10	4.1	4.1e-07	3.0	1.9e-07	3.1	1.7e-13
$k=4$								
7.2e+02	8	3.5e-05	0.0	2.8e-03	0.0	7.8e-04	0.0	6.0e-17
2.4e+03	28	2.2e-06	4.0	1.7e-04	4.0	7.5e-05	3.4	6.9e-17
1.2e+04	152	2.6e-08	6.4	5.4e-06	5.0	2.3e-06	5.0	5.9e-17
4.6e+04	570	6.9e-10	5.2	3.0e-07	4.2	1.2e-07	4.2	5.8e-17
1.9e+05	2346	2.0e-11	5.1	1.8e-08	4.1	7.2e-09	4.1	5.9e-17
7.7e+05	9520	6.8e-13	4.9	1.1e-09	4.0	4.5e-10	4.0	5.9e-17

Table 3.1: Errors and rates of convergence for the set problem up in Section 3.5.1, in Ω^s for the velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$.

In addition, we investigate the effect of changing various parameters of the problem. First, we consider the value of the viscosity μ that shows up in the Stokes equations and the interface conditions but not directly in the dual-porosity equations. We observe that the pressure error in the Stokes region changes accordingly but pressure errors in the dual-porosity region are not affected. But as the value of μ increases in the Stokes region, the pressure error increases.

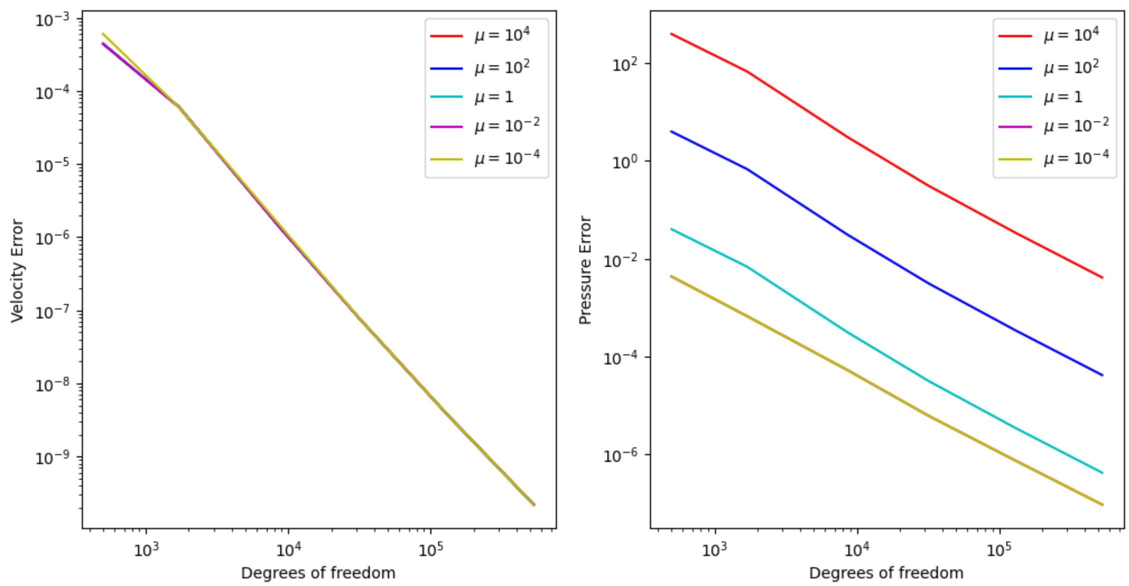
In microfractures in Ω^d

df	Cells	$\|e_u\|_{\Omega^d}$	Rate	$\|e_p\|_{\Omega^d}$	Rate	$\|\nabla\cdot e_u\|_{\Omega^d}$	Rate	$\|\Phi\|_{\Omega^d}$
$k=2$								
3.3e+02	8	1.6e-02	-	2.6e-02	-	2.5e-01	-	8.8e-06
1.1e+03	28	3.0e-03	2.5	8.7e-03	1.6	8.4e-02	1.6	8.1e-08
5.5e+03	152	1.9e-04	4.0	1.3e-03	2.7	1.3e-02	2.7	9.6e-10
2.0e+04	570	2.4e-05	3.0	3.5e-04	1.9	3.4e-03	1.9	1.2e-11
8.3e+04	2346	2.5e-06	3.2	8.0e-05	2.1	7.7e-04	2.1	3.8e-12
3.4e+05	9520	3.0e-07	3.1	1.9e-05	2.0	1.9e-04	2.0	1.6e-11
$k=3$								
5.0e+02	8	1.5e-03	-	3.4e-03	-	3.2e-02	-	1.2e-08
1.7e+03	28	1.2e-04	3.7	5.1e-04	2.7	4.9e-03	2.7	1.4e-11
8.6e+03	152	3.3e-06	5.1	3.7e-05	3.8	3.5e-04	3.8	8.2e-13
3.2e+04	570	2.1e-07	3.9	4.9e-06	2.9	4.7e-05	2.9	3.0e-12
1.3e+05	2346	1.2e-08	4.1	5.8e-07	3.1	5.6e-06	3.1	1.2e-11
5.3e+05	9520	7.1e-10	4.1	7.0e-08	3.0	6.8e-07	3.0	4.8e-11
$k=4$								
7.2e+02	8	1.2e-04	-	3.3e-04	-	3.2e-03	-	1.8e-11
2.4e+03	28	7.8e-06	4.0	4.2e-05	3.0	4.0e-04	3.0	1.6e-15
1.2e+04	152	8.9e-08	6.5	1.1e-06	5.3	1.0e-05	5.3	1.3e-15
4.6e+04	570	2.8e-09	5.0	6.8e-08	3.9	6.6e-07	3.9	1.3e-15
1.9e+05	2346	6.1e-11	5.5	3.3e-09	4.4	3.2e-08	4.4	1.3e-15
7.7e+05	9520	1.8e-12	5.1	1.9e-10	4.1	1.9e-09	4.1	1.3e-15

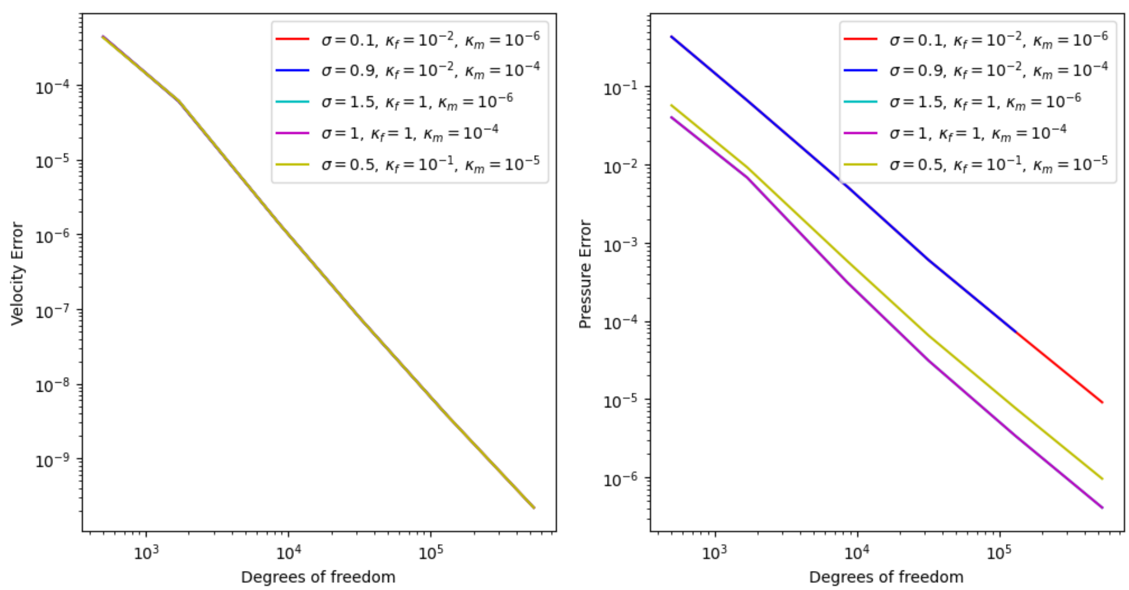
Table 3.2: Errors and rates of convergence, for the problem as set up in Section 3.5.1, in microfractures in Ω^d for the velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$. Here $e_u := u_h - u$, $e_p = p_h - p$, and $\Phi := \sigma\kappa_m(p_h^d - p_h^m) + \nabla\cdot u_h^d - \Pi_Q^d g$.

Upon changing the parameters that show up in the dual-porosity equations, the interface conditions but not directly in the Stokes equations, we found that the magnitude of the pressure errors in both subdomains changed.

In all of these runs, even though some parameter choices led to changes in pressure errors, we did not observe a corresponding change in velocity errors. This shows that the method possesses the pressure robustness property [55, 56] which is a consequence of strong mass conservation property of the scheme.

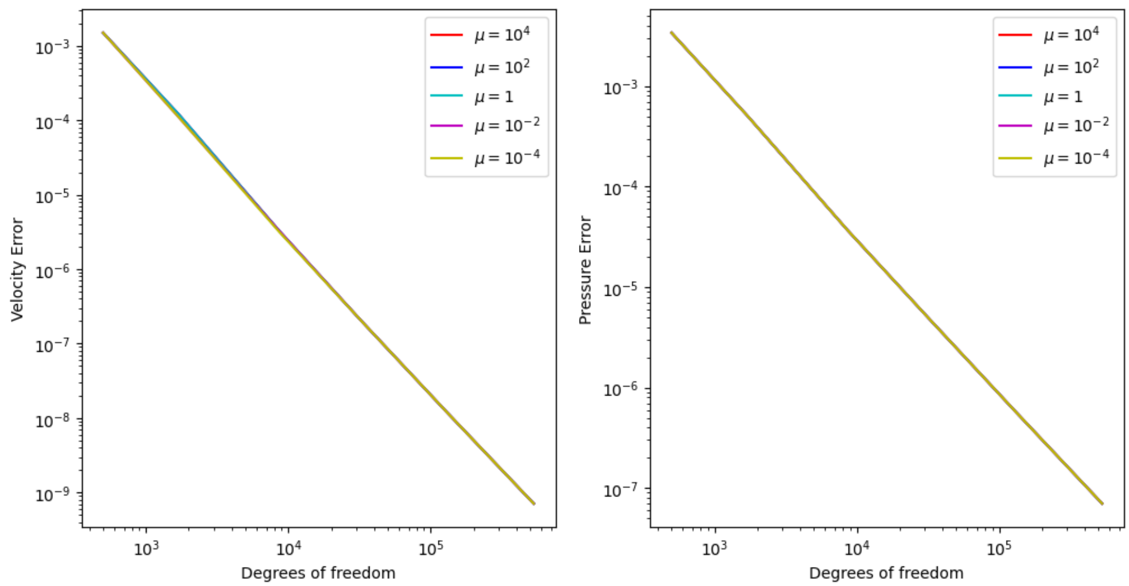


(a) Left: Error of velocity in Stokes domain, right: Error of Pressure in Stokes domain by changing only the value of μ.

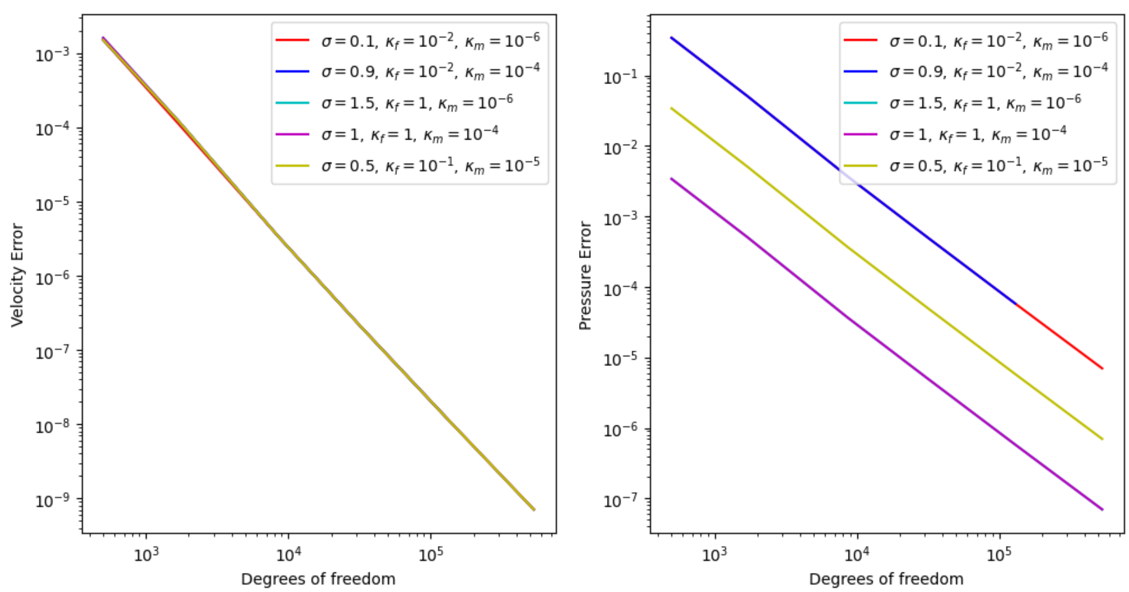


(b) Left: Error of velocity in Stokes domain, right: Error of Pressure in Stokes domain, changing the parameters in the graph.

Figure 3.2: The error of velocity and pressure in Stokes region for different parameters, as described in the graphs. Notice that some graphs overlap

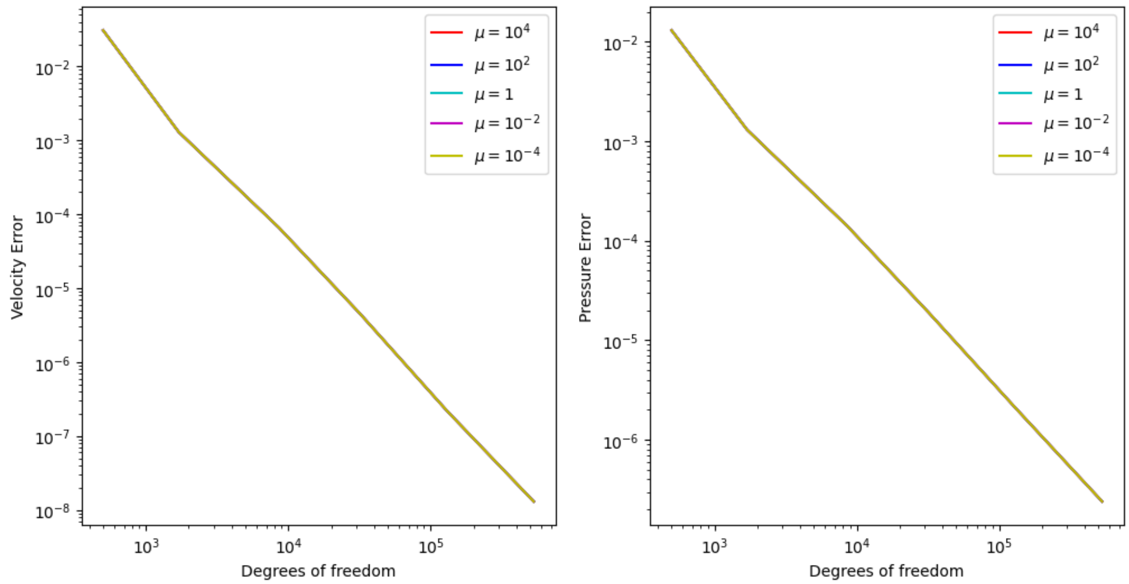


(a) Left: Error of velocity in microfracture, right: Error of Pressure in microfracture by changing only the value of μ.

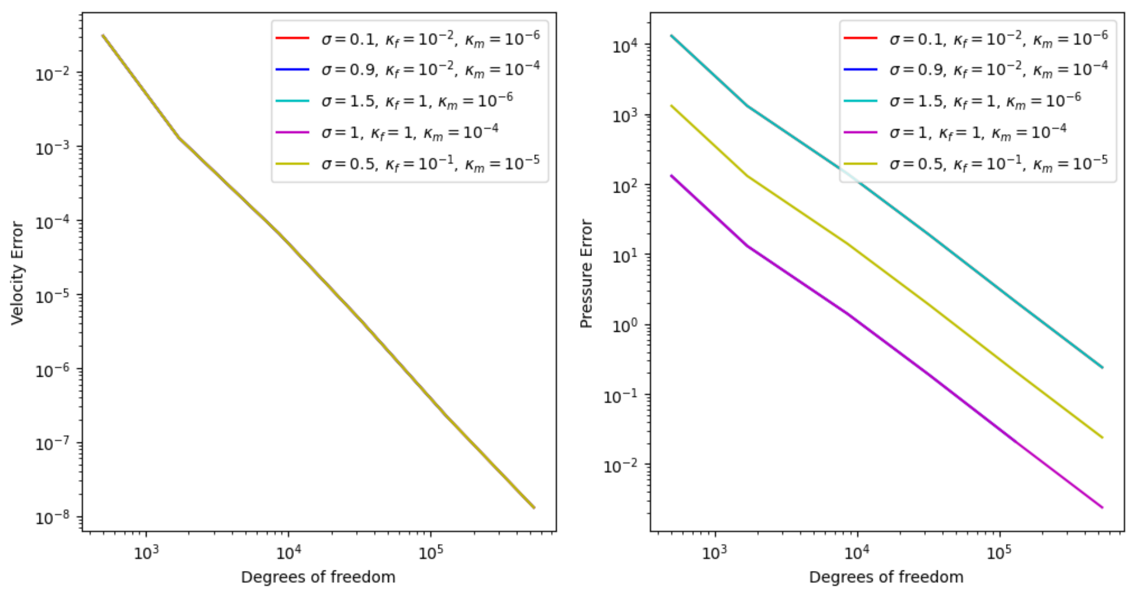


(b) Left: Error of velocity in microfracture, right: Error of Pressure in microfracture, changing the parameters in the graph.

Figure 3.3: The error of velocity and pressure in microfracture for different parameters, as described in the graphs.



(a) Left: Error of velocity in matrix, right: Error of pressure in matrix by changing only the value of μ.



(b) Left: Error of velocity in matrix, right: Error of pressure in matrix, changing the parameters in the graph.

Figure 3.4: The error of velocity and pressure in matrix for different parameters, as described in the graphs.

In the matrix in Ω^d

df	Cells	$\|e_u^m\|_{\Omega^d}$	Rate	$\|e_p^m\|_{\Omega^d}$	Rate	$\|\nabla\cdot e_u^m\|_{\Omega^d}$	Rate	$\|\Phi^m\|_{\Omega^d}$
$k=2$								
3.3e+02	8	9.7e-02	-	2.8e-02	-	2.5e-01	-	8.9e-04
1.1e+03	28	1.7e-02	2.5	1.1e-02	1.3	8.4e-02	1.6	7.1e-06
5.5e+03	152	1.7e-03	3.3	2.3e-03	2.3	1.3e-02	2.7	8.8e-08
2.0e+04	570	1.8e-04	3.2	5.4e-04	2.1	3.4e-03	1.9	9.8e-10
8.3e+04	2346	2.0e-05	3.2	1.3e-04	2.1	7.7e-04	2.1	9.7e-12
3.4e+05	9520	2.3e-06	3.1	3.0e-05	2.1	1.9e-04	2.0	5.3e-12
$k=3$								
5.0e+02	8	3.1e-02	-	1.3e-02	-	3.2e-02	-	1.2e-06
1.7e+03	28	1.3e-03	4.6	1.3e-03	3.4	4.9e-03	2.7	1.4e-09
8.6e+03	152	6.6e-05	4.3	1.4e-04	3.2	3.5e-04	3.8	8.0e-13
3.2e+04	570	4.5e-06	3.9	1.9e-05	2.9	4.7e-05	2.9	1.1e-12
1.3e+05	2346	2.2e-07	4.3	2.1e-06	3.2	5.6e-06	3.1	4.8e-12
5.3e+05	9520	1.3e-08	4.1	2.4e-07	3.1	6.8e-07	3.0	1.8e-11
$k=4$								
7.2e+02	8	5.8e-03	-	3.0e-03	-	3.2e-03	-	9.9e-10
2.4e+03	28	2.2e-04	4.7	2.4e-04	3.7	4.0e-04	3.0	6.0e-14
1.2e+04	152	5.2e-06	5.4	1.1e-05	4.4	1.0e-05	5.3	2.3e-15
4.6e+04	570	1.2e-07	5.5	5.4e-07	4.4	6.6e-07	3.9	2.1e-15
1.9e+05	2346	3.1e-09	5.3	3.0e-08	4.2	3.2e-08	4.4	2.1e-15
7.7e+05	9520	7.2e-11	5.4	1.5e-09	4.3	1.9e-09	4.1	2.1e-15

Table 3.3: Errors and rates of convergence, for the problem as set up in Section 3.5.1, in the matrix in Ω^d for the velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$. Here $e_u^m := u_h^m - u^m$, $e_p^m = p_h^m - p^m$, and $\Phi^m := \sigma\kappa_m(p_h^m - p_h^d) + \nabla\cdot u_h^m$.

3.5.2 Example 2

In this section, we will be simulating fluid flow around wellbores with open-hole completion in a naturally fractured reservoir. We will present two cases: (i) a vertical production wellbore; and (ii) a horizontal production wellbore, both with open-hole completion [57]. The examples provided below are inspired by [23, Section 6.4], [24, Section 6.2], and [26, Examples 5.3, 5.4].

3.5.2.1 A vertical production wellbore For this example, we define $\Omega^s = (1/2,1) \times (1/2,3/2)$ and $\Omega^d = (0,3/2)^2 \setminus \overline{\Omega}^s$, and then we define the boundaries and the interface:

$$\Gamma^s := \{(x,y) : y = 3/2 \text{ and } x \in (1/2,1)\},$$
$$\Gamma^d := \{(x,y) : x = 0 \text{ or } y = 0 \text{ or } x = 3/2 \text{ or } y = 3/2\} \setminus \Gamma^s,$$

and

$$\Gamma^I := \{(x,y) : x = 1/2 \text{ or } y = 1/2 \text{ or } x = 1\},$$

on which we impose the following boundary conditions:

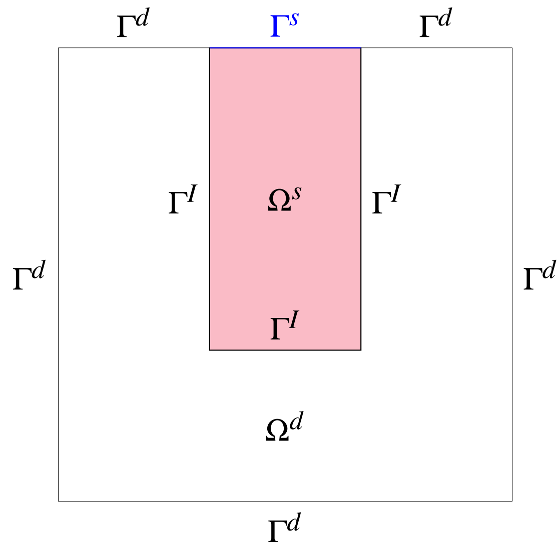
$$(-p^s I + \mu \varepsilon(u^s))n = 0 \qquad \text{on } \Gamma^s, \tag{3.58a}$$
$$p^m = 5 \times 10^4 \qquad \text{on } \Gamma^d, \tag{3.58b}$$
$$p^d = 10^4 \qquad \text{on } \Gamma^d. \tag{3.58c}$$

See Figure 3.5 for a depiction of the domain and its boundaries. Note that eq. (3.58a) imposes an outflow boundary condition for the free flow in Ω^s. The parameters for this problem are chosen as

$$h = 1/64,\ \kappa_m = 10^{-5},\ \kappa_f = 10^{-1},\ \mu = 10^{-3},\ \sigma = 0.9,\ f = 0,\ g = 0. \tag{3.59}$$

Figure 3.6a shows the pressure field in the wellbore and microfractures while Figure 3.6b shows the pressure field in the matrix. The pressure difference between p_h^m and p_h in Ω^d results in fluid flow from the matrix to the microfractures, while the pressure difference between p_h in Ω^d and p_h in Ω^s drives the fluid from the microfractures into the wellbore. The latter is observed in Figure 3.6c in which we plot the streamlines and magnitude of the velocity of the fluid flow in the microfractures and the wellbore. We

Figure 3.5: A plot of the domain used in Section 3.5.2.1.



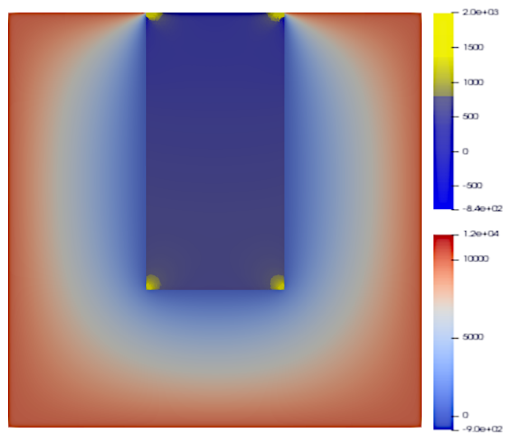
furthermore observe that once in the wellbore the velocity magnitude of the fluid is significantly higher than in the surrounding microfractures and that the fluid is driven towards the outflow boundary Γ^s. We plot the streamlines and magnitude of the velocity of the fluid in the matrix in Figure 3.6d. Here we observe that although fluid is driven towards the wellbore, there is no fluid exchange between the matrix and the wellbore as expected from the no-exchange interface condition on the matrix velocity imposed by the model.

Moreover, we remark that mass is conserved pointwise on the elements as predicted by eq. (3.33). Indeed, we compute:

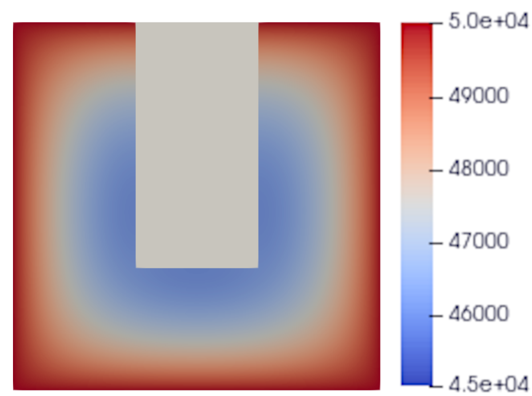
$$\begin{aligned}\|\nabla \cdot u_h^s\|_{\Omega^s} &= 1.0 \cdot 10^{-11},\\ \|\sigma\kappa_m(p_h^d - p_h^m) + \nabla \cdot u_h^d\|_{\Omega^d} &= 2.4 \cdot 10^{-11},\\ \|\sigma\kappa_m(p_h^m - p_h^d) + \nabla \cdot u_h^m\|_{\Omega^d} &= 1.2 \cdot 10^{-16}.\end{aligned}$$

Figure 3.6: The pressure and velocity solutions for the problem as set up in Section 3.5.2.1 with $\sigma = 0.9$. Left column: Pressure p_h and velocity u_h fields in Ω. Right column: Pressure p_h^m and velocity u_h^m fields in the matrix in Ω^d.

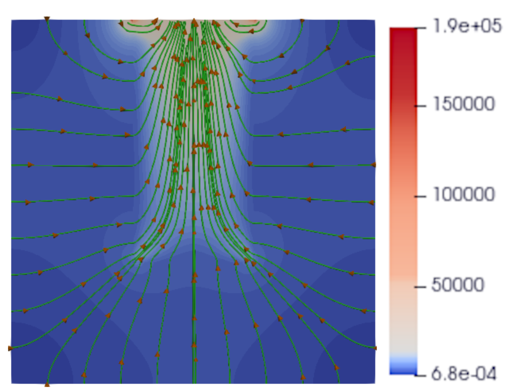
(a) Pressure p_h in Ω.



(b) Pressure p_h^m in the matrix in Ω^d.



(c) Velocity field u_h in Ω.



(d) Velocity field u_h^m in the matrix in Ω^d.

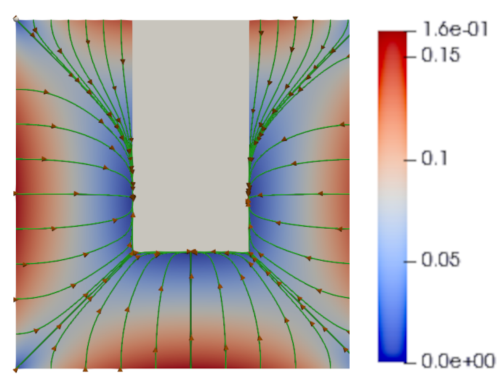
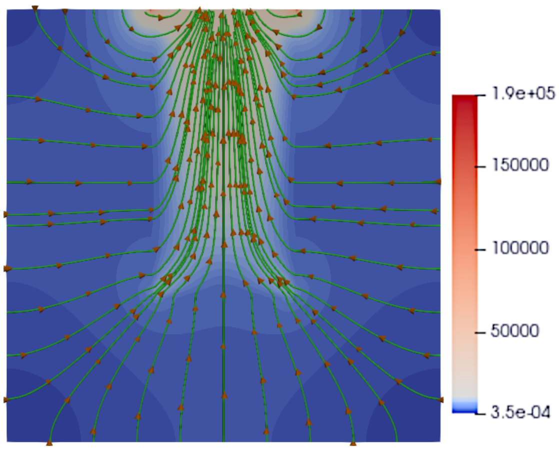
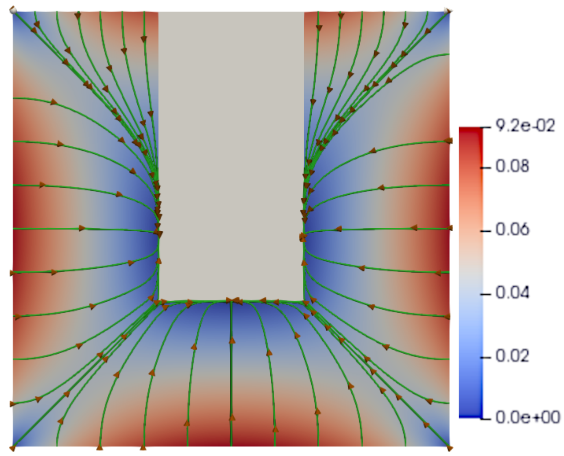


Figure 3.7: The velocity fields u_h in Ω and u_h^m in the matrix in Ω^d for the problem as set up in Section 3.5.2.1 with $\sigma = 0.5$.

(a) Velocity field u_h in Ω.



(b) Velocity field u_h^m in the matrix in Ω^d.



Finally, Figures 3.7 and 3.8 show the velocity fields obtained using shape factors $\sigma = 0.5$ and $\sigma = 0.1$, respectively, with all other parameters the same as before. As expected, the velocity magnitude decreases with decreasing σ.

3.5.2.2 Horizontal production wellbore For this example, we change the domain such that

$$\Omega^s = (1/4,1)\times(1/4,1/2)\cup[1,5/4)\times(1/4,1/2]\cup(1,5/4)\times(1/2,3/2),$$

and $\Omega^d = (0,3/2)^2 \setminus \overline{\Omega}^s$. The boundaries are defined as

$$\Gamma^s := \{(x,y) : y = 3/2 \text{ and } x \in (1,5/4)\},$$

$$\Gamma^d := \{(x,y) : x = 0 \text{ or } y = 0 \text{ or } x = 3/2 \text{ or } y = 3/2\} \setminus \Gamma^s,$$

$$\Gamma^I := \{(x,y) : x = 1/4 \text{ or } x = 1 \text{ or } x = 5/4 \text{ or } y = 1/4 \text{ or } y = 1/2\}.$$

See Figure 3.9 for a depiction of the domain and its boundaries. The parameters and boundary conditions are as in eqs. (3.58) and (3.59).

Figure 3.8: The velocity fields u_h in Ω and u_h^m in the matrix in Ω^d for the problem as set up in Section 3.5.2.1 with $\sigma = 0.1$.

(a) Velocity field u_h in Ω. (b) Velocity field u_h^m in the matrix in Ω^d.

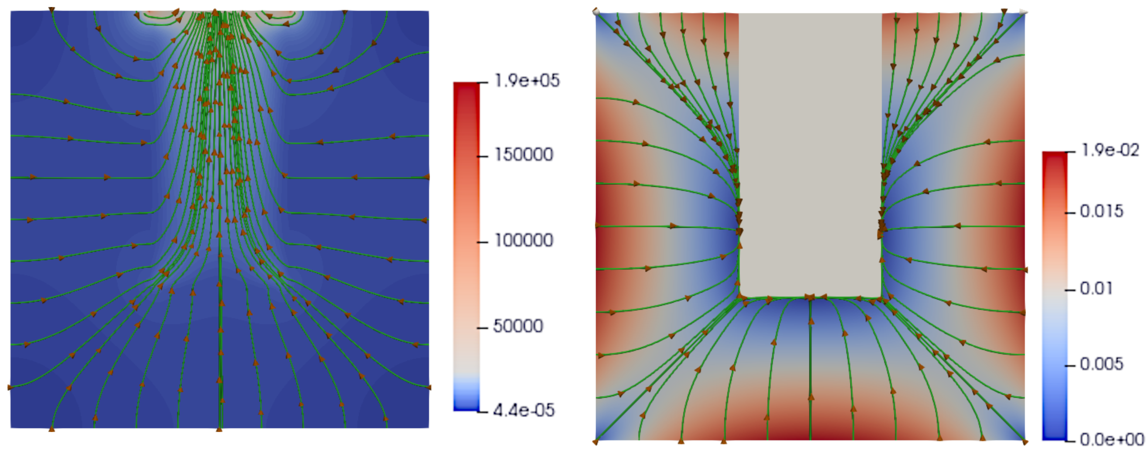
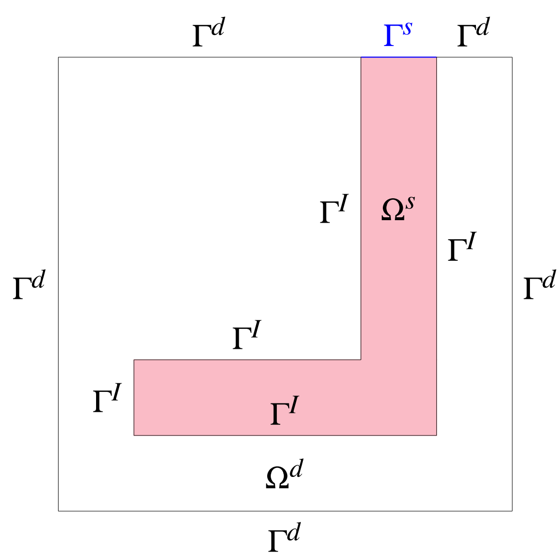


Figure 3.9: A plot of the domain used in Sections 3.5.2.2 and 3.5.3.



The pressure field in the wellbore and microfractures is shown in Figure 3.10a while the pressure field in the matrix is shown in Figure 3.10b. The streamlines and the magnitude of the velocity of the fluid flow in the microfractures and the wellbore and in the matrix for $\sigma = 0.9$, 0.5, and 0.1 are depicted in Figures 3.10c and 3.10d, Figure 3.11, and Figure 3.12, respectively. As in Section 3.5.2.1, the difference between the matrix pressure and the pressure in the microfractures drives the fluid from the matrix to the microfractures. The difference between the pressure in the microfractures and the pressure in the wellbore drives the fluid from the microfractures to the wellbore. We also observe that the fluid leaves the domain at Γ^s with a velocity magnitude larger than elsewhere in the domain and, as in Section 3.5.2.1, that the magnitude of the fluid velocity decreases as the shape parameter σ is decreased. As a final remark to this section, mass is conserved pointwise on the elements with

$$\begin{aligned} \|\nabla \cdot u_h^s\|_{\Omega^s} &= 3.2 \cdot 10^{-11}, \\ \|\sigma \kappa_m (p_h^d - p_h^m) + \nabla \cdot u_h^d\|_{\Omega^d} &= 1.5 \cdot 10^{-10}, \\ \|\sigma \kappa_m (p_h^m - p_h^d) + \nabla \cdot u_h^m\|_{\Omega^d} &= 6.7 \cdot 10^{-15}. \end{aligned}$$

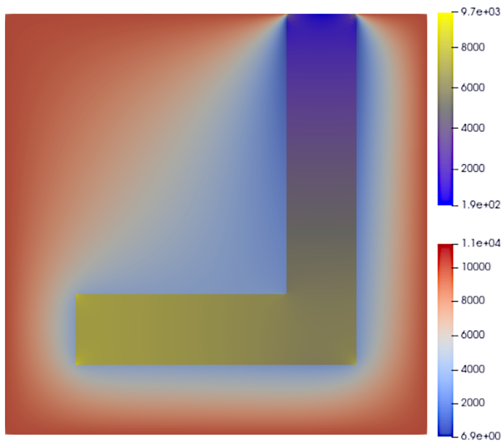
3.5.3 Example 3

Next, we show that the discretization also performs well on discontinuous data. We consider the same setup as in sections 3.5.2.1 and 3.5.2.2, changing only the permeabilities. In each element of the mesh, the values of κ_f and κ_m are now random distributed constants: $\kappa_f \in [10^{-2}, 1]$ and $\kappa_m \in [10^{-6}, 10^{-4}]$ (see Figures 3.13 and 3.15 for a plot of permeabilities).

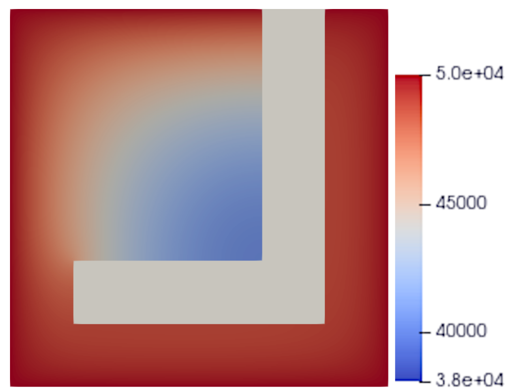
Our results in Figure 3.16 shows the dependence of the flow on the permeability: the fluid follows a non-uniform flow pattern in the dual-porosity region Ω^d as opposed to the uniform flow field observed in Figure 3.10. The fluid flows from Ω^d into the wellbore

Figure 3.10: The pressure and velocity solutions for the problem as set up in Section 3.5.2.2 with $\sigma = 0.9$. Left column: Pressure p_h and velocity u_h fields in Ω. Right column: Pressure p_h^m and velocity u_h^m fields in the matrix in Ω^d.

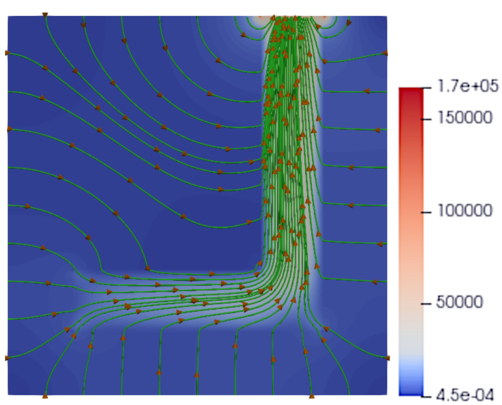
(a) Pressure p_h in Ω.



(b) Pressure p_h^m in the matrix in Ω^d.



(c) Velocity field u_h in Ω.



(d) Velocity field u_h^m in the matrix in Ω^d.

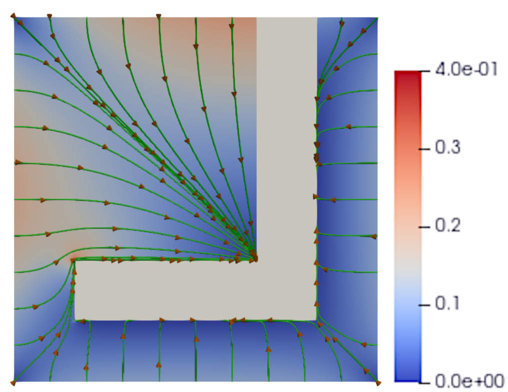
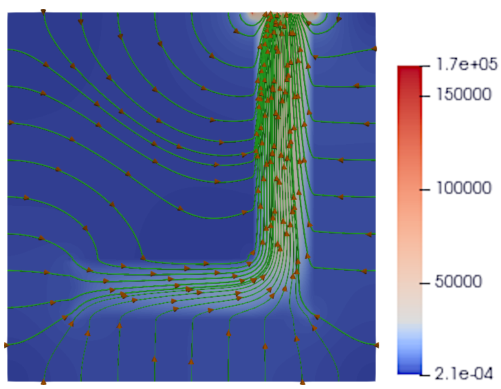


Figure 3.11: The velocity solutions for the problem as set up in Section 3.5.2.2 with $\sigma = 0.5$. Left column: Velocity u_h field in Ω. Right column: Velocity u_h^m field in the matrix in Ω^d.

(a) Velocity field u_h in Ω.



(b) Velocity field u_h^m in the matrix in Ω^d.

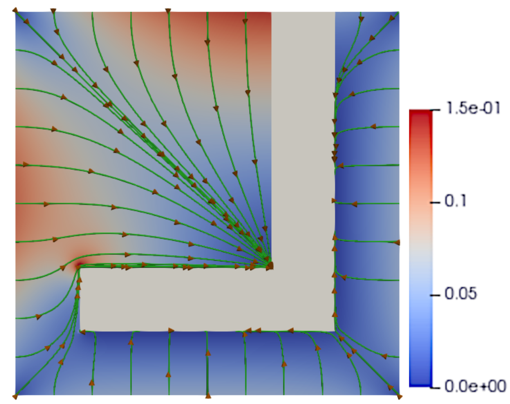
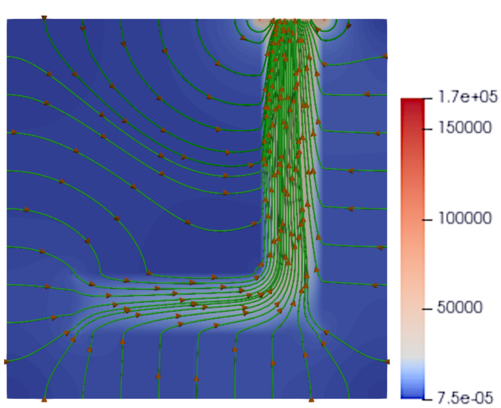


Figure 3.12: The velocity solutions for the problem as set up in Section 3.5.2.2 with $\sigma = 0.1$. Left column: Velocity u_h field in Ω. Right column: Velocity u_h^m field in the matrix in Ω^d.

(a) Velocity field u_h in Ω.



(b) Velocity field u_h^m in the matrix in Ω^d.

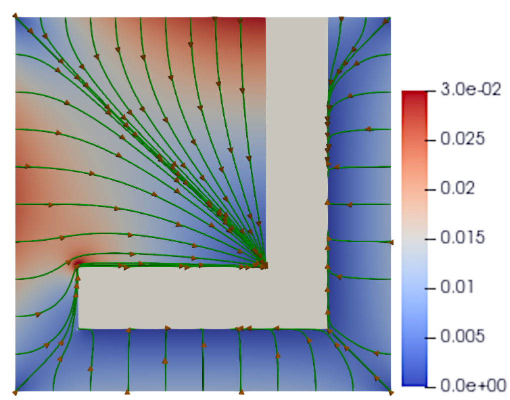
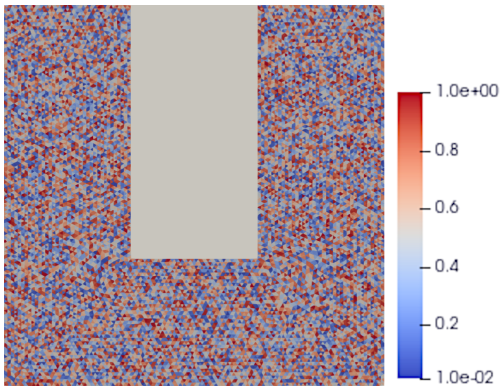
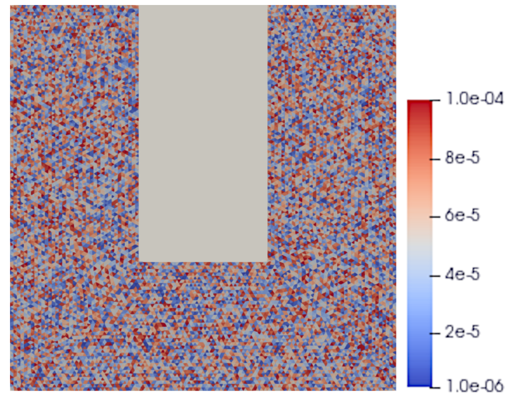


Figure 3.13: The permeabilities used for the problem as described in Section 3.5.3, $\sigma = 0.5$. Left, the random permeability field $\kappa_f \in [10^{-2}, 1]$ in the microfractures, and right, the random permeability field $\kappa_m \in [10^{-6}, 10^{-4}]$ in the matrix.

(a) Permeability κ_f in microfractures.



(b) Permeability κ_m in the matrix.



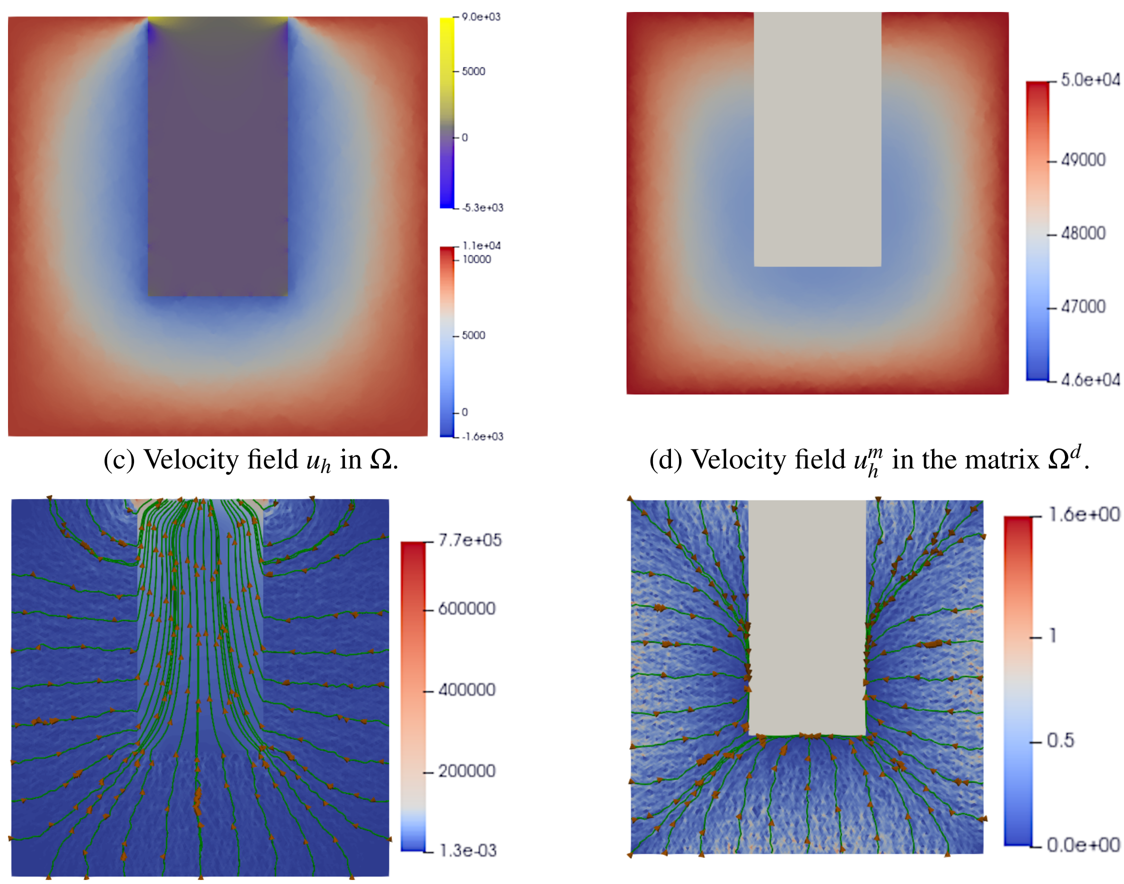
region Ω^s, avoiding low permeability regions in Ω^d, but flowing freely in regions with high permeability. As in Section 3.5.2.2, the fluid once again leaves the domain through Γ^s. Finally, as shown by the results below, mass is also conserved pointwise on the elements when dealing with discontinuous permeabilities:

$$\begin{aligned} \|\nabla \cdot u_h^s\|_{\Omega^s} &= 3.5 \cdot 10^{-11}, \\ \|\sigma \kappa_m (p_h^d - p_h^m) + \nabla \cdot u_h^d\|_{\Omega^d} &= 1.7 \cdot 10^{-10}, \\ \|\sigma \kappa_m (p_h^m - p_h^d) + \nabla \cdot u_h^m\|_{\Omega^d} &= 3.8 \cdot 10^{-15}. \end{aligned}$$

3.5.4 Example 4

In this numerical example, the free fluid subdomain is fully immersed inside of the dual-porosity subdomain. The simulation domain is the square $\Omega = [0, 10]^2$. Three macro-fractures (the union of which is Ω^s) in different directions are embedded in Ω, and the part of the domain excluding the macrofractures is the dual-porosity region Ω^d. This

Figure 3.14: The pressure and velocity solutions for the problem as set up in Section 3.5.3. Left column: Pressure p_h and velocity u_h fields in Ω. Right column: Pressure p_h^m and velocity u_h^m fields in the matrix in Ω^d.



(a) Pressure p_h in Ω.

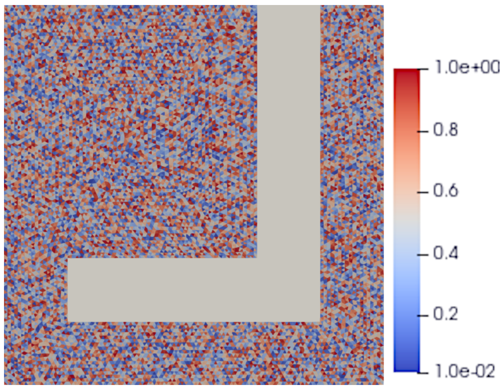
(b) Pressure p_h^m in the matrix in Ω^d.

(c) Velocity field u_h in Ω.

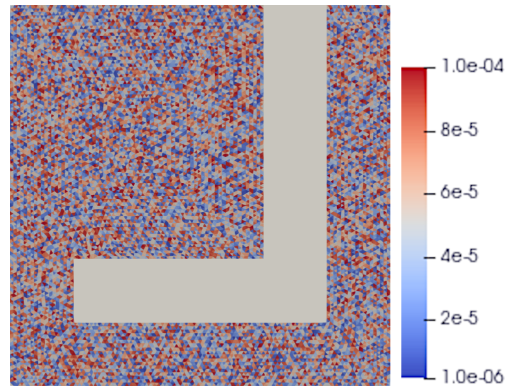
(d) Velocity field u_h^m in the matrix Ω^d.

Figure 3.15: The permeabilities used for the problem as described in Section 3.5.3, $\sigma = 0.5$. Left, the random permeability field $\kappa_f \in [10^{-2}, 1]$ in the microfractures, and right, the random permeability field $\kappa_m \in [10^{-6}, 10^{-4}]$ in the matrix.

(a) Permeability κ_f in microfractures.



(b) Permeability κ_m in the matrix.



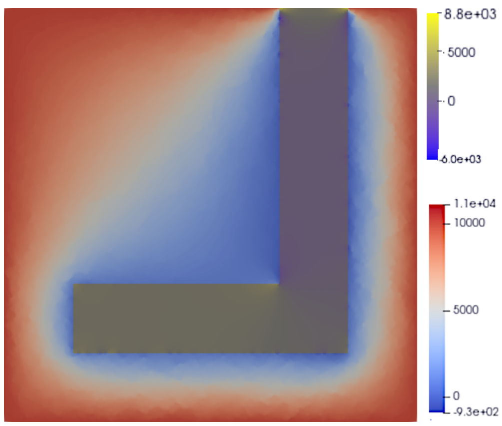
domain is depicted in Figure 3.17. Note that in this example, $\Gamma^s = \emptyset$. Although this case is not covered by our analysis, our numerical method performs well.

$\Omega^s = (1,4)\times(1,2)\cup(4.5,5.5)\times(3,7)\cup(3,7)\times(4.5,5.5)\cup(8,9)\times(5.5,9)$, $\Omega^d = \Omega\backslash\Omega^s$, $\Gamma^s = \emptyset$, $\Gamma^I = \overline{\Omega^s}\cap\overline{\Omega^d}$, and $\Gamma^{d,1} = \{0 \le x \le 3, y = 3\}\cup\{0 \le y \le 3, x = 0\}$,$\Gamma^{d,2} = \{7 \le x \le 10, y = 10\}\cup\{7 \le y \le 10, x = 10\}$, and $\Gamma^{d,3} = \partial\Omega\backslash(\Gamma^{d,1}\cup\Gamma^{d,2})$. Using the same parameters as in [28], we set $\kappa_f = 10^{-3}$, $\kappa_m = 10^{-7}$, $\alpha = 0.5$, $\beta = 10k^2$ and we choose $p_m = 10^4$, $p_f = 2\times 10^3 \in \Gamma^{d,1}$, and $p_m = p_f = 10^3 \in \Gamma^{d,2}$. The boundary conditions in $\Gamma^{d,3}$ are $u^m \cdot n = u \cdot n = 0$. The mesh size is chosen $h = 1/16$.

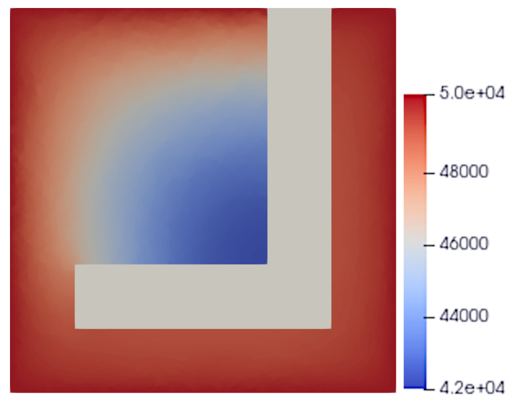
The results are displayed in fig. 3.18. From figures figs. 3.18c and 3.18d, it is clear that the flow inside the conduits is higher than in the microfracture. Additionally, there is no direct flow from the matrix to the conduits. These results align with those in [28].

Figure 3.16: The pressure and velocity solutions for the problem as set up in Section 3.5.3. Left column: Pressure p_h and velocity u_h fields in Ω. Right column: Pressure p_h^m and velocity u_h^m fields in the matrix in Ω^d.

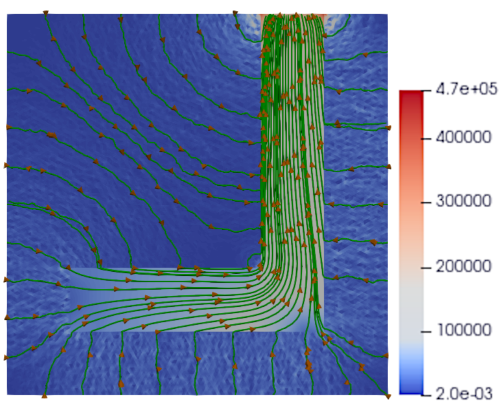
(a) Pressure p_h in Ω.



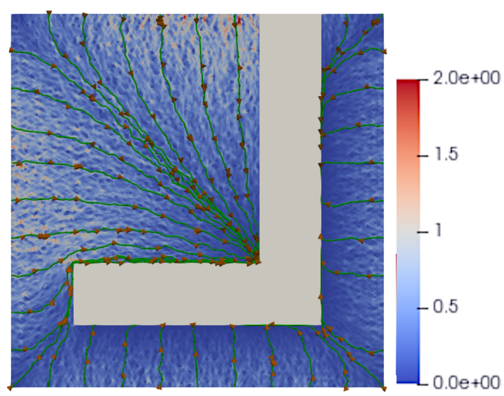
(b) Pressure p_h^m in the matrix in Ω^d.



(c) Velocity field u_h in Ω.



(d) Velocity field u_h^m in the matrix Ω^d.



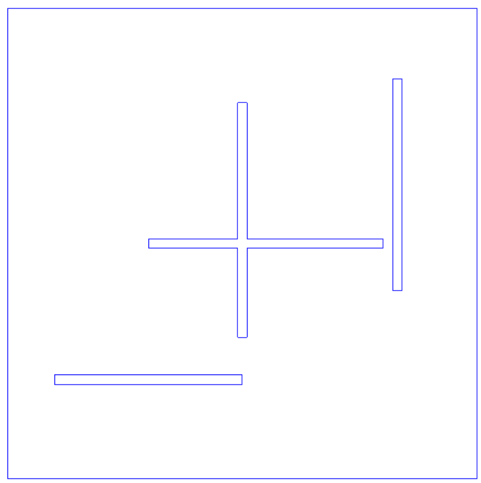
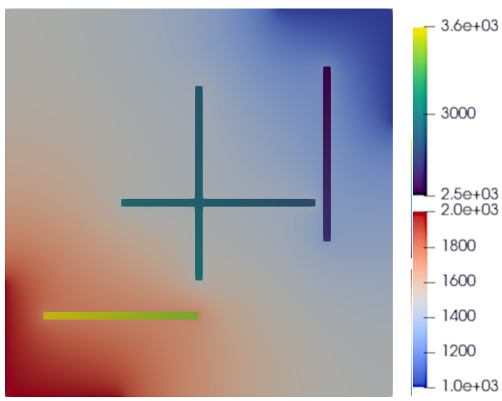


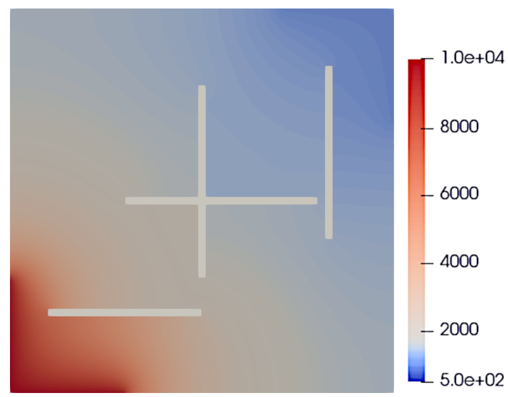
Figure 3.17: Domain of the problem in example 4: Three macrofractures immersed inside the porous media domain.

Figure 3.18: The pressure and velocity solutions for the problem as set up in Section 3.5.4. Left column: Velocity u_h and pressure p_h in Ω. Right column: Velocity u_h^m and pressure p_h^m in the matrix in Ω^d.

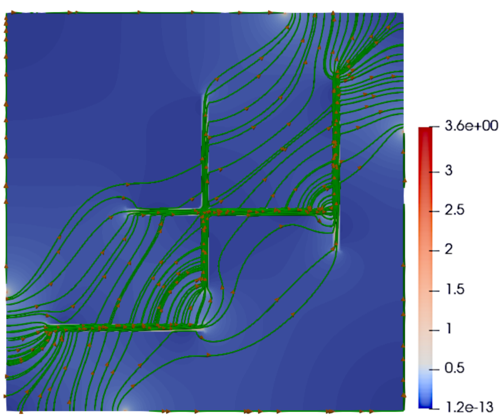
(a) Pressure p_h in Ω.



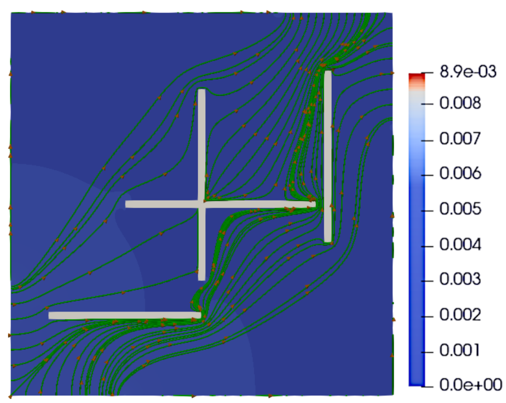
(b) Pressure p_h^m in the matrix in Ω^d.



(c) Velocity field u_h in Ω.



(d) Velocity field u_h^m in the matrix Ω^d.



CHAPTER FOUR

DUAL-POROSITY-NAVIER-STOKES- PROBLEM

4.1 The model

In this chapter, we investigate the dual-porosity-Navier-Stokes problem within the same domain as discussed in Section 3.1. The main difference compared to the model in Section 3.1 is the incorporation of the nonlinear component that models the convective transport of momentum in Ω^s. In Ω^s, instead of Stokes equations in Equation (3.1), we consider the following conservative formulation of the incompressible Navier-Stokes equations:

$$\nabla \cdot (u \otimes u) - 2\mu \nabla \cdot \varepsilon(u) + \nabla p = f \qquad \text{in} \quad \Omega^s, \tag{4.1a}$$

$$\nabla \cdot u = 0 \qquad \text{in} \quad \Omega^s, \tag{4.1b}$$

$$u = 0 \qquad \text{in} \quad \Gamma^s. \tag{4.1c}$$

In Ω^d, we consider the dual porosity model in (3.2) and in Γ^I, we consider Equation (3.3) for the interface conditions.

4.2 Weak formulation

In this section, we will analyze a weak formulation for the system of eqs. (3.2), (3.3) and (4.1).

The spaces and norms used in this section are defined in eqs. (3.4) and (3.5). The weak formulation for dual-porosity-Navier-Stokes model is the same as in eq. (3.11) except for an additional trilinear form that handles the new term $\nabla \cdot (u \otimes u)$ in the equations. This trilinear form comes from the fact that $\big(\nabla \cdot (u \otimes u), v\big)_{\Omega^s} = \big((u \cdot \nabla u, v\big)_{\Omega^s}$ if $\nabla \cdot u = 0$ in Ω^s. Therefore, the weak formulation of the system of eqs. (3.2), (3.3) and (4.1) can be stated as follows: Find $(u, u^m) \in Z = V \times V^m$, $(p, p^m) \in P = Q \times Q^m$

such that

$$t(u;u,v)+a\left((u,u^m),(v,v^m)\right)+b\left((v,v^m),(p,p^m)\right)=(f,v)_{\Omega^s} \qquad \forall (v,v^m)\in Z, \quad (4.2a)$$

$$b\left((u,u^m),(q,q^m)\right)-c\left((p,p^m),(q,q^m)\right)=-(g,q)_{\Omega^d} \quad \forall (q,q^m)\in P, \quad (4.2b)$$

where the bilinear forms $a(\cdot,\cdot), b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ are defined as in eq. (3.12) and the trilinear form is defined as follows:

$$t(w;u,v):=\left((w\cdot\nabla u),v\right)_{\Omega^s}. \tag{4.3}$$

Lemma 36. *[48, Lemma 6.32]. There exists a constant $\tau_s>0$ such that, for all $w,u,v\in V^s$,*

$$t(w,u,v)\leq \tau_s\|w\|_{V^s}\|u\|_{V^s}\|v\|_{V^s}, \tag{4.4}$$

where $\|\cdot\|_{V^s}:=\|\varepsilon(\cdot)\|_{\Omega^s}$.

In order to demonstrate that the weak formulation of the dual-porosity Navier-Stokes equation is well-posed, we first establish the well-posedness of the problem described in eq. (4.6), and then create a self-contraction mapping that has at least one fixed point under some data condition. This will provide us with a unique solution under specific conditions for the problem in eq. (4.2).

To prove the well-posedness of Equation (4.2), we introduce the subspace of V that consists of divergence-free vectors:

$$W=\{v\in V^s:\nabla\cdot v=0 \text{ in } \Omega^s\}.$$

Then for a fixed $w\in W$, we define the new bilinear form

$$A\left(w;(u,u^m),(v,v^m)\right):=t(w;u,v)+a\left((u,u^m),(v,v^m)\right), \quad \forall\left((u,u^m),(v,v^m)\right)\in Z\times Z \tag{4.5}$$

and introduce the following intermediate Oseen-type linear problem:

Given $w \in W$, find $\big((u,u^m),(p,p^m)\big) \in Z \times P$ such that:

$$A\left(w;(u,u^m),(v,v^m)\right) + b\left((v,v^m),(p,p^m)\right) = (f,v)_{\Omega^s} \qquad \forall (v,v^m) \in Z, \tag{4.6a}$$

$$b\left((u,u^m),(q,q^m)\right) - c((p,p^m),(q,q^m)) = -(g,q)_{\Omega^d} \qquad \forall (q,q^m) \in P. \tag{4.6b}$$

This problem is a saddle point problem with regular perturbation containing a nonsymmetrical bilinear form. Therefore, we apply case 2 in Theorem 17 to show well-posedness. This theorem requires $A(w;\cdot,\cdot)$ to be coercive in $\operatorname{Ker} B$, where B is defined in (3.14). However, this coercivity can only be achieved under some condition on w that we will specify in the following lemma.

Lemma 37 (Continuity and coercivity of $A(w;\cdot,\cdot)$)**.** *For any* $w \in W$*,*

$$A\left(w;(u,u^m),(v,v^m)\right) \le \tilde{C}_{nb} \|(u,u^m)\|_Z \|(v,v^m)\|_Z \quad \forall (u,u^m),(v,v^m) \in Z \times Z, \tag{4.7}$$

where $\tilde{C}_{nb} = \max\{\tau_s \|\varepsilon(w)\|_{\Omega^s} + 2\mu, \kappa_f^{-1}, \alpha\mu\kappa_f^{-1/2}, \kappa_m^{-1}\}$*. Furthermore, if*

$$\|w \cdot n\|_{\Gamma^I} \le \frac{4\mu\delta}{C_{\mathrm{tr},4}^2 C_K^2}, \tag{4.8}$$

$$A(w;(u,u^m),(u,u^m)) \ge 2(1-\delta)\mu \|\varepsilon(u)\|_{\Omega^s}^2 \qquad \forall (u,u^m) \in Z, \tag{4.9}$$

$$A\left(w;(u,u^m),(u,u^m)\right) \ge \tilde{C}_{nc} \|(u,u^m)\|_Z^2 \qquad \forall (u,u^m) \in \operatorname{Ker} B, \tag{4.10}$$

where $0 < \delta < 1$ *and* $\tilde{C}_{nc} = \min\{2\mu(1-\delta), \kappa_f^{-1}, \kappa_m^{-1}, \alpha\mu\kappa_f^{-1/2}\}$.

Proof. Continuity of $A(w;\cdot,\cdot)$ follows directly from eqs. (3.15a) and (4.4). By definition of $A(w;\cdot,\cdot)$ and eq. (3.15b), we have

$$\begin{aligned} &A\left(w;(u,u^m),(u,u^m)\right) \\ &\ge t(w;u,u) + 2\mu \|\varepsilon(u)\|_{\Omega^s}^2 + \kappa_f^{-1} \|u\|_{\Omega^d}^2 + \kappa_m^{-1} \|u^m\|_{\Omega^d}^2 + \alpha\mu\kappa_f^{-1/2} \|(u^s)^t\|_{\Gamma^I}^2 . \end{aligned} \tag{4.11}$$

Since $w \in W$ and $u = 0$ on Γ^s, by the divergence theorem, we obtain:

$$t(w;u,u) = -(\nabla \cdot w, |u|^2)_{\Omega^s} + \frac{1}{2}\langle w \cdot n, |u|^2\rangle_{\Gamma^s \cup \Gamma^I} = \frac{1}{2}\langle w \cdot n, |u|^2\rangle_{\Gamma^I}.$$

Then, using the Cauchy-Schwarz inequality, the trace inequality in eq. (2.5b), and eq. (2.7) leads to the following inequality:

$$|t(w;u,u)| \leq \frac{1}{2}\|w \cdot n\|_{\Gamma^I}\|u\|^2_{L^4(\Gamma^I)} \leq \frac{1}{2}C^2_{\mathrm{tr},4}C^2_K\|w \cdot n\|_{\Gamma^I}\|\varepsilon(u)\|^2_{\Omega^s}.$$

Consequently, we have:

$$t(w;u,u) \geq -\frac{1}{2}C^2_{\mathrm{tr},4}C^2_K\|w \cdot n\|_{\Gamma^I}\|\varepsilon(u)\|^2_{\Omega^s}. \tag{4.12}$$

Therefore, (4.9) follows by (4.8). This, eq. (4.11), and imply

$$\begin{aligned}
A\left(w;(u,u^m),(u,u^m)\right) &\geq (2\mu - \frac{1}{2}C^2_{\mathrm{tr},4}C^2_K\|w \cdot n\|_{\Gamma^I})\|\varepsilon(u)\|^2_{\Omega^s} \\
&\quad + \kappa_f^{-1}\|u\|^2_{\Omega^d} + \kappa_m^{-1}\|u^m\|^2_{\Omega^d} + \alpha\mu\kappa_f^{-1/2}\|(u^s)^t\|^2_{\Gamma^I} \\
&\geq 2\mu(1-\delta)\|\varepsilon(u)\|^2_{\Omega^s} + \kappa_f^{-1}\|u\|^2_{\Omega^d} + \kappa_m^{-1}\|u^m\|^2_{\Omega^d} + \alpha\mu\kappa_f^{-1/2}\|(u^s)^t\|^2_{\Gamma^I}.
\end{aligned} \tag{4.13}$$

Therefore, eq. (4.9) follows.

If in addition $(u,u^m) \in \mathrm{Ker\,B}$, then $\nabla \cdot u = \nabla \cdot u^m = 0$ and we have

$$\|(u,u^m)\|_Z = \|\varepsilon(u)\|^2_{\Omega^s} + \|u\|^2_{\Omega^d} + \|u^m\|^2_{\Omega^d} + \|(u^s)^t\|^2_{\Gamma^I}.$$

Equation (4.10) then follows from eq. (4.13). ■

Theorem 38. *Let $w \in W$ be such that* (4.8) *holds. Then, there exists a unique solution $\left((u,u^m),(p,p^m)\right) \in Z \times P$ to Equation* (4.6) *such that $u^s := u|_{\Omega^s} \in W$, and the following inequalities hold:*

$$\|(u,u^m)\|_Z \leq C_1\|f\|_{\Omega^s} + C_2\|g\|_{\Omega^d}, \tag{4.14a}$$

$$\|(p,p^m)\|_P \leq C_3\|f\|_{\Omega^s} + C_4\|g\|_{\Omega^d}, \tag{4.14b}$$

where the constants C_1, C_2, C_3, *and* $C_4 > 0$ *are given in eqs.* (A.7), (A.8), (A.11) *and* (A.12), *which depend only on* $\tilde{C}_{\text{nb}}$ *and* $\tilde{C}_{\text{nc}}$, *the continuity and coercivity constants of the bilinear form* $A(w;\cdot,\cdot)$, $\sigma\kappa_m$, *the continuity constant of* $c(\cdot,\cdot)$, *and the inf-sup condition constant of* $b(\cdot,\cdot)$ *given in Lemma 18.*

Proof. The existence and uniqueness of the solution, together with the inequalities in eq. (4.14) follow by Theorem 17, Case 2 since for any $w \in W$ such that $\|w\cdot n\|_{\Gamma^I} \leq \frac{4\mu\delta}{C_{\text{tr},4}^2 C_K^2}$, the problem given in Equation (4.6) satisfies the conditions of this theorem by Lemmas 18, 19, 21 and 37 . Choosing $(q,q^m) = ((\nabla\cdot u)\chi_{\Omega^s}, 0)$ in eq. (4.6b) shows that $u^s \in W$. ■

To prove the existence of a solution to (4.2), we follow [58] and define the following space:

$$X := \left\{ v \in W : \|v\|_{V^s} \leq C_1\|f\|_{\Omega^s} + C_2\|g\|_{\Omega^d} \right\}, \tag{4.15}$$

where $C_1\|f\|_{\Omega^s} + C_2\|g\|_{\Omega^d}$ is the left-hand side of (4.14a). We define the mapping $T : X \mapsto X$ such that $T(w) = u^s$, for any $w \in X$, where $u^s := u|_{\Omega^s}$ denotes the restriction of the first component of the solution $((u,u^m),(p,p^m))$ of eq. (4.6) to Ω^s. Observe that if u^s is a fixed point of T, then $((u,u^m),(p,p^m))$ is a solution of (4.2).

The following three lemmas prove that the mapping T satisfies the conditions of Theorem 14.

Lemma 39. *Let* $f \in [L^2(\Omega^s)]^{\dim}$ *and* $g \in L^2(\Omega^s)$ *be such that*

$$C_1\|f\|_{\Omega^s} + C_2\|g\|_{\Omega^d} \leq \frac{4\delta\mu}{C_{\text{tr},2}C_{\text{tr},4}^2 C_K^3}, \tag{4.16}$$

for some $0 < \delta < 1$. *Then the mapping* T *is well-defined in* X.

Proof. First, we observe that if $w \in X$, using eqs. (2.5a) and (2.7)

$$\|w\cdot n\|_{\Gamma^I} \leq C_K C_{\text{tr},2}\|\varepsilon(w)\|_{\Omega^s} = C_K C_{\text{tr},2}\|w\|_{V^s} \leq C_K C_{\text{tr},2}(C_1\|f\|_{\Omega^s} + C_2\|g\|_{\Omega^d}).$$

The data assumption eq. (4.16) then guarantees that (4.8) holds. Thus, based on Theorem 38, problem (4.6) has a unique solution $((u,u^m),(p,p^m)) \in Z \times P$ such that $u \in W$. This implies that $T(w) = u^s \in X$ is well-defined. ■

Lemma 40. *Let* $f \in [L^2(\Omega^s)]^{\dim}$ *and* $g \in L^2(\Omega^s)$ *be such that* (4.16) *holds. Then the mapping* T *is continuous in* X *and satisfies the following equation:*

$$\|T(w_2) - T(w_1)\|_{V^s} \leq \frac{\tau_s}{2\mu(1-\delta)} \|T(w_2)\|_{V^s} \|w_2 - w_1\|_{V^s} \quad \forall w_1, w_2 \in X, \tag{4.17}$$

for some $0 < \delta < 1$. *Furthermore,*

$$\|T(w_2) - T(w_1)\|_{V^s} \leq \frac{2\delta\tau_s}{(1-\delta)C_{\mathrm{tr},2}C_{\mathrm{tr},4}^2 C_K^3} \|w_2 - w_1\|_{V^s} \quad \forall w_1, w_2 \in X. \tag{4.18}$$

Proof. Let $w_i \in X$, $i = 1,2$. By Lemma 39, there exist $((u_i,u_i^m),(p_i,p_i^m)) \in Z \times P$ that satisfy Equation (4.6) for the fixed w_i, such that $T(w_i) = u_i^s \in W$. Then for any $\big((v,v^m),(q,q^m)\big) \in Z \times P$, the following holds:

$$\begin{aligned}
& t(w_1;u_1,v) + a\left((u_1,u_1^m),(v,v^m)\right) + b\left((v,v^m),(p_1,p_1^m)\right) \\
& \qquad = t(w_2;u_2,v) + a\left((u_2,u_2^m),(v,v^m)\right) + b\left((v,v^m),(p_2,p_2^m)\right), \\
& b\left((u_1,u_1^m),(q,q^m)\right) - c\left((p_1,p_1^m),(q,q^m)\right) \\
& \qquad = b\left((u_2,u_2^m),(q,q^m)\right) - c\left((p_2,p_2^m),(q,q^m)\right).
\end{aligned}$$

We can simplify the equations above by subtracting the right-side terms from both sides and using bilinearity of the forms a, b, and c as follows:

$$\begin{aligned}
& t(w_1;u_1,v) - t(w_2;u_2,v) \\
& + a\left((u_1-u_2,u_1^m-u_2^m),(v,v^m)\right) + b\left((v,v^m),(p_1-p_2,p_1^m-p_2^m)\right) = 0, \\
& b\left((u_1-u_2,u_1^m-u_2^m),(q,q^m)\right) - c\left((p_1-p_2,p_1^m-p_2^m),(q,q^m)\right) = 0,
\end{aligned}$$

for all $((v,v^m),(q,q^m)) \in Z \times P$.

Testing the equations above with $(v, v^m) = (u_1 - u_2, u_1^m - u_2^m)$ and $(q, q^m) = (p_1 - p_2, p_1^m - p_2^m)$ and subtracting the second equation from the first one, we get:

$$\begin{aligned} t(w_1; u_1, u_1 - u_2) - t(w_2; u_2, u_1 - u_2) + a\left((u_1 - u_2, u_1^m - u_2^m), (u_1 - u_2, u_1^m - u_2^m)\right) \\ + c\left((p_1 - p_2, p_1^m - p_2^m), (p_1 - p_2, p_1^m - p_2^m)\right) = 0. \end{aligned} \tag{4.19}$$

By adding and subtracting the term $t(w_2; u_1, u_1 - u_2)$ from eq. (4.19), we can simplify the nonlinear part as follows:

$$\begin{aligned} & t(w_1; u_1, u_1 - u_2) - t(w_2; u_2, u_1 - u_2) \\ & \qquad = t(w_1 - w_2; u_2, u_1 - u_2) + t(w_1; u_1 - u_2, u_1 - u_2). \end{aligned}$$

Using this in (4.19) and recalling the definition of $A(\cdot;\cdot,\cdot)$ in Equation (4.5), we obtain the following result:

$$\begin{aligned} t(w_1 - w_2; u_2, u_1 - u_2) + A\left(w_1; (u_1 - u_2, u_1^m - u_2^m), (u_1 - u_2, u_1^m - u_2^m)\right) \\ + c\left((p_1 - p_2, p_1^m - p_2^m), (p_1 - p_2, p_1^m - p_2^m)\right) = 0. \end{aligned} \tag{4.20}$$

Observe that since $w_1 \in X$, and by (4.16), w_1 satisfies (4.8). Then since $c(\cdot,\cdot)$ is positive semi-definite, using (4.9), (4.4), and (4.14a) in (4.20), and the assumption on the data, we obtain:

$$\begin{aligned} & 2(1-\delta)\mu \|\varepsilon(u_2 - u_1)\|_{\Omega^s}^2 \leq A\left(w_1; (u_1 - u_2, u_1^m - u_2^m), (u_1 - u_2, u_1^m - u_2^m)\right) \\ & + c\left((p_1 - p_2, p_1^m - p_2^m), (p_1 - p_2, p_1^m - p_2^m)\right) = -t(w_1 - w_2; u_2, u_2 - u_1) \\ & \leq \tau_s \|w_2 - w_1\|_{V^s} \|u_2\|_{V^s} \|u_2 - u_1\|_{V^s} = \tau_s \|w_2 - w_1\|_{V^s} \|u_2\|_{V^s} \|\varepsilon(u_2 - u_1)\|_{\Omega^s}. \end{aligned} \tag{4.21}$$

After cancelling $\|\varepsilon(u_2 - u_1)\|_{\Omega^s}$ on both sides of the inequality above and noting that $\|T(w_2) - T(w_1)\|_{V^s} = \|\varepsilon(u_2 - u_1)\|_{\Omega^s}$, we arrive at (4.17). Inequality (4.18) then follows from (4.17) by (4.16) since u_2 satisfies (4.14a).

■

Lemma 41. *Let $f \in [L^2(\Omega^s)]^{\dim}$ and $g \in L^2(\Omega^s)$ be such that* (4.16) *holds. Then $\overline{T(X)}$ is compact in X.*

Proof. The proof follows [59, Lemma 3.8]. Let $\{w_k\}$ be a sequence in X. By the definition of X, $\{w_k\}_{k\in\mathbb{N}}$ is bounded in $[H^1(\Omega^s)]^{\dim}$ and therefore has a convergent subsequence (still denoted by $\{w_k\}$) with limit $w \in [H^1(\Omega^s)]^{\dim}$. Since by Proposition 12, $[H^1(\Omega^s)]^{\dim}$ is compactly embedded in $[L^4(\Omega^s)]^{\dim}$, w_k converges to w in $[L^4(\Omega^s)]^{\dim}$. Since X is closed, $w \in X$. Next, using the inequality in Equation (4.21) for $w_2 = w_k$, $w_1 = w$ and then following the proof of the continuity of $t(\cdot;\cdot,\cdot)$ in [48, Lemma 6.32] we arrive at

$$\begin{aligned} \|T(w_k) - T(w)\|^2_{V^s} = \|u_k - u\|^2_{V^s} &\leq \frac{1}{2(1-\delta)\mu} |t(w_k - w; u_k, u_k - u)| \\ &\leq \frac{\dim^{1/2} \mathrm{C}_{(2,4)} \mathrm{C}_{\mathrm{K}}^2}{2(1-\delta)\mu} \|w_k - w\|_{L^4(\Omega^s)} \|u\|_{V^s} \|u_k - u\|_{V^s}. \end{aligned} \tag{4.22}$$

Therefore, $T(w_k)$ converges to $T(w)$ in $H^1(\Omega^s)$, and therefore $\overline{T(X)}$ is compact in X. ■

Having established that the conditions of Theorem 14 are satisfied, we can conclude the existence of a solution to (4.2). The theorem is stated below.

Theorem 42. *Let $f \in [L^2(\Omega^s)]^{\dim}$ and $g \in L^2(\Omega^s)$ be such that* (4.16) *holds. Then Equation* (4.2) *has at least one solution that satisfies* (4.14).

We need a stronger assumption on the data that guarantees that T is a contraction mapping to ensure the uniqueness of the solution for Equation (4.2).

Theorem 43. *Let $f \in [L^2(\Omega^s)]^{\dim}$ and $g \in L^2(\Omega^s)$ and $0 < \delta, \theta < 1$ be such that*

$$C_1 \|f\|_{L^2(\Omega^s)} + C_2 \|g\|_{L^2(\Omega^d)} \leq \min\left\{ \frac{2\mu(1-\delta)\theta}{\tau_s}, \frac{4\mu\delta}{C_{\mathrm{tr},2} C^2_{\mathrm{tr},4} C^3_K} \right\} \tag{4.23}$$

where C_1,C_2 are given in Theorem 17, Case 2. Then Equation (4.2) *has a unique solution that satisfies the stability result* (4.14).

Proof. For any two points $w_1, w_2 \in X$, let $u_1 = T(w_1)$, and $u_2 = T(w_2)$. Since (4.23) implies (4.16), by (4.17), since $u_2 \in X$,

$$\|T(w_2) - T(w_1)\|_{V^s} \leq \frac{\tau_s}{2\mu(1-\delta)} \|u_2\|_{V_s} \|w_2 - w_1\|_{V^s} \tag{4.24}$$

$$\leq \frac{\tau_s}{2\mu(1-\delta)} (C_1 \|f\|_{\Omega^s} + C_2 \|g\|_{\Omega^d}) \|w_2 - w_1\|_{V^s}. \tag{4.25}$$

Therefore, by (4.23), T is a contraction mapping and Theorem 16 establishes the existence of a unique fixed point $u^s \in X$ of T. This means that a unique solution to (4.2) exists and it satisfies Equation (4.14). ■

4.3 The HDG method

In this section, we will introduce an HDG method to solve the dual-porosity-Navier-Stokes problem defined by (4.1), (3.2), and (3.3). The discretization of the linear part of the problem, the discrete spaces, and their norms are identical to the ones defined in Section 3.3. In addition, we introduce a new norm on $\boldsymbol{Z}_h$ inspired by [15] as follows:

$$\|\boldsymbol{v}_h\|^2_{Z_h^\#} := \|\boldsymbol{v}_h\|^2_{Z_h} + \|\nabla \cdot v_h^d\|^2_{\Omega^d} + \|\nabla \cdot v_h^m\|^2_{\Omega^d} + \sum_{K \in \mathscr{T}^d} h_K^{-1} \|(v_h - \bar{v}_h) \cdot n\|^2_{\partial K \cap \Gamma^I}$$
$$+ \sum_{F \in \mathscr{F}^d \setminus \mathscr{F}^I} h_F^{-1} \left\| [\![v_h \cdot n]\!] \right\|^2_F + \sum_{F \in \mathscr{F}^d} h_F^{-1} \left\| [\![v_h^m \cdot n]\!] \right\|^2_F,$$

for any $\boldsymbol{v}_h \in \boldsymbol{Z}_h$.

The discretization of the nonlinear advective term $\nabla \cdot (u \otimes u)$ in Ω^s as introduced in [15] is as follows:

$$
\begin{aligned}
t_h(w;u,v_h) = & -(u\otimes w, \nabla v_h)_{\Omega^s} + \frac{1}{2}\left\langle w\cdot n(u+\bar{u}), v_h - \bar{v}_h\right\rangle_{\partial\mathscr{T}^s} \\
& + \frac{1}{2}\left\langle |w\cdot n|(u-\bar{u}), v_h-\bar{v}_h\right\rangle_{\partial\mathscr{T}^s} + \left\langle (w\cdot n)\bar{u}, \bar{v}_h\right\rangle_{\Gamma^I}. \quad (4.26)
\end{aligned}
$$

This trilinear form is obtained by choosing the following advective numerical flux from [60]:

$$
\hat{\sigma}_a^s = u_h \otimes u_h + (\bar{u}_h - u_h)\otimes \lambda u_h, \quad (4.27)
$$

where

$$
\lambda = \begin{cases} 1 & \text{if } u_h\cdot n < 0, \\ 0 & \text{if } u_h \cdot n \geq 0. \end{cases}
$$

The HDG method for the dual-porosity-Navier-Stokes problem is a combination of (4.26) and (3.29):

Find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$ such that:

$$
t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h) + a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{p}_h,\boldsymbol{v}_h) = (f, v_h)_{\Omega^s} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \quad (4.28a)
$$

$$
b_h(\boldsymbol{q}_h,\boldsymbol{u}_h) - c_h(\boldsymbol{p}_h,\boldsymbol{q}_h) = -(g, q_h)_{\Omega^d} \qquad \forall \boldsymbol{q}_h \in \boldsymbol{P}_h. \quad (4.28b)
$$

Lemma 44 (Consistency)**.** *Let* $((u,u^m),(p,p^m))$ *be the solution of the dual-porosity-Navier-Stokes problem defined by eqs.* (3.2)*,* (3.3) *and* (4.1)*. Then,* $\boldsymbol{u} = (u, \gamma(u), u^m)$*,* $\boldsymbol{p} = (p, \gamma(p^s), \gamma(p^d), p^m, \gamma(p^m))$ *satisfies* (4.28)*, where* $\gamma(\cdot)$ *represents the trace operator applied to the respective functions within the skeleton of the mesh corresponding to their defined domains.*

Proof. The proof is an extension of the proof of Lemma 24. First, since $\gamma(u) = u$ on ∂K,

$$
t_h(u;\boldsymbol{u},\boldsymbol{v}_h) = -(\nabla v_h, u\otimes u)_{\Omega^s} + \langle (u\cdot n)u, v_h - \bar{v}_h\rangle_{\partial\mathscr{T}^s} + \langle u\cdot n, u\cdot \bar{v}_h\rangle_{\Gamma^I}.
$$

Then, applying integration by parts, using the smoothness of u, single valuedness of $\bar{v}_h$, and that $u = 0$ on Γ^s yields:

$$t_h(u;\boldsymbol{u},\boldsymbol{v}_h) = (\nabla\cdot(u\otimes u), v_h)_{\Omega^s} - \langle (u\cdot n)u, \bar{v}_h)\rangle_{\partial\mathscr{T}^s} + \langle (u\cdot n)u, \cdot\bar{v}_h\rangle_{\Gamma^I}$$
$$= (\nabla\cdot(u\otimes u), v_h)_{\Omega^s}.$$

Next, by combining the result in Lemma 24 with the equation above we get the following equation:

$$\begin{aligned}
&t_h(u;\boldsymbol{u},\boldsymbol{v}_h) + a_h(\boldsymbol{u},\boldsymbol{v}_h) + \sum_{j=s,d}[b_h^j(\boldsymbol{p},v_h) + b_h^{I,j}(p^j,\bar{v}_h)] + b_h^d(\boldsymbol{p}^m, v_h^m)\\
&\quad + \sum_{j=s,d}\left(b_h^j(\boldsymbol{q}_h,u) + b_h^{I,j}(\bar{q}_h^j,u)\right) + b_h^d(\boldsymbol{q}_h^m,u^m) - c_h(\boldsymbol{p},\boldsymbol{q}_h)\\
=&(\nabla\cdot(u\otimes u),v_h)_{\Omega^s} - (\nabla\cdot(2\mu\varepsilon(u)) - \nabla p, v_h)_{\Omega^s} + \langle 2\mu(n^s\cdot\varepsilon(u)n^s) + p^d - p^s), n^s\cdot\bar{v}_h\rangle_{\Gamma^I}\\
&+ (\kappa_f^{-1}u + \nabla p, v_h)_{\Omega^d} + (\kappa_m^{-1}u^m + \nabla p^m, v_h^m)_{\Omega^d} - (q_h, \nabla\cdot u + \sigma\kappa_m(p-p^m))_{\Omega^d}\\
&\quad - (q_h^m, \nabla\cdot u^m + \sigma\kappa_m(p^m - p))_{\Omega^d}.
\end{aligned}$$

Recalling eqs. (3.2), (3.3) and (4.1) leads to:

$$\begin{aligned}
t_h(u;\boldsymbol{u},\boldsymbol{v}_h) &+ a_h(\boldsymbol{u},\boldsymbol{v}_h) + \sum_{j=s,d}[b_h^j(\boldsymbol{p},v_h) + b_h^{I,j}(p^j,\bar{v}_h)] + b_h^d(\boldsymbol{p}^m, v_h^m)\\
&+ \sum_{j=s,d}\left(b_h^j(\boldsymbol{q}_h,u) + b_h^{I,j}(\bar{q}_h^j,u)\right) + b_h^d(\boldsymbol{q}_h^m,u^m) - c_h(\boldsymbol{p},\boldsymbol{q}_h)\\
&= (f, v_h)_{\Omega^s} + (g, q_h)_{\Omega^d}.
\end{aligned}$$

■

Similar to the continuous case the well-posedness of (4.28) will be conducted in two-steps:

First we prove that the following linear problem for appropriately chosen conditions on $w \in \boldsymbol{V}^s(h)$ is well-posed using the saddle point theory: Given $w \in \boldsymbol{V}^s(h)$, find $\boldsymbol{u}_h \in \boldsymbol{Z}_h$, $\boldsymbol{p}_h \in \boldsymbol{P}_h$ such that:

$$t_h(w;\boldsymbol{u}_h,\boldsymbol{v}_h)+a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{p}_h,\boldsymbol{v}_h)=(f,v_h)_{\Omega^s} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{Z}_h, \tag{4.29a}$$

$$b_h(\boldsymbol{q}_h,\boldsymbol{u}_h)-c_h(\boldsymbol{p}_h,\boldsymbol{q}_h)=-(g,q_h)_{\Omega^d} \qquad \forall \boldsymbol{q}_h \in \boldsymbol{P}_h. \tag{4.29b}$$

In the next step, we prove that eq. (4.28) is well-posed using the fixed point theory and the well-posedness of eq. (4.29).

We start by proving an inf-sup condition on the bilinear form $b_h(\cdot,\cdot)$. Note that the proof of Theorem 31 needs to be updated to account for the norm change from $\|\cdot\|_{Z_h}$ to $\|\cdot\|_{Z_h^\#}$.

The following two lemmas are the updated versions of Lemmas 29 and 30.

Lemma 45. *There exists a constant $C>0$ independent of the mesh size h such that for any $(q_h,q_h^m)\in Q_h\times Q_h^m$,*

$$C(\|q_h\|_\Omega^2+\|q_h^m\|_{\Omega^d}^2)^{1/2} \le \sup_{\boldsymbol{v}_h\in \boldsymbol{Z}_h^0,\boldsymbol{v}_h\neq 0} \frac{b_h^1(\boldsymbol{u}_h,(q_h,q_h^m))}{\|\boldsymbol{v}_h\|_{Z_h^\#}}. \tag{4.30}$$

Proof. Let $(q_h,q_h^m)\in Q_h\times Q_h^m$. Then q_h and q_h^m are, respectively, elements of $L_0^2(\Omega^d)$ and $L_0^2(\Omega^d)$. As in the proof of Lemma 29, there exist $v\in[H_0^1(\Omega)]^{\dim}$ and $v^m\in[H_0^1(\Omega^d)]^{\dim}$ such that:

$$\begin{aligned} -\nabla\cdot v &= q_h \text{ in } \Omega, & C\|v\|_{1,\Omega} &\le \|q_h\|_\Omega,\\ -\nabla\cdot v^m &= q_h^m \text{ in } \Omega^d, & C\|v^m\|_{1,\Omega^d} &\le \|q_h^m\|_{\Omega^d}. \end{aligned}$$

As proved in eq. (3.43), $(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\in \boldsymbol{Z}_h^0$ and

$$\|(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\|_{Z_h}^2 \le C(\|v\|_{1,\Omega^s}^2+\|v^m\|_{1,\Omega^d}^2+\|v\|_{1,\Omega^d}^2). \tag{4.31}$$

Now we will discuss the rest of the terms in $\|(\Pi_V v,\bar{\Pi}_V v,\Pi_V^m v^m)\|_{Z_h^\#}^2$.

By triangle inequality and Lemma 28 we obtain

$$\|\nabla\cdot\Pi_V v\|_{\Omega^d}^2 \le 2(\|\nabla\cdot(v-\Pi_V v)\|_{\Omega^d}^2 + \|\nabla\cdot v\|_{\Omega^d}^2) \le C\|\nabla\cdot v\|_{\Omega^d}^2,$$

and similarly

$$\|\nabla\cdot\Pi_V^m v^m\|_{\Omega^d}^2 \le C\|\nabla\cdot v^m\|_{\Omega^d}^2.$$

Since $\Pi_V v \in H_0(\mathrm{div};\Omega)$ and $\Pi_V^m v^m \in H_0(\mathrm{div};\Omega^d)$,

$$\begin{aligned} \left\|[\![\Pi_V v\cdot n]\!]\right\|_F^2 &= 0 \quad \forall F\in\mathscr{F}^d\backslash\mathscr{F}^I \\ \left\|[\![\Pi_V^m v^m\cdot n]\!]\right\|_F^2 &= 0 \quad \forall F\in\mathscr{F}^d. \end{aligned}$$

By (3.41b),

$$\sum_{K\in\mathscr{T}^d} h_K^{-1}\|(\Pi_V v-\bar{\Pi}_V v)\cdot n\|_{\partial K\cap\Gamma^I}^2 \le C\sum_{K\in\mathscr{T}^d} h_K^{-1}h_K\|v\|_{1,K}^2 = C\|v\|_{1,\Omega^d}^2.$$

Therefore,

$$\begin{aligned} \|\nabla\cdot\Pi_V v\|_{\Omega^d}^2 + \|\nabla\cdot\Pi_V v^m\|_{\Omega^d}^2 + \sum_{K\in\mathscr{T}^d} h_K^{-1}\|(\Pi_V v-\bar{\Pi}_V v)\cdot n\|_{\partial K\cap\Gamma^I}^2 \\ + \sum_{F\in\mathscr{F}^d\backslash\mathscr{F}^I} h_F^{-1}\left\|[\![\Pi_V v\cdot n]\!]\right\|_F^2 + \sum_{F\in\mathscr{F}^d} h_F^{-1}\left\|[\![\Pi_V^m v^m\cdot n]\!]\right\|_F^2 \\ \le C(C\|\nabla\cdot v\|_{\Omega^d}^2 + C\|\nabla\cdot v^m\|_{\Omega^d}^2 + \|v\|_{1,\Omega^d}^2). \end{aligned}$$

This together with (4.31) implies that

$$\|(\Pi_V v, \bar{\Pi}_V v, \Pi_V v^m)\|_{\boldsymbol{Z}_h^\#} \le C(\|v\|_{1,\Omega}^2 + \|v^m\|_{1,\Omega^d}^2)^{1/2}.$$

The rest of the proof follows as in Lemma 29 by replacing $\|\boldsymbol{v}_h\|_{Z_h}$ by $\|\boldsymbol{v}_h\|_{Z_h^\#}$ in (3.44). ∎

Lemma 46. *There exists a constant $C \geq 0$, independent of h, such that for any $(\bar{q}_h^s, \bar{q}_h^d, \bar{q}_h^m) \in \bar{Q}_h^s \times \bar{Q}_h^d \times \bar{Q}_h^d$,*

$$C\left(\sum_{j=s,d} \sum_{K \in \mathscr{T}^j} h_K \|\bar{q}_h^j\|_K^2 + \sum_{K \in \mathscr{T}^d} h_K \|\bar{q}_h^m\|_K^2 \right)^{1/2} \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{Z}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{b_h^2(\boldsymbol{v}_h, (\bar{q}_h^s, \bar{q}_h^d, \bar{q}_h^m))}{\|\boldsymbol{v}_h\|_{Z_h^\#}}.$$

Proof. Let $(\bar{q}_h^s, \bar{q}_h^d, \bar{q}_h^m) \in \bar{Q}_h^s \times \bar{Q}_h^d \times \bar{Q}_h^d$ and let $L\bar{q}_h$ and $L\bar{q}_h^m$ be defined as in the proof of Lemma 30. Then by (3.47), (3.46), and single valuedness of $\bar{q}_h, \bar{q}_h^m$,

$$\begin{aligned}
\|(L\bar{q}_h, 0, L\bar{q}_h^m)\|_{\boldsymbol{Z}_h^\#}^2 &= \|(L\bar{q}_h, 0, L\bar{q}_h^m)\|_{\boldsymbol{Z}_h}^2 + \|\nabla \cdot L\bar{q}_h^d\|_{\Omega^d}^2 + \|\nabla \cdot L\bar{q}_h^m\|_{\Omega^d}^2 \\
&+ \sum_{K \in \mathscr{T}^d} h_K^{-1} \|L\bar{q}_h^d\|_{\partial K \cap \Gamma^I}^2 + \sum_{F \in \mathscr{F}^d \setminus \mathscr{F}^I} h_F^{-1} \|[\![L\bar{q}_h \cdot n]\!]\|_F^2 + \sum_{F \in \mathscr{F}^d} h_F^{-1} \|[\![L\bar{q}_h^m \cdot n]\!]\|_F^2 \\
&\leq C\Big(\sum_{j=s,d} \sum_{K \in \mathscr{T}^j} h_K^{-1} \|\bar{q}_h^j\|_{\partial K}^2 + \sum_{K \in \mathscr{T}^d} h_K^{-1} \|\bar{q}_h^m\|_{\partial K}^2 \Big).
\end{aligned}$$

The rest of the proof is the same as in Lemma 30 with a simple change of $\|(L\bar{q}_h, 0, L\bar{q}_h^m)\|_{\boldsymbol{Z}_h}^2$ to $\|(L\bar{q}_h, 0, L\bar{q}_h^m)\|_{\boldsymbol{Z}_h^\#}^2$.

■

As a consequence of Lemmas 45 and 46 and [51, Theorem 31.1], we have the following:

Theorem 47 (Inf-sup condition)**.** *There exists a constant $c_{bb} \geq 0$, independent of h, such that for any $\boldsymbol{q}_h = (q_h, \bar{q}_h^s, \bar{q}_h^d, q_h^m, \bar{q}_h^m) \in \boldsymbol{P}_h$.*

$$c_{bb} \|\boldsymbol{q}_h\|_{P_h} \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{Z}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{b_h(\boldsymbol{p}_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{Z_h^\#}}.$$

■

Next, we prove the boundedness of the bilinear form $b_h(\cdot,\cdot)$.

Lemma 48. *There is a positive constant c_{bc}, independent of meshsize h, such that for all $\boldsymbol{v}_h \in \boldsymbol{Z}_h$ and $\boldsymbol{p}_h \in \boldsymbol{P}_h$,*

$$|b_h(\boldsymbol{v}_h, \boldsymbol{p}_h)| \leq c_{bc} \|\boldsymbol{v}_h\|_{Z_h^\#} \|\boldsymbol{p}_h\|_{P_h}. \tag{4.32}$$

Proof. Recall that

$$b_h(\boldsymbol{q}_h,\boldsymbol{v}_h) = [b_h^s(\boldsymbol{q}_h,v_h)+b_h^{I,s}(\bar{q}_h^s,\bar{v}_h)]+[b_h^d(\boldsymbol{q}_h,v_h)+b_h^{I,d}(\bar{q}_h^d,\bar{v}_h)]+b_h^d(\boldsymbol{q}_h^m,v_h^m)$$
$$=: I_1+I_2+I_3. \quad (4.33)$$

Bounds for I_1 and I_2 are given in [15, Lemma 3]. So it is sufficient to bound I_3.

$$\begin{aligned}|I_3| =& b_h^d(\boldsymbol{q}_h^m,v_h^m) = -(q_h^m,\nabla\cdot v_h^m)_{\Omega^d}+\langle\bar{q}_h^m,v_h^m\cdot n\rangle_{\partial\mathscr{T}^d}\\ =& -(q_h^m,\nabla\cdot v_h^m)_{\Omega^d}+\langle\bar{q}_h^m,[\![v_h^m\cdot n]\!]\rangle_{\mathscr{F}^d}\\ \le& C\Big(\|\nabla\cdot v_h^m\|_{\Omega^d}^2+\sum_{F\in\mathscr{F}^d}h_F^{-1}\|[\![v_h^m\cdot n]\!]\|_F^2\Big)\Big(\|q_h^m\|_{\Omega^d}^2+\sum_{K\in\mathscr{T}^d}h_K\|\bar{q}_h^m\|_{\partial K}^2\Big). \quad (4.34)\end{aligned}$$

Using this, the bounds for I_1 and I_2 in [15, Lemma 3], and Hölder's inequality, we have

$$\begin{aligned}&|b_h(\boldsymbol{q}_h,\boldsymbol{v}_h)|\\ &\le C\Big\{\Big(\sum_{K\in\mathscr{T}^s}\|\nabla v_h\|_K^2+\sum_{K\in\mathscr{T}^s}h_K^{-1}\|v_h-\bar{v}_h\|_{\partial K}^2\Big)\Big(\|q_h\|_{\Omega^s}^2+\sum_{K\in\mathscr{T}^s}h_K\|\bar{q}_h^s\|_{\partial K}\Big)\\ &+\Big(\|\nabla\cdot v_h\|_{\Omega^d}^2+\sum_{F\in\mathscr{F}^d\setminus\mathscr{F}^I}h_F^{-1}\|[\![v_h\cdot n]\!]\|_F^2+\sum_{F\in\mathscr{F}^I}h_F^{-1}\|(v_h^d-\bar{v}_h)\cdot n\|_F^2\Big)\\ &\quad\Big(\|q_h\|_{\Omega^d}^2+\sum_{K\in\mathscr{T}^d}h_K\|\bar{q}_h^d\|_{\partial K}^2\Big)\\ &+\Big(\|\nabla\cdot v_h^m\|_{\Omega^d}^2+\sum_{F\in\mathscr{F}^d}h_F^{-1}\|[\![v_h^m\cdot n]\!]\|_F^2\Big)\Big(\|q_h^m\|_{\Omega^d}^2+\sum_{K\in\mathscr{T}^d}h_K\|\bar{q}_h^m\|_{\partial K}^2\Big)\Big\}\end{aligned}$$

■

As a consequence of (3.38), and the definition of the norm $\|\cdot\|_{Z_h^\#}$, we derive the following lemma:

Lemma 49. *The bilinear form $a_h(\cdot,\cdot)$ is a continuous bilinear form with respect to the $\|\cdot\|_{Z_h^\#}$ norm. Furthermore, for any $\boldsymbol{u}_h$, $\boldsymbol{v}_h\in\boldsymbol{Z}_h$,*

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)\le C_c\|\boldsymbol{u}_h\|_{Z_h^\#}\|\boldsymbol{v}_h\|_{Z_h^\#},$$

where $C_c := \max(2\mu(1+C_{\mathrm{tr}}), 2\mu(\beta+C_{\mathrm{tr}}), \kappa_f^{-1}, \kappa_m^{-1}, \alpha\mu\kappa_f^{-1/2})$. *Furthermore, as a consequence of corollary 27 the bilinear form* $a_h(\cdot,\cdot)$ *is a continuous bilinear form with respect to the* $\|\cdot\|_{\boldsymbol{Z}_h^{\#}}$ *norm, with only changing the trace inequality to continuous trace inequality.*

Next, we will require the following known properties of the trilinear form $t_h(\cdot;\cdot,\cdot)$.

Lemma 50. *[15, Lemma 4] Let* $w_j \in \boldsymbol{V}^s(h) \cap H(\mathrm{div},\Omega^s)$*, be such that* $\nabla\cdot w_j = 0$ *on each* $K \in \mathscr{T}^s$ *for* $j = 1,2$*. Then for all* $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}^s(h)$*, there exists a constant* $c_w > 0$ *such that:*

$$|t_h(w_1;\boldsymbol{u}_h,\boldsymbol{v}_h) - t_h(w_2;\boldsymbol{u}_h,\boldsymbol{v}_h)| \le c_w \|w_1 - w_2\|_{1,h,\Omega^s} \|\boldsymbol{u}_h\|_{V_h^s} \|\boldsymbol{v}_h\|_{V_h^s}, \tag{4.35}$$

where $\|w\|_{1,h,\Omega^s} = \|\left(w, \{\{w\}\}\right)\|_{V_h^s}$.

Remark. *From [45, Theorem 4.4 and Proposition 4.5] the following holds:*

$$\|w\|_{1,h,\Omega^s} \le \|\boldsymbol{w}\|_{V_h^s} \tag{4.36}$$

for any $w \in V^s(h) = \widetilde{V}^s + V_h^s$*, where* $\widetilde{V}^s := \{v \in [H^2(\Omega^s)]^{dim} : v = 0 \text{ on } \Gamma^s\}$.

Remark. *In the previous lemma, setting* $w_2 = 0$ *results in the continuity of the trilinear form* $t_h(\cdot;\cdot,\cdot)$*, that is, for any* $w \in \boldsymbol{V}^s(h) \cap H(\mathrm{div},\Omega^s)$*, such that* $\nabla\cdot w = 0$ *on each* $K \in \mathscr{T}^s$*,*

$$|t_h(w;\boldsymbol{u}_h,\boldsymbol{v}_h)| \le c_w \|w\|_{1,h,\Omega^s} \|\boldsymbol{u}_h\|_{V_h^s} \|\boldsymbol{v}_h\|_{V_h^s} \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}^s(h). \tag{4.37}$$

The remark above, in conjunction with Lemma 49 and Corollary 27 yields:

Corollary 51. *For any* $w \in \boldsymbol{V}^s(h) \cap H(\mathrm{div},\Omega^s)$ *such that* $\nabla\cdot w = 0$ *on each* $K \in \mathscr{T}^s$*, the sum of the two bilinear forms* $t_h(w;\cdot,\cdot) + a_h(\cdot,\cdot)$ *is bounded, that is,*

$$t_h(w;\boldsymbol{u}_h,\boldsymbol{v}_h) + a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \le C_{nb} \|\boldsymbol{u}_h\|_{\boldsymbol{Z}_h^{\#}} \|\boldsymbol{v}_h^s\|_{\boldsymbol{Z}_h^{\#}} \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{Z}_h, \tag{4.38}$$

where $C_{nb} := \max(2\mu\max(1+C_{\mathrm{tr}}, \beta+C_{\mathrm{tr}}) + c_w\|w\|_{1,h,\Omega^s}, \kappa_f^{-1}, \kappa_m^{-1}, \alpha\mu\kappa_f^{-1/2})$*, where the* C_{tr} *is the continuous trace inequality.* ■

The following lemma states that the form $t_h(w;\cdot,\cdot)+a_h(\cdot,\cdot)$ is also coercive but only under an appropriate assumption on the convective velocity. This condition is analogous to the eq. (4.8) in the continuous case.

Lemma 52. *Let $w \in \boldsymbol{V}^s(h)\cap H(\mathrm{div},\Omega^s)$ such that $\nabla\cdot w = 0, \forall K \in \mathscr{T}^s$, then*

$$t_h(w;\boldsymbol{u}_h,\boldsymbol{u}_h) \geq -(c_{h,4}^2 + c_{\mathrm{tr},4}^2 C_K^2)\|w\cdot n\|_{\Gamma^I}\,\|\boldsymbol{u}_h\|_{V_h^s}^2 \quad \forall \boldsymbol{u}_h \in \boldsymbol{V}_h, \tag{4.39}$$

where $C_{h,4}$ and $C_{\mathrm{tr},4}$ are the same as in [15, Lemma 6]. Furthermore, if

$$\|w\cdot n\|_{\Gamma^I} \leq \mu c_{ae}^s \delta (c_{h,4}^2 + c_{\mathrm{tr},4}^2 C_K^2)^{-1}, \tag{4.40}$$

where $0<\delta<1$, then, for all $\boldsymbol{u}_h \in \boldsymbol{Z}_h$,

$$t_h(w;\boldsymbol{u}_h,\boldsymbol{u}_h) + a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq C_{nc}\|\boldsymbol{u}_h\|_{Z_h}^2, \tag{4.41}$$

and for all $\boldsymbol{u}_h \in \mathrm{Ker}(b_h) := \{\boldsymbol{v}_h \in \boldsymbol{Z}_h : b_h(\boldsymbol{v}_h,\boldsymbol{q}_h) = 0 \quad \forall \boldsymbol{q}_h \in \boldsymbol{P}_h\}$,

$$t_h(w;\boldsymbol{u}_h,\boldsymbol{u}_h) + a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq C_{nc}\,\|\boldsymbol{u}_h\|_{Z_h^\#}^2, \tag{4.42}$$

where $C_{nc} = \min\{(1-\delta)\mu c_{ae}^s, \kappa_f^{-1}, \alpha\kappa_f^{-1/2}, \kappa_m^{-1}\} > 0$ and c_{ae}^s is the constant in Lemma 25.

Proof. The proof of Equation (4.39) is given in [15, Lemma 6]. Equation (4.41) follows from (4.39), (4.40), and (3.36) in lemma 25.

Now, let us prove (4.42). Given that $\boldsymbol{u}_h \in \mathrm{Ker}(b_h)$, $\nabla\cdot u_h = 0$ in Ω^d, $[\![u_h\cdot n]\!] = 0$ in $\mathscr{F}^d\backslash\mathscr{F}^I$, $[\![u_h^m\cdot n]\!] = 0$ on any $F\in\mathscr{F}^d$, and $u_h^d\cdot n = \bar{u}_h\cdot n$ on Γ^I. Therefore, we deduce the subsequent result:

$$\begin{aligned}\|\boldsymbol{u}_h\|_{Z_h^\#}^2 &= \|\boldsymbol{u}_h\|_{Z_h}^2 + \|\nabla\cdot u_h\|_{\Omega^d}^2 + \sum_{K\in\mathscr{T}^d} h_K^{-1}\|(u_h-\bar{u}_h)\cdot n\|_{\partial K\cap\Gamma^I}^2 \\ &\quad + \sum_{F\in\mathscr{F}^d\backslash\mathscr{F}^I} h_F^{-1}\big\|[\![u_h\cdot n]\!]\big\|_F^2 + \|\nabla\cdot u_h^m\|_{\Omega^d}^2 + \sum_{F\in\mathscr{F}^d} h_F^{-1}\big\|[\![u_h^m\cdot n]\!]\big\|_F^2 = \|\boldsymbol{u}_h\|_{Z_h}^2.\end{aligned}$$

Then (4.42) follows directly from eq. (4.41). ■

The following theorem proves the well-posedness of eq. (4.29) under the condition on the convective velocity w that we introduced in Lemma 52 to achieve coercivity of $t_h(w;\cdot,\cdot)+a_h(\cdot,\cdot)$.

Theorem 53. *Let $w \in \boldsymbol{V}^s(h) \cap H(\mathrm{div},\Omega^s)$ such that $\nabla \cdot w = 0$ for all $K \in \mathscr{T}^s$. If w satisfies* (4.40)*, then eq.* (4.29) *is well-posed. Furthermore,*

$$\|\boldsymbol{u}_h\|_{Z_h^\#} \leq C_1^h \|f\|_{\Omega^s} + C_2^h \|g\|_{\Omega^d}, \tag{4.43}$$

$$\|\boldsymbol{p}_h\|_{P_h} \leq C_3^h \|f\|_{\Omega^s} + C_4^h \|g\|_{\Omega^d}. \tag{4.44}$$

where $C_1^h, C_2^h, C_3^h, C_4^h > 0$ are constants that are defined Theorem 17, Case 2, depending on the coercivity constant C_{nc} in Lemma 52, the continuity constant C_{nb} of the bilinear form $t_h(w;\cdot,\cdot)+a_h(\cdot,\cdot)$ in Corollary 51, the continuity constant $\sigma\kappa_m$ of the bilinear form $c(\cdot,\cdot)$, and the inf-sup constant c_{bb} in Theorem 47.

Proof. Let $w \in \boldsymbol{V}^s(h) \cap H(\mathrm{div},\Omega^s)$ such that $\nabla \cdot w = 0$ for all $K \in \mathscr{T}^s$. Then by Theorem 47, lemmas 48 and 52, and corollary 51, the bilinear forms $a(\cdot,\cdot) := t_h(w;\cdot,\cdot)+a_h(\cdot,\cdot)$, $b(\cdot;\cdot) := b_h(\cdot,\cdot)$, and $c(\cdot,\cdot) = c_h(\cdot,\cdot)$ in (4.29) satisfy the conditions of Theorem 17, Case 2. Therefore, (4.29) is well-posed and (4.43) holds with the constants $C_1^h, C_2^h, C_3^h, C_4^h$ defined by (2.16). ■

Now, we advance to the second phase of our analysis where we will apply the fixed point theory to conclude that eq. (4.28) is well-posed. For that purpose, we define the following subspace in $\boldsymbol{V}_h^s$:

$$\boldsymbol{B}_h^s := \Big\{ \boldsymbol{v}_h^s \in \boldsymbol{V}_h^s : b_h^s(\boldsymbol{q}_h, v_h^s) + b_h^{I,s}(\bar{q}_h^s, \bar{v}_h) = 0 \quad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^s \text{ and } \\ \|\boldsymbol{v}_h^s\|_{V_h^s} \leq C_1^h \|f\|_{\Omega^s} + C_2^h \|g\|_{\Omega^d} \Big\},$$

where $C_1^h, C_2^h, C_3^h, C_4^h$ are the constants in Theorem 53.

Theorem 54. *Let* $f \in [L^2(\Omega^s)]^{\dim}$ *and* $g \in L^2(\Omega^d)$ *be such that*

$$C_1^h \|f\|_{\Omega^s} + C_2^h \|g\|_{\Omega^d} \leq \frac{\mu c_{ae}^s \delta}{c_{\mathrm{tr},2} C_K (c_{\mathrm{tr},4}^2 + c_{h,4}^2 C_K^2)}, \tag{4.45}$$

where $0 < \delta < 1$ *and* $C_1^h, C_2^h, C_3^h, C_4^h$ *are the constants in Theorem 53. Then there exists at least one solution* $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$ *to eq.* (4.28) *that satisfies the following bounds:*

$$\|\boldsymbol{u}_h\|_{Z_h^\#} \leq C_1^h \|f\|_{\Omega^s} + C_2^h \|g\|_{\Omega^d}, \tag{4.46a}$$

$$\|\boldsymbol{p}_h\|_{P_h} \leq C_3^h \|f\|_{\Omega^s} + C_4^h \|g\|_{\Omega^d}. \tag{4.46b}$$

Furthermore, if

$$C_1^h \|f\|_{\Omega^s} + C_2^h \|g\|_{\Omega^d} \leq \mu c_{ae}^s \min\{(1-\delta)\theta c_w^{-1}, \frac{\delta}{c_{\mathrm{tr},2} C_K (c_{\mathrm{tr},4}^2 + c_{h,4}^2 C_K^2)}\}, \tag{4.47}$$

where $0 < \theta < 1$*, then the solution to eq.* (4.28) *is unique.*

Proof. Suppose that (4.45) holds. Following the approach in [61], we introduce the operator $\boldsymbol{T}_h : \boldsymbol{B}_h^s \mapsto \boldsymbol{B}_h^s$ such that: $\boldsymbol{T}_h(\boldsymbol{w}_h^s) = \boldsymbol{u}_h^s$ for all $\boldsymbol{w}_h^s = (w_h^s, \bar{w}_h^s) \in \boldsymbol{B}_h^s$. Here, $\boldsymbol{u}_h^s := (u_h|_{\Omega^s}, \bar{u}_h) \in \boldsymbol{V}_h^s$ where $(u_h, \bar{u}_h)$ is the first two components of the velocity solution $\boldsymbol{u}_h \in \boldsymbol{Z}_h$ of (4.29) corresponding to w_h^s. We will first prove that $\boldsymbol{T}_h$ is well-defined and bounded. *Well-definition of* $\boldsymbol{T}_h$*:*

For $\boldsymbol{w}_h^s \in \boldsymbol{B}_h^s$, by eq. (3.33d), we have that $\nabla \cdot w_h^s = 0$ in Ω^s, and by eq. (3.33a) and [48, Lemma 1.24], $w_h^s \in H(\mathrm{div}, \Omega^s)$. Furthermore, by the inequality [45, Proposition 4.5], the definition of $\bar{w}_h^s$, and the results of eqs. (3.33a) and (3.33b) the following inequality holds:

$$\|w_h^s \cdot n\|_{\Gamma^I} \leq c_{\mathrm{tr},2} \|w_h^s\|_{\Omega^s} \leq c_{\mathrm{tr},2} C_K \|\boldsymbol{w}_h^s\|_{V_h^s}.$$

Thus by eq. (4.45) we obtain

$$\left\|w_h^s \cdot n\right\|_{\Gamma^I} \leq \mu c_{ae}^s \delta (c_{h,4}^2 + c_{\mathrm{tr},4}^2 C_K^2)^{-1}.$$

Therefore by Theorem 53, eq. (4.29) has a unique solution $(\boldsymbol{u}_h, \boldsymbol{p}_h)$ satisfying eq. (4.43). This implies that $\boldsymbol{u}_h^s \in \boldsymbol{B}_h^s$ and concludes that the operator $\boldsymbol{T}_h$ is well-defined.

Boundedness of the operator $\boldsymbol{T}_h$:

Let $\boldsymbol{w}_{h1}^s, \boldsymbol{w}_{h2}^s \in \boldsymbol{B}_h^s$.

In what follows, we will bound $\boldsymbol{T}_h(\boldsymbol{w}_{h2}^s) - \boldsymbol{T}_h(\boldsymbol{w}_{h1}^s)$ using eq. (4.42). By definition of $\boldsymbol{T}_h$, there exists unique $(\boldsymbol{u}_{1h}, \boldsymbol{p}_{1h}), (\boldsymbol{u}_{2h}, \boldsymbol{p}_{2h}) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$ such that $\boldsymbol{T}_h(\boldsymbol{w}_{h1}^s) = u_{1h}^s$ and $\boldsymbol{T}_h(\boldsymbol{w}_{h2}^s) = u_{2h}^s$.

$$
\begin{aligned}
t_h(\boldsymbol{w}_{h1}^s; \boldsymbol{u}_{1h}, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_{1h}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{p}_{1h}) + b_h(\boldsymbol{u}_{1h}, \boldsymbol{q}_h) - c_h(\boldsymbol{p}_{1h}, \boldsymbol{q}_h) = \\
t_h(\boldsymbol{w}_{h2}^s; \boldsymbol{u}_{2h}, \boldsymbol{v}_h) + a_h(\boldsymbol{u}_{2h}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{p}_{2h}) + b_h(\boldsymbol{u}_{2h}, \boldsymbol{q}_h) - c_h(\boldsymbol{p}_{2h}, \boldsymbol{q}_h).
\end{aligned}
$$

We subtract the two equations above, and test with $\boldsymbol{v}_h = \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}$, and $\boldsymbol{q}_h = \boldsymbol{p}_{1h} - \boldsymbol{p}_{2h}$, to obtain:

$$
\begin{aligned}
t_h(\boldsymbol{w}_{h1}^s; \boldsymbol{u}_{1h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) - t_h(\boldsymbol{w}_{h2}^s; \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) \\
+ a_h(\boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) + c_h(\boldsymbol{p}_{2h} - \boldsymbol{p}_{1h}, \boldsymbol{p}_{2h} - \boldsymbol{p}_{1h}) = 0
\end{aligned}
$$

Following a similar approach as in the continuous case, we add and subtract $t_h(\boldsymbol{w}_{h1}^s; \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h})$, use eq. (4.41), and simplify using the semi-positiveness of $c(\cdot,\cdot)$:

$$
\begin{aligned}
&(1-\delta)\mu c_{ae}^s \|\boldsymbol{T}_h(\boldsymbol{w}_{h2}^s) - \boldsymbol{T}_h(\boldsymbol{w}_{h1}^s)\|_{V_h^s}^2 = (1-\delta)\mu c_{ae}^s \|\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}\|_{V_h^s}^2 = \\
&\le t_h(\boldsymbol{w}_{h1}^s; \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) + a_h(\boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) + c_h(\boldsymbol{p}_{2h} - \boldsymbol{p}_{1h}, \boldsymbol{p}_{2h} - \boldsymbol{p}_{1h}) \\
&\qquad = t_h(\boldsymbol{w}_{h2}^s; \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) - t_h(\boldsymbol{w}_{h1}^s; \boldsymbol{u}_{2h}, \boldsymbol{u}_{1h} - \boldsymbol{u}_{2h}) \\
&\qquad\qquad \le c_w \|\boldsymbol{w}_{h1}^s - \boldsymbol{w}_{h2}^s\|_{V_h^s} \|\boldsymbol{u}_{2h}\|_{V_h^s} \|\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}\|_{V_h^s},
\end{aligned}
$$

thus,

$$\|\boldsymbol{T}_h(\boldsymbol{w}^s_{h2}) - \boldsymbol{T}_h(\boldsymbol{w}^s_{h1})\|_{V^s_h} \leq \frac{c_w}{(1-\delta)\mu c^s_{ae}} \|\boldsymbol{w}^s_{h1} - \boldsymbol{w}^s_{h2}\|_{V^s_h} \|\boldsymbol{u}_{2h}\|_{V^s_h}$$
$$\leq \frac{c_w}{(1-\delta)\mu c^s_{ae}} (C^h_1 \|f^s\|_{\Omega^s} + C^h_2 \|g\|_{\Omega^d}) \|w_1 - w_2\|_{V^s_h}. \quad (4.48a)$$

Therefore the operator $\boldsymbol{T}_h$ is bounded on $\boldsymbol{B}^s_h$.

Since the $\boldsymbol{B}^s_h$ is a finite-dimensional space and $\boldsymbol{T}_h$ is well-defined and bounded, by theorem 15 we deduce that the operator T_h has at least one fixed point. Therefore, the discrete problem of the dual-porosity-Navier-Stokes given in Equation (4.28) has at least one solution.

The rest of the proof establishes the uniqueness of the solution. From inequality eq. (4.48), $\boldsymbol{T}_h$ is a contraction if $(C^h_1 \|f^s\|_{\Omega^s} + C^h_2 \|g\|_{\Omega^d}) \leq \frac{(1-\delta)\theta\mu c^s_{ae}}{c_w}$, where $0 < \theta < 1$. This is satisfied under the assumption (4.47) and therefore uniqueness follows by theorem 16. ■

4.4 A priori error analysis

We follow with the prior error analysis of the HDG method presented in Section 4.3. For this analysis, we will extend the proof in Section 3.4, by considering the non-linear term. To perform the analysis, we use the same notation, operators, and projections as in Section 3.4. The following lemma presents the error equations for the dual-porosity-Navier-Stokes model.

Lemma 55 (Error equations)**.** *For any $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{Z}_h \times \boldsymbol{P}_h$, the following holds:*

$$t_h(u; \boldsymbol{e}^h_u, \boldsymbol{v}_h) + a_h(\boldsymbol{e}^h_u, \boldsymbol{v}_h) + b_h(\boldsymbol{e}^h_p, \boldsymbol{v}_h) = t_h(u; \boldsymbol{e}^I_u, \boldsymbol{v}_h) + t_h(u; \boldsymbol{u}_h, \boldsymbol{v}_h) - t_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)$$
$$+ a^s_h(\boldsymbol{e}^I_u, \boldsymbol{v}_h) + a^d_h(e^I_u, v_h) + a^m_h(e^I_{u^m}, v^m_h), \quad (4.49a)$$

$$b_h(\boldsymbol{q}_h, \boldsymbol{e}^h_u) - c_h(\boldsymbol{e}^h_p, \boldsymbol{q}_h) = 0. \quad (4.49b)$$

Proof. Equation (4.49b) follows from (3.53b). As a result of Lemma 24 and (4.28), we obtain:

$$t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)+a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{p}_h,\boldsymbol{v}_h)=t_h(u;\boldsymbol{u},\boldsymbol{v}_h)+a_h(\boldsymbol{u},\boldsymbol{v}_h)+b_h(\boldsymbol{p},\boldsymbol{v}_h), \tag{4.50}$$

for all $\boldsymbol{v}_h\in\boldsymbol{Z}_h, \boldsymbol{q}_h\in\boldsymbol{P}_h$. Subtracting $t_h(u;\boldsymbol{\Pi}u,\boldsymbol{v}_h)-t_h(u;\boldsymbol{u}_h,\boldsymbol{v}_h)+a_h(\boldsymbol{\Pi}u,\boldsymbol{v}_h)+b_h(\boldsymbol{\Pi}p,\boldsymbol{v}_h)$ from both sides of the equation above, we arrive at the following equation.

$$\begin{aligned} t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)&+t_h(u;\boldsymbol{u}_h,\boldsymbol{v}_h)-t_h(u;\boldsymbol{\Pi}u,\boldsymbol{v}_h)+a_h(\boldsymbol{e}_u^h,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p^h,\boldsymbol{v}_h)\\ &=t_h(u;\boldsymbol{u},\boldsymbol{v}_h)+t_h(u;\boldsymbol{u}_h,\boldsymbol{v}_h)-t_h(u;\boldsymbol{\Pi}u,\boldsymbol{v}_h)+a_h(\boldsymbol{e}_u^I,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p^I,\boldsymbol{v}_h).\end{aligned}$$

Then, due to the linearity of $t_h(\cdot;\cdot,\cdot)$ with respect to the second component, we can group terms together and obtain:

$$\begin{aligned} t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)&+t_h(u;\boldsymbol{e}_u^h,\boldsymbol{v}_h)+a_h(\boldsymbol{e}_u^h,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p^h,\boldsymbol{v}_h)\\ &=t_h(u;\boldsymbol{e}_u^I,\boldsymbol{v}_h)+t_h(u;\boldsymbol{u}_h,\boldsymbol{v}_h)+a_h(\boldsymbol{e}_u^I,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p^I,\boldsymbol{v}_h).\end{aligned}$$

After subtracting the term $t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)$ from both sides and using Equation (3.53a) to simplify the right-hand side of the equation above, we obtain the result in Equation (4.49a). ∎

The following theorem gives the main result of this section.

Theorem 56. *Let* $(\boldsymbol{u},\boldsymbol{p})\in\boldsymbol{Z}\times\boldsymbol{P}$ *be the solution of the dual-porosity-Navier-Stokes problem eqs.* (3.2)*,* (3.3) *and* (4.1)*, such that* $u^s\in[H^{k+1}(\Omega^s)]^{\dim}$*,* $u^d\in[H^k(\Omega^d)]^{\dim}$*,* $u^m\in[H^k(\Omega^d)]^{\dim}$*, and let* $(\boldsymbol{u}_h,\boldsymbol{p}_h)\in\boldsymbol{Z}_h\times\boldsymbol{P}_h$ *be the solution of the corresponding HDG method given in eq.* (4.28)*, where the source functions* f *and* g *satisfy the conditions* *below:*

$$C_1\,\|f\|_{L^2(\Omega^s)} + C_2\,\|g\|_{L^2(\Omega^d)} \le 2\mu \min\Big\{(1-\delta)\theta\tau_s^{-1}, \frac{2\delta}{C_{\mathrm{tr},2}C_{\mathrm{tr},4}^2 C_K^3}\Big\},$$

$$C_1^h\|f\|_{\Omega^s} + C_2^h\|g\|_{\Omega^d} \le \mu c_{ae}^s \min\{(1-\delta)\theta c_w^{-1}, \frac{\delta}{c_{\mathrm{tr},2}C_K(c_{\mathrm{tr},4}^2 + c_{h,4}^2 C_K^2)}\},$$

where, $0<\delta,\theta<1$*, and the other constants are defined in* (4.23) *and* (4.47)*. Then the following inequalities hold:*

$$(\|\boldsymbol{e}_u^h\|_{V_h^s}^2 + \|e_u^h\|_{\Omega^d}^2 + \|e_{u^m}^h\|_{\Omega^d}^2)^{1/2} \le C C_{ner} h^k\Big(\|u\|_{k+1,\Omega^s} + \|u\|_{k+1,\Omega^d} + \|u^m\|_{k+1,\Omega^d}\Big), \tag{4.52a}$$

$$\|\boldsymbol{e}_p^h\|_{P_h} \le C\beta_p^{-1} C_{per} h^k\Big(\|u\|_{k+1,\Omega^s} + \|u\|_{k+1,\Omega^d} + \|u^m\|_{k+1,\Omega^d}\Big), \tag{4.52b}$$

where $C>0$ *is a generic constant independent of* h,σ,μ,κ_f*, and* κ_m*,*
$C_{ner} = \dfrac{\max\Big(2\mu(1-\delta)\tau_s^{-1}c_w\theta, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}, \mu c_{ae}^s(1-\delta)\theta\Big)}{\max\Big(\mu c_{ae}^s(1-\delta)(1-\theta), \kappa_f^{-1}, \kappa_m^{-1}\Big)}$,
$C_{per} = \max\Big(CC_{nb}C_{ner}, \mu c_{ae}^s(1-\delta)\theta, 2\mu(1-\delta)c_w\theta\tau_s^{-1}, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}\Big)$*,* c_{bb} *is the inf-sup constant in Theorem 47,* $C_c^{s,*}$ *is the new boundedness constant in corollary 27.*

Proof. The proof is similar to the proof of Theorem 34 except for the need to handle the nonlinear term. First, we test eq. (4.49) with $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{e}_u^h, -\boldsymbol{e}_p^h)$. Then, using Equation (4.41) and the positive semi-definiteness of c_h, we obtain:

$$\begin{aligned}
&\mu c_{ae}^s(1-\delta)\,\|\boldsymbol{e}_u^h\|_{V_h^s}^2 + \kappa_f^{-1}\|e_u^h\|_{\Omega^d}^2 + \kappa_m^{-1}\|e_{u^m}^h\|_{\Omega^d}^2 \\
&\le t_h(u;\boldsymbol{e}_u^h,\boldsymbol{e}_u^h) + a_h(\boldsymbol{e}_u^h,\boldsymbol{e}_u^h) + c_h(\boldsymbol{e}_p^h,\boldsymbol{e}_p^h) \\
=&\Big(t_h(u;\boldsymbol{e}_u^I,\boldsymbol{e}_u^h) + a_h^s(\boldsymbol{e}_u^I,\boldsymbol{e}_u^h) + a_h^d(e_u^I,e_u^h) + a_h^m(e_{u^m}^I,e_{u^m}^h)\Big) \\
&+\Big(t_h(u;\boldsymbol{u}_h,\boldsymbol{e}_u^h) - t_h(u_h;\boldsymbol{u}_h,\boldsymbol{e}_u^h)\Big) = I_1 + I_2.
\end{aligned} \tag{4.53}$$

By using (4.35), (3.56), and (4.36) we bound I_1:

$$
\begin{aligned}
I_1 =& t_h(u;\boldsymbol{e}_u^I,\boldsymbol{e}_u^h)+a_h^s(\boldsymbol{e}_u^I,\boldsymbol{e}_u^h)+a_h^d(e_u^I,e_u^h)+a_h^m(e_{u^m}^I,e_{u^m}^h)\\
&\le \max(c_w\|u\|_{1,h,\Omega^s},C_c^{s,*},\kappa_f^{-1},\kappa_m^{-1})(\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}^2+\|e_u^I\|_{\Omega^d}^2+\|e_{u^m}^I\|_{\Omega^d}^2)^{1/2}\\
&\quad\times(\|\boldsymbol{e}_u^h\|_{V_h^s}^2+\|e_u^h\|_{\Omega^d}^2+\|e_{u^m}^h\|_{\Omega^d}^2)^{1/2}.
\end{aligned}
$$

Since $u^s\in H^{k+1}(\Omega^s)$ then $\|u^s\|_{1,h,\Omega^s}=\|\varepsilon(u^s)\|_{\Omega^s}$, then using the stability of the dual-porosity-Navier-Stokes problem stated in (4.14a) we obtain

$$
\begin{aligned}
I_1 \le& \max(2\mu(1-\delta)\tau_s^{-1}c_w\theta,C_c^{s,*},\kappa_f^{-1},\kappa_m^{-1})(\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}^2+\|e_u^I\|_{\Omega^d}^2+\|e_{u^m}^I\|_{\Omega^d}^2)^{1/2}\\
&\times(\|\boldsymbol{e}_u^h\|_{V_h^s}^2+\|e_u^h\|_{\Omega^d}^2+\|e_{u^m}^h\|_{\Omega^d}^2)^{1/2}.
\end{aligned}
\tag{4.54}
$$

To bound the second term we first use Lemma 50, then eq. (4.36), apply the triangle inequality, and then use the stability results in eq. (4.46a) and the condition on the data to obtain:

$$
\begin{aligned}
I_2 &\le |t_h(u;\boldsymbol{u}_h,\boldsymbol{e}_u^h)-t_h(u_h;\boldsymbol{u}_h,\boldsymbol{e}_u^h)|\\
&\le c_w\,\|u-u_h\|_{1,h,\Omega^s}\,\|\boldsymbol{u}_h\|_{V_h^s}\,\|\boldsymbol{e}_u^h\|_{V_h^s}\le c_w\,\|u-u_h\|_{V_h^s}\,\|\boldsymbol{u}_h\|_{V_h^s}\,\|\boldsymbol{e}_u^h\|_{V_h^s}\\
&\le c_w(\|\boldsymbol{e}_u^h\|_{V_h^s}+\|\boldsymbol{e}_u^I\|_{V_h^s})\,\|\boldsymbol{u}_h\|_{V_h^s}\,\|\boldsymbol{e}_u^h\|_{V_h^s}\\
&\le c_w(C_1^h\|f\|_{\Omega^s}+C_2^h\|g\|_{\Omega^d})(\|\boldsymbol{e}_u^h\|_{V_h^s}^2+\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}\,\|\boldsymbol{e}_u^h\|_{V_h^s})\\
&\le \mu c_{ae}^s(1-\delta)\theta(\|\boldsymbol{e}_u^h\|_{V_h^s}^2+\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}\,\|\boldsymbol{e}_u^h\|_{V_h^s})\\
&\le \mu c_{ae}^s(1-\delta)\theta\,\|\boldsymbol{e}_u^h\|_{V_h^s}^2+\mu c_{ae}^s(1-\delta)\theta\|\boldsymbol{e}_u^I\|_{V_h^{s,*}}\|\boldsymbol{e}_u^h\|_{V_h^s}.
\end{aligned}
\tag{4.55}
$$

Combining eqs. (4.53) to (4.55), we obtain

$$
\begin{aligned}
&\mu c_{ae}^{s}(1-\delta)\,\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \kappa_f^{-1}\|e_u^h\|^2_{\Omega^d} + \kappa_m^{-1}\|e_{u^m}^h\|^2_{\Omega^d} - \mu c_{ae}^{s}(1-\delta)\theta\,\|\boldsymbol{e}_u^h\|^2_{V_h^s} \\
&\le \max\Big(2\mu(1-\delta)\tau_s^{-1}c_w\theta, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}, \mu c_{ae}^{s}(1-\delta)\theta\Big) \qquad (4.56)\\
&\qquad \times (\|\boldsymbol{e}_u^I\|^2_{V_h^{s,*}} + \|e_u^I\|^2_{\Omega^d} + \|e_{u^m}^I\|^2_{\Omega^d})^{1/2}(\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d})^{1/2}.
\end{aligned}
$$

Since $0<\theta<1$, $\mu c_{ae}^{s}(1-\delta) - \mu c_{ae}^{s}(1-\delta)\theta > 0$. Thus, letting $C_{ner} = \dfrac{\max\Big(2\mu(1-\delta)\tau_s^{-1}c_w\theta, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}, \mu c_{ae}^{s}(1-\delta)\theta\Big)}{\min\Big(\mu c_{ae}^{s}(1-\delta)(1-\theta), \kappa_f^{-1}, \kappa_m^{-1}\Big)}$

and using the interpolation estimate [62, Lemma 7] for the velocity in Ω^s, and the BDM interpolation estimates given in Lemma 28 for the velocities in microfractures and in the matrix we get the following estimate:

$$
\begin{aligned}
(\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d})^{1/2} &\le C_{ner}(\|\boldsymbol{e}_u^I\|^2_{V_h^{s,*}} + \|e_u^I\|^2_{\Omega^d} + \|e_{u^m}^I\|^2_{\Omega^d})^{1/2} \\
&\le CC_{ner}h^k\Big(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}\Big).
\end{aligned}
\tag{4.57}
$$

The next step is to prove eq. (4.52b). Using eq. (4.49a), Lemma 26, and the result in eq. (3.56), and eq. (4.55) we obtain:

$$
\begin{aligned}
|b_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h)| &\le |t_h(u;\boldsymbol{e}_u^h,\boldsymbol{v}_h) + a_h(\boldsymbol{e}_u^h,\boldsymbol{v}_h)| + |t_h(u;\boldsymbol{u}_h,\boldsymbol{v}_h) - t_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)| \\
&\quad + |t_h(u;\boldsymbol{e}_u^I,\boldsymbol{v}_h) + a_h^s(\boldsymbol{e}_u^I,\boldsymbol{v}_h^s) + a_h^d(e_u^I,v_h^d) + a_h^m(e_{u^m}^I,v_h^m)| = I_3 + I_4 + I_5.
\end{aligned}
$$

The terms I_4, and I_5 are bounded using identical steps to eqs. (4.54) and (4.55) just replacing $\boldsymbol{e}_u^h$ with $\boldsymbol{v}_h$:

$$
\begin{aligned}
I_4 + I_5 \leq \max &\left(\mu c_{ae}^s(1-\delta)\theta, 2\mu(1-\delta)c_w\theta\tau_s^{-1}, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}\right) \\
&\times (\|\boldsymbol{e}_u^I\|^2_{V_h^{s,*}} + \|e_u^I\|^2_{\Omega^d} + \|e_{u^m}^I\|^2_{\Omega^d})^{1/2}(\|\boldsymbol{v}_h\|^2_{V_h^s} + \|v_h\|^2_{\Omega^d} + \|v_h^m\|^2_{\Omega^d})^{1/2} \\
&\leq C\max\left(\mu c_{ae}^s(1-\delta)\theta, 2\mu(1-\delta)c_w\theta\tau_s^{-1}, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}\right) h^k \\
&\qquad \times \left(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}\right) \|\boldsymbol{v}_h\|_{\boldsymbol{Z}_h^{\#}}.
\end{aligned}
$$

Now, we bound I_3. Using the continuity result in eqs. (3.15a) and (4.37), and (4.56) we get the following

$$
\begin{aligned}
I_3 &= |t_h(u;\boldsymbol{e}_u^h, \boldsymbol{v}_h) + a_h(\boldsymbol{e}_u^h, \boldsymbol{v}_h)| \ \leq \\
&\leq C_{nb}(\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d} + \|(\bar{e}_u^h)^t\|^2_{\Gamma^I})^{1/2} \\
&\leq CC_{nb}(\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d})^{1/2} \\
&\leq CC_{nb}C_{ner}h^k\left(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}\right)\|\boldsymbol{v}_h\|_{Z_h^{\#}}.
\end{aligned}
$$

Any two norms in the finite dimensional spaces are equivalent, thus in $\boldsymbol{Z}_h$, C is the equivalence constant between two norms: $\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d} + \|(\bar{e}_u^h)^t\|^2_{\Gamma^I})^{1/2} \leq C(\|\boldsymbol{e}_u^h\|^2_{V_h^s} + \|e_u^h\|^2_{\Omega^d} + \|e_{u^m}^h\|^2_{\Omega^d})^{1/2}$. Combining the above inequalities with the inf-sup condition in Theorem 47, Lemma 28, eq. (3.41), and eq. (3.54a) we obtain:

$$
\begin{aligned}
\|\boldsymbol{e}_p^h\|_{P_h} &\leq c_{bb}^{-1} \sup_{\boldsymbol{v}_h \in \boldsymbol{Z}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{\boldsymbol{b}_h(\boldsymbol{e}_p^h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{Z}_h^{\#}}} \\
&\leq Cc_{bb}^{-1}C_{per}h^k\left(\|u\|_{k+1,\Omega^s} + \|u\|_{k,\Omega^d} + \|u^m\|_{k,\Omega^d}\right),
\end{aligned}
$$

where $C_{per} = \max\left(CC_{nb}C_{ner}, \mu c_{ae}^s(1-\delta)\theta, 2\mu(1-\delta)c_w\theta\tau_s^{-1}, C_c^{s,*}, \kappa_f^{-1}, \kappa_m^{-1}\right)$. Hence, the result is proved. ■

By applying the triangle inequality and following identical steps as in Corollary 35, we arrive at the following a priori error estimates.

Corollary 57. *Under the assumptions of Theorem 56,*

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{V_h^s}+&\|u-u_h\|_{\Omega^d}+\|u^m-u_h^m\|_{\Omega^d} &\text{(4.58a)}\\
&\leq C(C_{ner}+1)h^k\left(\|u\|_{k+1,\Omega^s}+\|u\|_{k,\Omega^d}+\|u^m\|_{k,\Omega^d}\right),\\
\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{P}_h}\leq& Ch^k(\|p\|_{k,\Omega^s}+\|p\|_{k,\Omega^d}+\|p^m\|_{k,\Omega^d}) &\text{(4.58b)}\\
&+Cc_{bb}^{-1}C_{per}h^k(\|u\|_{k+1,\Omega^s}+\|u\|_{k,\Omega^d}+\|u^m\|_{k,\Omega^d}),
\end{aligned}$$

where $\boldsymbol{u}:=(u,u^s|_{\Gamma_0^s},u^m)$*,* $\boldsymbol{p}:=(p,p^s|_{\Gamma_0^s},p^d|_{\Gamma_0^d},p^m,p^m|_{\Gamma_0^m})$*, and* C *is a constant which depends on the shape regularity of the mesh and the polynomial degree* k *but is independent of the mesh size* h *and the problem parameters* σ,μ,κ_f*, and* κ_m*.*

■

4.5 Numerical examples

The NGSolve library [53, 54] is used in the following simulations where the mesh is unstructured and the penalty parameter is chosen as $\beta=10k^2$ as in Section 3.5.

4.5.1 Example 1

In order to validate our theoretical results, we test the HDG method defined by eq. (4.28) on the manufactured solution described in Section 3.5.1. We have chosen the following parameters for the experiment: $\kappa_m=1$, $\kappa_f=1$, $\sigma=0.9$, $\mu=1$, and $\alpha=\mu\sqrt{\kappa_f}(1+4\pi^2)/2$ and present the result for $k=2,3,4$ in Tables 4.1 to 4.3.

Note that since our problem is nonlinear we use Picard iterations with tolerance $\varepsilon=10^{-12}$. Table 4.1 presents the L^2-errors and convergence rates for u and p in Ω^s and Tables 4.2 and 4.3 presents the L^2-errors and convergence rates for u and p in Ω^d, and u^m and p^m in Ω^d together with a column demonstrating mass conservation. We observe that the convergence rates are at least k or higher, thereby supporting Corollary 35 and eqs. (3.33d) to (3.33f). Note that the tables actually show that $\|u_h - u\|_{\Omega^s}$, $\|u_h - u\|_{\Omega^d}$, and $\|u_h^m - u^m\|_{\Omega^d}$ converge with an asymptotic rate of convergence of $k+1$, even though this is not shown by our error analysis.

In the Navier-Stokes region in Ω^s

dof	Cells	$\|e_u\|_{\Omega^s}$	Rate	$\|e_p\|_{\Omega^s}$	Rate	$\|\nabla e_u\|_{\Omega^s}$	Rate	$\|\nabla \cdot u_h\|_{\Omega^s}$
$k=2$								
3.3e+02	8	4.4e-03	-	1.6e-01	-	7.4e-02	-	8.2e-16
1.1e+03	28	7.4e-04	2.6	3.8e-02	2.1	2.0e-02	1.9	7.2e-16
5.5e+03	152	5.2e-05	3.8	7.7e-03	2.3	3.8e-03	2.4	2.0e-15
2.0e+04	570	4.9e-06	3.4	1.9e-03	2.0	8.2e-04	2.2	3.6e-15
8.3e+04	2346	5.7e-07	3.1	4.9e-04	2.0	2.0e-04	2.0	7.1e-15
3.4e+05	9520	6.8e-08	3.1	1.3e-04	2.0	4.9e-05	2.0	1.3e-14
$k=3$								
5.0e+02	8	4.4e-04	-	4.0e-02	-	9.9e-03	-	4.6e-15
1.7e+03	28	6.1e-05	2.8	6.8e-03	2.5	2.1e-03	2.2	1.1e-14
8.6e+03	152	1.4e-06	5.4	3.1e-04	4.4	1.3e-04	4.0	2.4e-14
3.2e+04	570	7.5e-08	4.3	3.1e-05	3.3	1.5e-05	3.2	4.3e-14
1.3e+05	2346	3.8e-09	4.3	3.4e-06	3.2	1.6e-06	3.2	8.6e-14
5.3e+05	9520	2.2e-10	4.1	4.1e-07	3.0	1.9e-07	3.1	1.7e-13
$k=4$								
7.2e+02	8	3.5e-05	-	2.8e-03	-	7.8e-04	-	5.0e-17
2.4e+03	28	2.2e-06	4.0	1.7e-04	4.0	7.5e-05	3.4	7.3e-17
1.2e+04	152	2.6e-08	6.4	5.4e-06	5.0	2.3e-06	5.0	6.3e-17
4.6e+04	570	6.9e-10	5.2	3.0e-07	4.2	1.2e-07	4.2	5.9e-17
1.9e+05	2346	2.0e-11	5.1	1.8e-08	4.1	7.2e-09	4.1	5.8e-17
7.7e+05	9520	6.8e-13	4.9	1.1e-09	4.0	4.5e-10	4.0	5.9e-17

Table 4.1: Errors and convergence rates in Ω^s for velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$, as set up in Section 4.5.1.

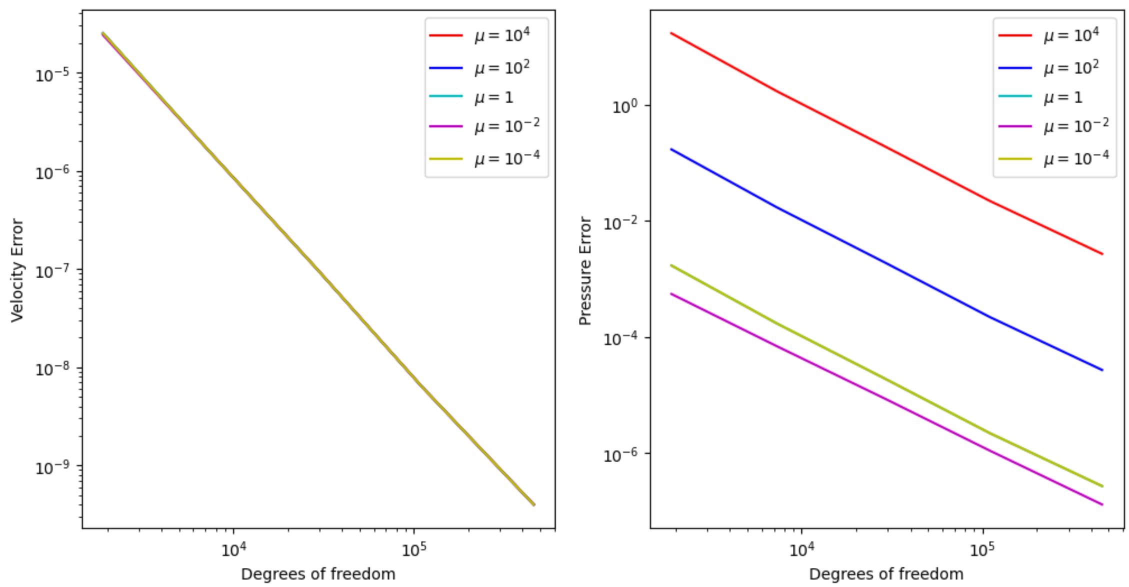
Next, as in Section 3.5.1, we explore the effect of changing various parameters of the problem in Figure 4.1-Figure 4.3. First, we consider the viscosity μ that appears in the Navier-Stokes equations and the interface conditions, but does not appear directly in the

In microfractures in Ω^d

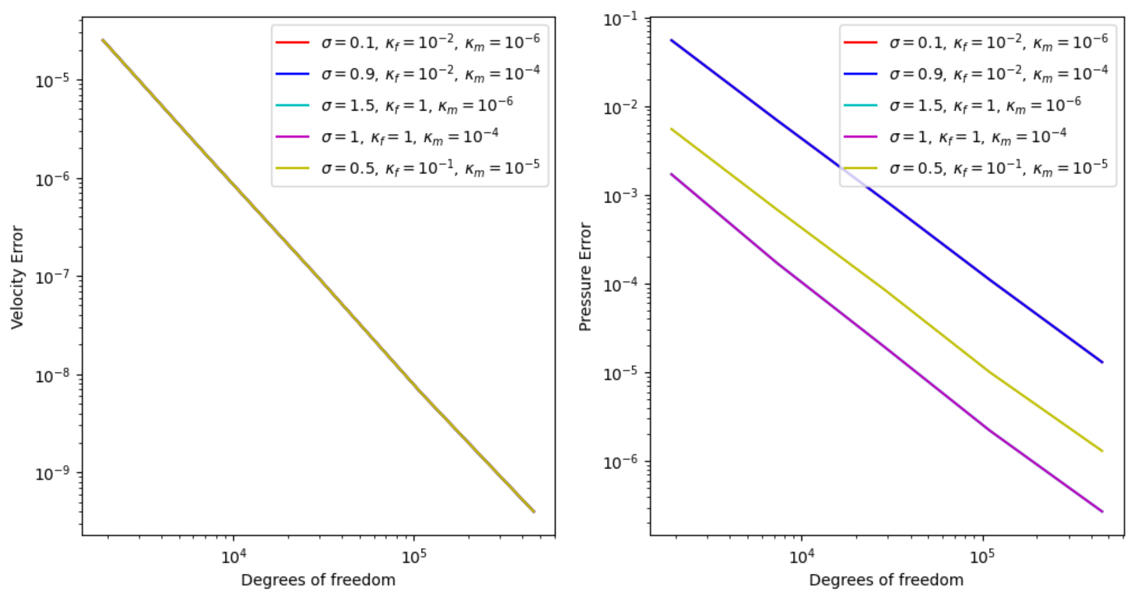
dof	Cells	$\|e_u\|_{\Omega^d}$	Rate	$\|e_p\|_{\Omega^d}$	Rate	$\|\nabla\cdot e_u\|_{\Omega^d}$	Rate	$\|\Phi\|_{\Omega^d}$
$k=2$								
3.3e+02	8	1.6e-02	-	2.6e-02	-	2.5e-01	-	8.8e-06
1.1e+03	28	3.0e-03	2.5	8.7e-03	1.6	8.4e-02	1.6	8.1e-08
5.5e+03	152	1.9e-04	4.0	1.3e-03	2.7	1.3e-02	2.7	9.6e-10
2.0e+04	570	2.4e-05	3.0	3.5e-04	1.9	3.4e-03	1.9	1.2e-11
8.3e+04	2346	2.5e-06	3.2	8.0e-05	2.1	7.7e-04	2.1	3.9e-12
3.4e+05	9520	3.0e-07	3.1	1.9e-05	2.0	1.9e-04	2.0	1.6e-11
$k=3$								
5.0e+02	8	1.5e-03	-	3.4e-03	-	3.2e-02	-	1.2e-08
1.7e+03	28	1.2e-04	3.7	5.1e-04	2.7	4.9e-03	2.7	1.4e-11
8.6e+03	152	3.3e-06	5.1	3.7e-05	3.8	3.5e-04	3.8	8.1e-13
3.2e+04	570	2.1e-07	3.9	4.9e-06	2.9	4.7e-05	2.9	3.0e-12
1.3e+05	2346	1.2e-08	4.1	5.8e-07	3.1	5.6e-06	3.1	1.2e-11
5.3e+05	9520	7.1e-10	4.1	7.0e-08	3.0	6.8e-07	3.0	4.8e-11
$k=4$								
7.2e+02	8	1.2e-04	-	3.3e-04	-	3.2e-03	-	1.8e-11
2.4e+03	28	7.8e-06	4.0	4.2e-05	3.0	4.0e-04	3.0	1.6e-15
1.2e+04	152	8.9e-08	6.5	1.1e-06	5.3	1.0e-05	5.3	1.3e-15
4.6e+04	570	2.8e-09	5.0	6.8e-08	3.9	6.6e-07	3.9	1.3e-15
1.9e+05	2346	6.1e-11	5.5	3.3e-09	4.4	3.2e-08	4.4	1.3e-15
7.7e+05	9520	1.8e-12	5.1	1.9e-10	4.1	1.9e-09	4.1	1.3e-15

Table 4.2: Errors and rates of convergence in microfractures in Ω^d, for the velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$ as set up in Section 4.5.1.

dual-porosity equations. When we increase the value of μ, we notice that the pressure error in the Navier-Stokes region increases accordingly, whereas the pressure errors in the dual-porosity region remain unaffected. However, there is no change in the velocity error, which is consistent with the scheme's pressure robustness. For the parameters that appear in the dual-porosity equations but not directly in the Navier-Stokes equations, we found that the magnitude of the pressure errors changed in both subdomains. Although some parameter choices led to changes in pressure errors, we did not observe corresponding changes in velocity errors. It aligns with our method's pressure robustness property.

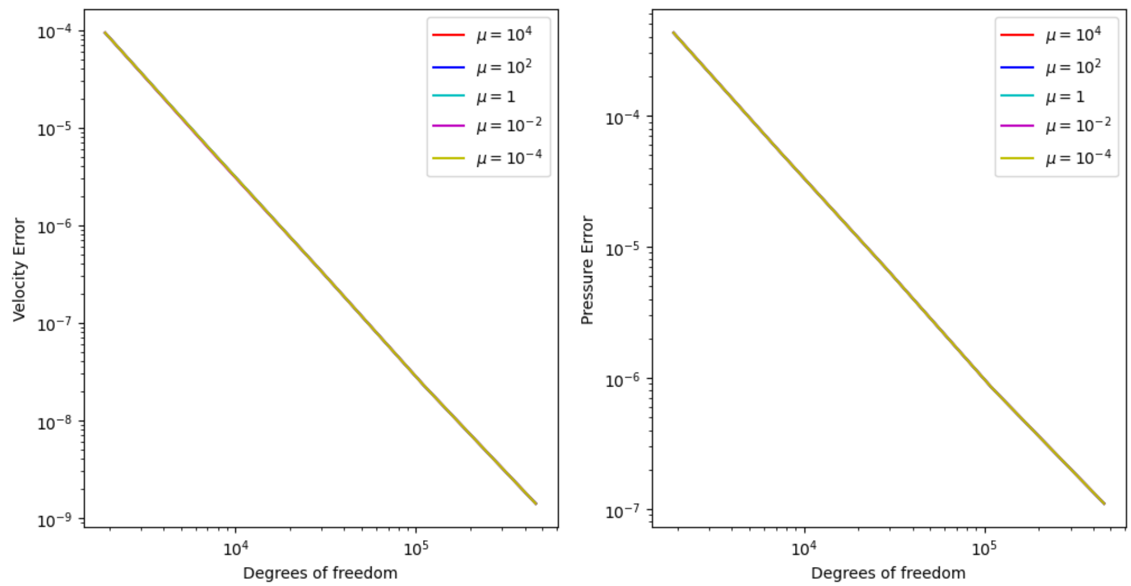


(a) Left: Error of velocity in Navier-Stokes domain, right: Error of Pressure in Navier-Stokes domain by changing only the value of μ.

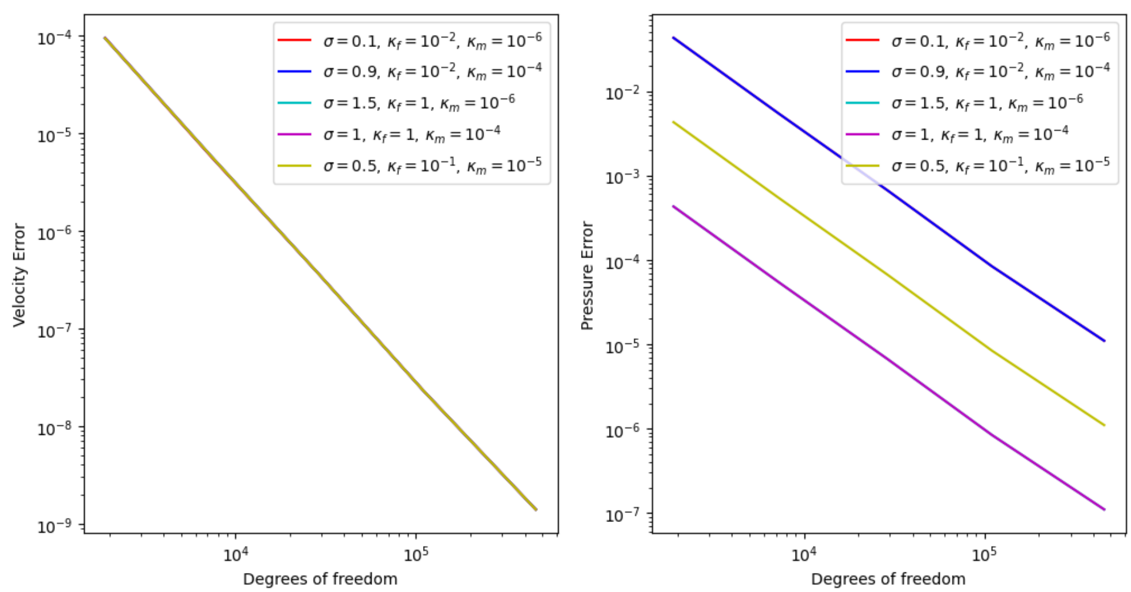


(b) Left: Error of velocity in Navier-Stokes domain, right: Error of Pressure in Navier-Stokes domain, changing the parameters in the graph.

Figure 4.1: The error of velocity and pressure in Navier-Stokes region for different parameters, as described in the graphs.

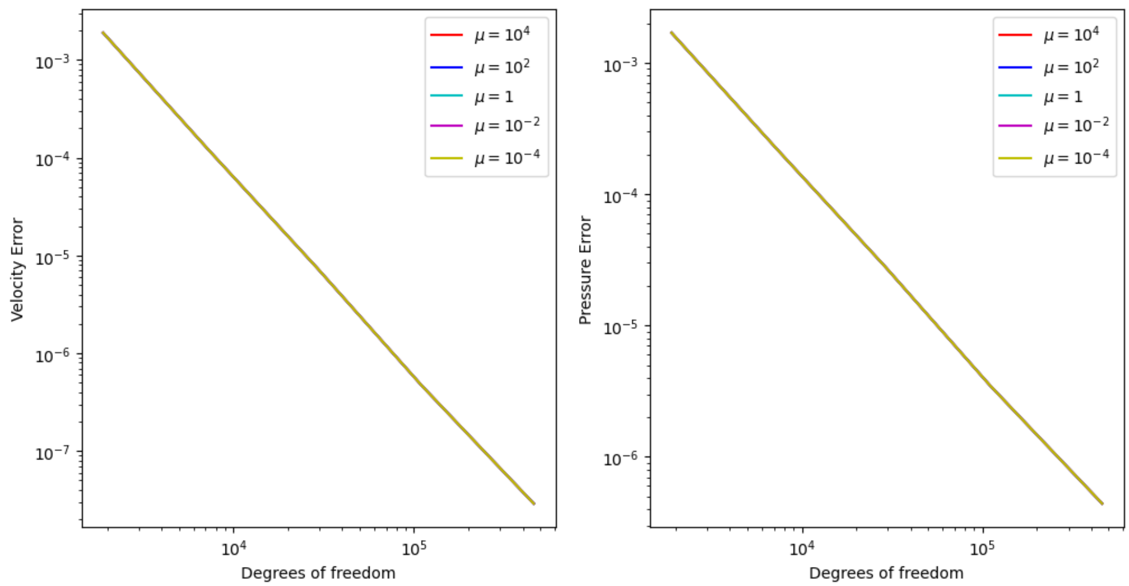


(a) Left: Error of velocity in microfracture, right: Error of Pressure in microfracture by changing only the value of μ.

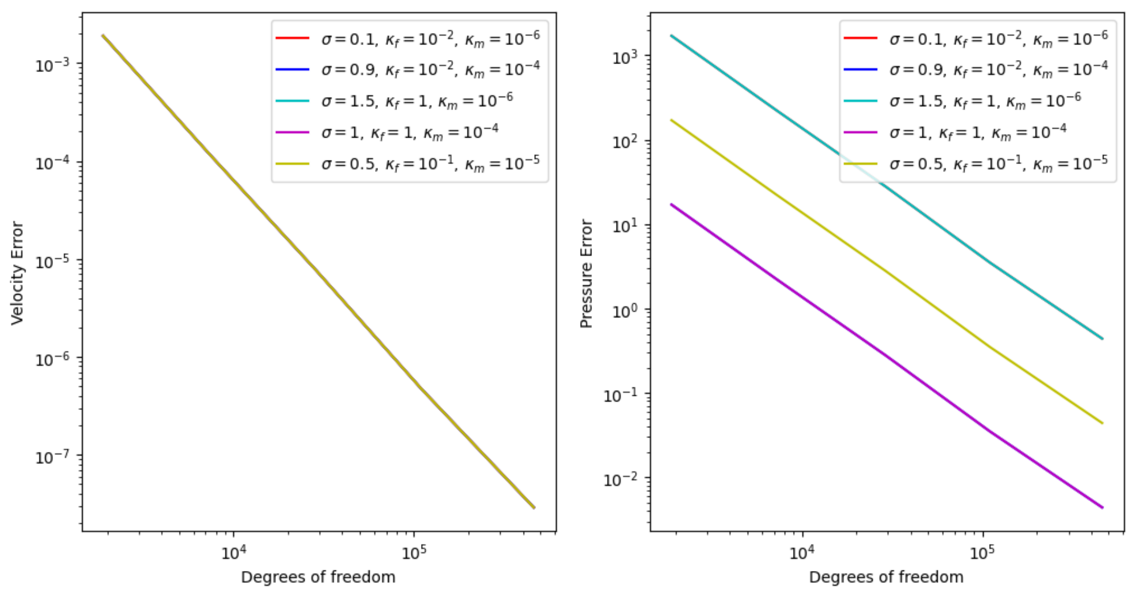


(b) Left: Error of velocity in microfracture, right: Error of Pressure in microfracture, changing the parameters in the graph.

Figure 4.2: The error of velocity and pressure in microfracture for different parameters, as described in the graphs.



(a) Left: Error of velocity in matrix, right: Error of pressure in matrix by changing only the value of μ.



(b) Left: Error of velocity in matrix, right: Error of pressure in matrix, changing the parameters in the graph.

Figure 4.3: The error of velocity and pressure in matrix for different parameters, as described in the graphs.

In the matrix in Ω^d								
dof	Cells	$\|e_u^m\|_{\Omega^d}$	Rate	$\|e_p^m\|_{\Omega^d}$	Rate	$\|\nabla \cdot e_u^m\|_{\Omega^d}$	Rate	$\|\Phi^m\|_{\Omega^d}$
$k=2$								
3.3e+02	8	9.7e-02	-	2.8e-02	-	2.5e-01	-	8.9e-04
1.1e+03	28	1.7e-02	2.5	1.1e-02	1.3	8.4e-02	1.6	7.1e-06
5.5e+03	152	1.7e-03	3.3	2.3e-03	2.3	1.3e-02	2.7	8.8e-08
2.0e+04	570	1.8e-04	3.2	5.4e-04	2.1	3.4e-03	1.9	9.8e-10
8.3e+04	2346	2.0e-05	3.2	1.3e-04	2.1	7.7e-04	2.1	9.7e-12
3.4e+05	9520	2.3e-06	3.1	3.0e-05	2.1	1.9e-04	2.0	5.3e-12
$k=3$								
5.0e+02	8	3.1e-02	-	1.3e-02	-	3.2e-02	-	1.2e-06
1.7e+03	28	1.3e-03	4.6	1.3e-03	3.4	4.9e-03	2.7	1.4e-09
8.6e+03	152	6.6e-05	4.3	1.4e-04	3.2	3.5e-04	3.8	8.0e-13
3.2e+04	570	4.5e-06	3.9	1.9e-05	2.9	4.7e-05	2.9	1.1e-12
1.3e+05	2346	2.2e-07	4.3	2.1e-06	3.2	5.6e-06	3.1	4.8e-12
5.3e+05	9520	1.3e-08	4.1	2.4e-07	3.1	6.8e-07	3.0	1.8e-11
$k=4$								
7.2e+02	8	5.8e-03	-	3.0e-03	-	3.2e-03	-	9.9e-10
2.4e+03	28	2.2e-04	4.7	2.4e-04	3.7	4.0e-04	3.0	6.0e-14
1.2e+04	152	5.2e-06	5.4	1.1e-05	4.4	1.0e-05	5.3	2.3e-15
4.6e+04	570	1.2e-07	5.5	5.4e-07	4.4	6.6e-07	3.9	2.1e-15
1.9e+05	2346	3.1e-09	5.3	3.0e-08	4.2	3.2e-08	4.4	2.1e-15
7.7e+05	9520	7.2e-11	5.4	1.5e-09	4.3	1.9e-09	4.1	2.2e-15

Table 4.3: Errors and rates of convergence in the matrix in Ω^d, for the velocity and pressure fields using polynomial degrees $k=2$, $k=3$ and $k=4$ as set up in Section 4.5.1.

4.5.2 Example 2

In this example, we simulate the flow in a horizontal open-hole wellbore with a vertical production wellbore. The set up of this example is from [32, Example 3]. Horizontal open-hole wellbores with a vertical production wellbore are commonly used for extracting oil and gas from naturally fractured reservoirs [63]. The subdomains, their boundary and the interface are given by:

$$\Omega^s = \{(x,y) : 1.8 < x < 2.2,\ 1.8 < y < 6\}$$

$$\Gamma^s = \{(x,y) : x = 1.8, 2 \leq y \leq 6\} \cup \{(x,y) : x = 2.2, 2 \leq y \leq 6\}$$
$$\cup \{(x,y) : 1.8 \leq x \leq 2.2,\ y = 1.8\}$$

$$\Omega^d = \{(x,y) : 0 < x < 1.8,\ 0 < y < 2\} \cup \{(x,y) : 1.8 < x < 2.2,\ 0 < y < 1.8\}$$
$$\cup \{(x,y) : 2.2 < x < 4,\ 0 < y < 2\}$$

$$\Gamma^d = \{(x,y) :\ 0 \leq x \leq 4,\ y = 0\} \cup \{(x,y) :\ x = 4,\ 0 \leq y \leq 2\}$$
$$\cup \{(x,y)P :\ 2.2 \leq x \leq 4,\ y = 2\} \cup \{(x,y) :\ 0 \leq x \leq 1.8,\ y = 2\} \cup \{(x,y) : x = 0,\ 0 \leq y \leq 2\},$$

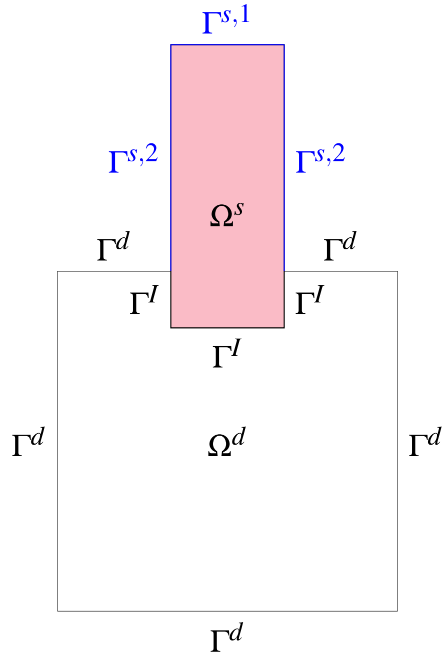
$$\Gamma^I = \{(x,y) : x = 1.8, 1.8 \leq y \leq 2\} \cup \{(x,y) : x = 2.2, 1.8 \leq y \leq 2\}$$
$$\cup \{(x,y) : 1.8 \leq x \leq 2.2,\ y = 1.8\}.$$

On the top boundary of the vertical wellbore, $\Gamma^{s1} = \{(x,y) : 1.8 \leq x \leq 2.2, y = 6\}$, we impose an outflow boundary condition $(-p^s I + \mu\varepsilon(u^s))n = 0$, and on $\Gamma^{s2} = \Gamma^s \backslash \Gamma^{s1}$, we impose $u \cdot n = 0$. The Dirichlet boundary conditions are imposed on Γ^d for the pressures, such that $p_f = 10^4$ and $p_m = 5 \times 10^4$. See Figure 4.4 for a sketch of the domain.

The model parameters used in the simulation are: $k_m = 10^{-8}$, $k_f = 10^{-3}$, $\mu = 1$, and $\sigma = 0.9$, $\alpha = 1$, and $\beta = 10^{-7}$. We choose the step size $h = 1/32$.

As shown in Figures 4.5a and 4.5b, the pressure in matrix is higher than the pressure in microfracture. This is expected as different microfracture and matrix pressures are imposed on the external boundary Γ^d. In the vertical well, the fluid pressure drops towards the upper boundary $\Gamma^{s,1}$ and the fluid flows much faster as seen in Figures 4.5c and 4.5d. We also see in Figure 4.5d that there is no fluid flow from the matrix to the vertical well as the model requires.

Figure 4.4: A plot of the domain used in Section 4.5.2.

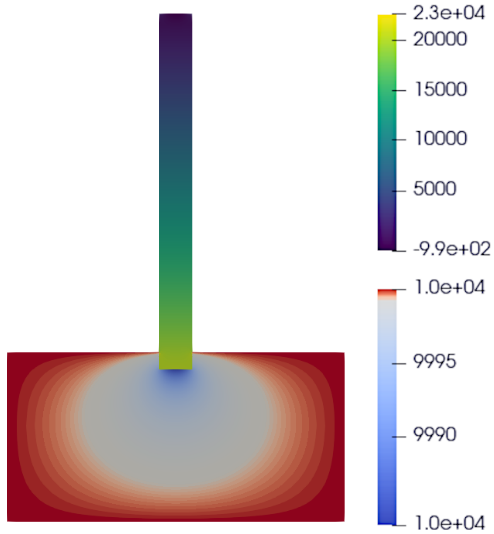


Furthermore, we demonstrate that the numerical scheme preserves mass conservation by calculating the following errors:

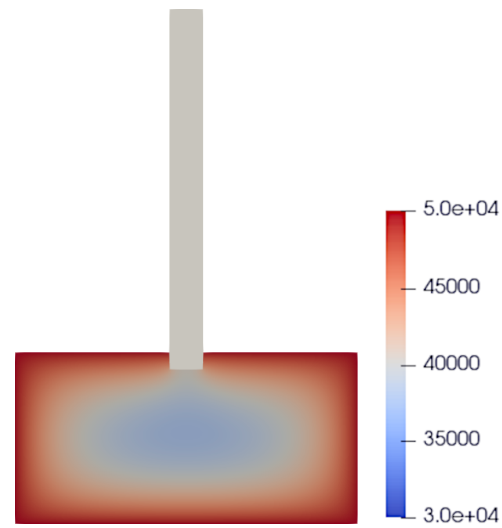
$$\begin{aligned}
\|\nabla \cdot u_h^s\|_{\Omega^s} &= 3.4 \cdot 10^{-14}, \\
\|\sigma \kappa_m (p_h^d - p_h^m) + \nabla \cdot u_h^d\|_{\Omega^d} &= 9.9 \cdot 10^{-10}, \\
\|\sigma \kappa_m (p_h^m - p_h^d) + \nabla \cdot u_h^m\|_{\Omega^d} &= 4.4 \cdot 10^{-14}.
\end{aligned}$$

Figure 4.5: The pressure and velocity solutions for the problem as set up in Section 4.5.2. Left column: Velocity u_h and pressure p_h in Ω. Right column: Velocity u_h^m and pressure p_h^m in the matrix in Ω^d.

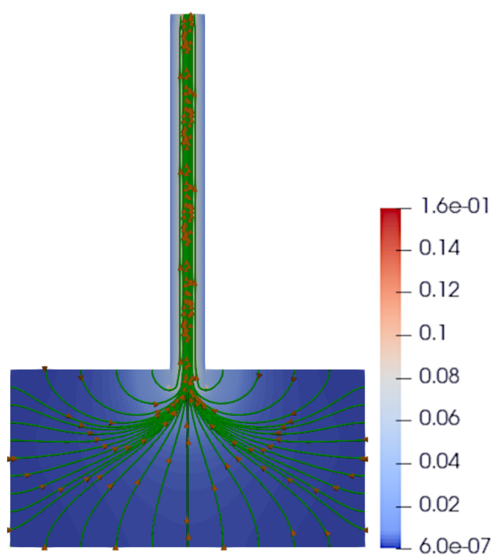
(a) Pressure p_h in Ω.



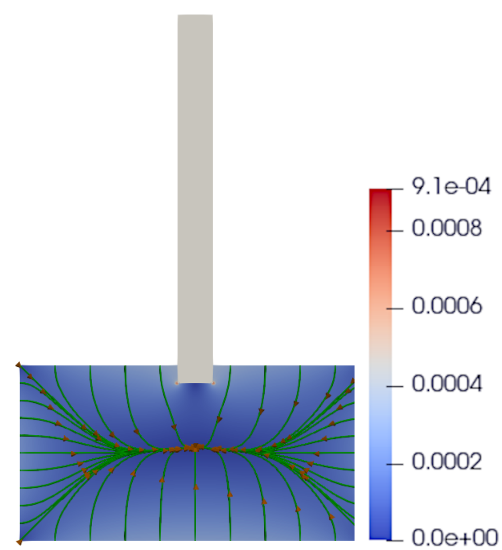
(b) Pressure p_h^m in the matrix in Ω^d.



(c) Velocity field u_h in Ω.



(d) Velocity field u_h^m in the matrix in Ω^d.



CHAPTER FIVE

CONCLUSIONS AND FUTURE WORK

In this chapter, we summarize our main results and discuss potential future projects. We note that the work in Chapter 3 is already published in [1].

In a dual-porosity-Navier-Stokes system, the flow in the conduits is governed by the Navier-Stokes equation. The main difference from the dual-porosity-Stokes model is the addition of a nonlinear term that represents the convective transport of momentum.

In Chapter 3, we introduce a weak formulation and the discretization of the dual-porosity-Stokes problem which describes free flow in macrofractures/conduits by the Stokes equations, while the flow in microfractures/matrix is governed by the dual-porosity equations. The weak formulation of these equations leads to a saddle-point problem with a regular perturbation term. Through the use of saddle point theory, we demonstrate that the weak formulation and the corresponding HDG discretization are well-posed, in addition, we provide an a priori error analysis showing optimal rates of convergence in the energy norm. We conclude the chapter with three examples; first we use a manufactured solution to validate the theoretical rates of convergence and mass conservation and investigate the effects of changing parameters on the discretization errors. In the second numerical example, we simulate the flow of the fluid around a wellbore with open-hole completion in a naturally fractured reservoir, providing velocity and pressure fields. In the last example, we select random permeability fields the in dual-porosity region, demonstrating that our solution is effective in discontinuous permeability conditions as well.

In Chapter 4, we propose and analyze the weak formulation and the hybridizable discontinuous Galerkin method for the dual-porosity-Navier-Stokes problem. In the macrofracture/conduits of this problem, there is a nonlinear term representing convection, which makes the well-posedness analysis and a priori error analysis challenging. Through the application of the fixed point theory and the saddle point theory with a perturbed nonsymmetric term, we demonstrate under certain data assumptions, that the weak formulation and the discrete problem are well-posed, and a prior error analysis of the scheme is provided. In the first numerical example, we show that the convergence rates and mass conversation errors results align with the theoretical findings.

Future research includes extending these results to the time-dependent case where flow and transport problems can be solved sequentially while time-lagging the concentration. Different time-stepping methods can be investigated. Alternatively, we can use a multirate scheme [64] where a larger time step size is used for the slower dual-porosity flow. This is expected to reduce the computational effort but requires an extra stabilization term.

A more complicated extension would be by adding an advection diffusion type transport equation to the system to investigate, for example, the transport of a chemical species via the dual-porosity- (Navier-)Stokes flow. In this problem, the coupling is two way; 1. the flow is affected by the concentration of the chemical, 2. the concentration of the chemical is affected by the (Navier-)Stokes dual porosity velocity. This makes the problem very challenging to analyze and requires certain regularity assumptions.

APPENDIX A

A PROOF OF THE WELL-POSEDNESS OF SADDLE POINT PROBLEMS WITH REGULAR PERTURBED TERMS.

In this section, we provide a complete proof of Theorem 17 second case. First, we state a well-known theorem:

Theorem 58 (Banach–Necas–Babuska)**.** *[42, Theorem 2.6],[65, 66] Let V be a Banach space and let W be a reflexive Banach space. Let $a : V \times W \to \mathbb{R}$ be a continuous bilinear form, and $f \in W'$. Then, the problem find $u \in V$ such that*

$$a(u,w) = \langle f, w \rangle \quad \forall w \in W$$

is well-posed, i.e., it has a unique solution $u \in V$ that satisfies

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{W'}.$$

if and only if the following holds:

- *If there exist a constant $\alpha > 0$ such that:*

$$\inf_{0 \neq v \in V} \sup_{0 \neq w \in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} \geq \alpha,$$

- *For any $w \in W$,*

$$\forall v \in V, \quad a(v,w) = 0 \implies w = 0.$$

Proof of Theorem 17, Case 2. Let $V \times Q$ be equipped with the norm

$$\|(v,q)\|_{V \times Q} = \|v\|_V + \|q\|_Q, \forall (v,q) \in V \times Q,$$

and let $(V \times Q)' = V' \times Q'$ be equipped with the norm

$$\|(f,g)\|_{(V \times Q)'} = \|f\|_{V'} + \|g\|_{Q'} \quad \forall (f,g) \in V' \times Q'.$$

Define the following operator: $\mathbf{L} : V \times Q \mapsto (V \times Q)'$ such that:

$$\forall (u,p) \in V \times Q, \quad \mathbf{L}(u,p) = (Au + Bp, B^T u - Cp).$$

Then finding $(u,p) \in V \times Q$ such that

$$\mathbf{L}(u,p) = (f,g)$$

is equivalent to finding $(u,p) \in V \times Q$ such that

$$\mathscr{L}\left((u,p),(v,q)\right) = \langle f,v\rangle + \langle g,q\rangle \quad \forall (v,q) \in V \times Q,$$

where $\mathscr{L}\left((u,p),(v,q)\right) := a(u,v) + b(v,p) - b(u,q) + c(p,q)$.

We proceed by showing that the conditions in Theorem 58 hold for the operator $\mathscr{L}$. Let us start by proving the second condition in theorem 58 as it requires less effort.

Let $(u,p) \in V \times Q$ and suppose that

$$\mathscr{L}\left((u,p),(v,q)\right) = 0 \quad \forall (v,q) \in V \times Q. \tag{A.1}$$

Choosing $v = u$ and $q = p$ in Equation (A.1) yields

$$a(u,u) + c(p,p) = 0 \quad \forall (v,q) \in V \times Q.$$

Then positive semi-definiteness of $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ imply

$$a(u,u) = c(p,p) = 0.$$

Furthermore, since $c(\cdot,\cdot)$ is symmetric positive semidefinite in Q, by [67, 4.2.29],

$$|c(p,q)| \leq \sqrt{c(p,p)}\sqrt{c(q,q)} \quad \forall q \in Q. \tag{A.2}$$

Thus,

$$c(p,q) = 0 \quad \forall q \in Q.$$

This and choosing $v = 0$ in Equation (A.1) imply $b(u,q) = 0$ for all $q \in Q$. Therefore

$$u \in Ker B.$$

Now since $a(u,u) = 0$, $u \in \mathrm{Ker}B$, and $a(\cdot,\cdot)$ is coercive in $\mathrm{Ker}B$ we conclude that $u = 0$. The result that $p = 0$ follows from the inf-sup condition on $b(\cdot,\cdot)$ in Equation (2.15). Thus, the bilinear form $\mathscr{L}(\cdot,\cdot)$ satisfies the second condition of Theorem 58.

To prove that the first condition in Theorem 58 holds, by [42, Theorem A.43], it is sufficient to show that there exists some constant C such that:

$$\|\mathbf{L}(u,p)\|_{(V\times Q)'} \geq C\|(u,p)\|_{V\times Q}.$$

Since the problem is linear, we can use the principle of superposition and consider the solution of eq. (2.10) as a sum of two problems obtained by setting g to zero (Case 1) and f to zero (Case 2) in eq. (2.11), respectively. Before considering these two cases, let us recall that since $V = \mathrm{Ker}B + \mathrm{Ker}B^{\perp}$, there exist $u_0 \in \mathrm{Ker}B, \bar{u} \in \mathrm{Ker}B^{\perp}$ such that $u = u_0 + \bar{u}$.

Case 1: $g = 0$.

First, since $Au + B^T p = f$, we have $B^T p = f - Au$. Therefore

$$\|p\|_Q \leq \frac{\|B^T p\|_{V'}}{\beta} = \frac{\|f - Au\|_{V'}}{\beta} \leq \frac{\|f\|_{V'} + C_a\|u\|_V}{\beta}, \tag{A.3}$$

where β is as in eq. (2.15).

By testing the first equation in eq. (2.10) with $v = u_0$, we obtain

$$a(u,u_0) + b(u_0,p) = \langle f, u_0\rangle.$$

Noting that $b(u_0,p) = 0$, we obtain

$$a(u,u_0) = \langle f, u_0\rangle.$$

Thus,eqs. (2.9) and (2.12) imply

$$\|u_0\|_V \leq \frac{C_a\|\bar{u}\|_V + \|f\|_{V'}}{\alpha}.$$

Thus, by the triangle inequality,

$$\|u\|_V \leq \left(1+\frac{C_a}{\alpha}\right)\|\bar{u}\|_V + \frac{\|f\|_{V'}}{\alpha}. \tag{A.4}$$

Testing eq. (2.10) where $g = 0$ with $v = u, c = -p$, we obtain:

$$a(u,u) + c(p,p) = \langle f, u \rangle.$$

Since $Bu = Cp$, [67, (4.2.30)] and positive semi-definiteness of $a(\cdot,\cdot)$ imply

$$\|Bu\|_{Q'}^2 = \|Cp\|_{Q'}^2 \leq C_c c(p,p) \leq C_c \langle f, u \rangle \leq C_c \|f\|_{V'} \|u\|_V.$$

Recalling the decomposition $u = \bar{u} + u_0$ such that $Bu_0 = 0$ and eq. (2.15) leads to

$$\beta^2 \|\bar{u}\|_V^2 \leq \|B\bar{u}\|_{Q'}^2 \leq C_c \|f\|_{V'} \|u\|_V. \tag{A.5}$$

Combining eqs. (A.4) and (A.5), we get

$$\|\bar{u}\|_V^2 \leq \frac{C_c}{\beta^2}\|f\|_{V'}\|u\|_V \leq \frac{C_c}{\beta^2}\|f\|_{V'}\left(\left(1+\frac{C_a}{\alpha}\right)\|\bar{u}\|_V + \frac{\|f\|_{V'}}{\alpha}\right).$$

This is of the form $t^2 \leq at + b$, where $t := \|\bar{u}\|_V$, $a := \frac{C_c}{\beta^2}\|f\|_{V'}\left(1+\frac{C_a}{\alpha}\right)$, and $b := \frac{C_c}{\beta^2\alpha}\|f\|_{V'}^2$. It is known that

$$\forall a,b,t > 0, \quad t^2 \leq at + b \implies t \leq a + \sqrt{b}. \tag{A.6}$$

Therefore,

$$\|\bar{u}\|_V \leq \|f\|_{V'}\left(\frac{C_c}{\beta^2}\left(1+\frac{C_a}{\alpha}\right) + \frac{\sqrt{C_c}}{\beta\sqrt{\alpha}}\right).$$

Finally, by plugging in $\|\bar{u}\|_V$ in eq. (A.4) and then the outcome in eq. (A.3), we obtain:

$$\|u\|_V \leq C_1 \|f\|_{V'} \tag{A.7}$$

$$\|p\|_Q \leq C_3 \|f\|_{V'} \tag{A.8}$$

where $C_1 := \left(\frac{C_c}{\beta^2}\left(1+\frac{C_a}{\alpha}\right)^2 + \frac{\sqrt{C_c}}{\beta\sqrt{\alpha}}\left(1+\frac{C_a}{\alpha}\right)+\frac{1}{\alpha}\right)$ and $C_3 := \frac{1+C_aC_1}{\beta}$.

Case 2: $f = 0$.

In this case, $Au + B^T p = 0$, therefore $\|Au\|_{V'} = \|B^T p\|_{V'}$. By the inf-sup condition eq. (2.15) and the continuity of A, eq. (2.9), we obtain

$$\|p\|_Q \leq \frac{C_a\|u\|_V}{\beta}. \tag{A.9}$$

Testing the first equation in eq. (2.10) where $f = 0$ with $v = u_0 \in \operatorname{Ker} B$, we get $a(u,u_0) = 0$. By eqs. (2.9) and (2.12), we get:

$$\|u_0\|_V \leq \frac{C_a\|\bar{u}\|_V}{\alpha}.$$

Thus, by the triangle inequality,

$$\|u\|_V \leq \left(1+\frac{C_a}{\alpha}\right)\|\bar{u}\|_V. \tag{A.10}$$

The proof will be complete if we bound $\|\bar{u}\|_V$. Testing eq. (2.10) where $f = 0$ with $v = u$ and $q = -p$ we obtain $a(u,u) + c(p,p) = \langle g, -p\rangle$. Therefore, using the positive semi definiteness of $a(\cdot,\cdot)$, [67, (4.2.30)], eq. (A.9), eq. (A.10) and the symmetry of $c(\cdot,\cdot)$ we obtain:

$$\begin{aligned}\|Cp\|_{Q'}^2 \leq C_c c(p,p) \leq C_c\|g\|_{Q'}\|p\|_Q \leq \frac{C_aC_c}{\beta}\|g\|_{Q'}\|u\|_V \\ \leq \frac{C_aC_c}{\beta}\left(1+\frac{C_a}{\alpha}\right)\|\bar{u}\|_V\|g\|_{Q'}.\end{aligned}$$

Now the second equation in eq. (2.11) gives $Bu = Cp + g$. Using this, the inequality $(a+b)^2 \leq 2(a^2+b^2)$, and that $Bu_0 = 0$, we arrive at

$$\begin{aligned}\beta^2\|\bar{u}\|_V^2 \leq \|B\bar{u}\|_{Q'}^2 = \|Bu\|_{Q'}^2 \leq 2(\|Cp\|_{Q'}^2 + \|g\|_{Q'}^2) \\ \leq 2\left(\frac{C_aC_c}{\beta}\left(1+\frac{C_a}{\alpha}\right)\|\bar{u}\|_V\|g\|_{Q'} + \|g\|_{Q'}^2\right).\end{aligned}$$

By eq. (A.6), letting $t = \beta\|\bar{u}\|_V$, $a = \frac{2C_aC_c}{\beta^2}\left(1+\frac{C_a}{\alpha}\right)\|g\|_{Q'}$, and $b = 2\|g\|^2_{Q'}$, we have

$$\beta\|\bar{u}\|_V \leq \frac{2C_aC_c}{\beta^2}\left(1+\frac{C_a}{\alpha}\right)\|g\|_{Q'} + \sqrt{2}\|g\|_{Q'},$$

thus,

$$\|u\|_V \leq C_2\|g\|_{Q'}, \tag{A.11}$$

$$\|p\|_Q \leq C_4\|g\|_{Q'}. \tag{A.12}$$

where $C_2 = \frac{C_a+\alpha}{\alpha\beta^3}\left(2C_aC_c\left(1+\frac{C_a}{\alpha}\right)+\sqrt{2}\beta^2\right)$, and $C_4 = \frac{C_aC_2}{\beta}$.

Gathering Equations (A.7), (A.8), (A.11) and (A.12) and using the property of the solution of the sum of two linear systems, we obtain the bounding property of **L**, therefore **L** is surjective, and $\mathscr{L}$ satisfies the first condition given in theorem 58, yielding the well-posedness of Equation (2.10). ■

REFERENCES

[1] A. Cesmelioglu, J. J. Lee, S. Rhebergen, and D. Tabaku, "A hybridizable discontinuous Galerkin method for the dual-porosity-Stokes problem," *Computers & Mathematics with Applications*, vol. 165, pp. 180–195, 2024.

[2] T. Arbogast and D. S. Brunson, "A computational method for approximating a Darcy–Stokes system governing a vuggy porous medium," *Comput. Geosci.*, vol. 11, no. 3, pp. 207–218, 2007.

[3] T. Arbogast and M. S. M. Gomez, "A discretization and multigrid solver for a Darcy-Stokes system of three dimensional vuggy porous media," *Computat. Geosci.*, vol. 13, no. 3, pp. 331–348, 2009.

[4] L. Badea, M. Discacciati, and A. Quarteroni, "Numerical analysis of the Navier–Stokes/Darcy coupling," *Numer. Math.*, vol. 115, no. 2, pp. 195–227, 2010.

[5] A. Çeşmelioğlu and B. Rivière, "Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow," *J. Sci. Comput.*, vol. 40, no. 1, pp. 115–140, 2009.

[6] M. Discacciati, "Domain decomposition methods for the coupling of surface and groundwater flows," 2004.

[7] M. Discacciati, E. Miglio, and A. Quarteroni, "Mathematical and numerical models for coupling surface and groundwater flows," *Appl. Numer. Math.*, vol. 43, no. 1, pp. 57 – 74, 2002. 19th Dundee Biennial Conference on Numerical Analysis.

[8] P. Chidyagwai and B. Rivière, "On the solution of the coupled Navier-Stokes and Darcy equations," *Comput. Methods Appl. Mech. and Eng.*, vol. 198, no. 47, pp. 3806–3820, 2009.

[9] B. Rivière and I. Yotov, "Locally conservative coupling of Stokes and Darcy flows," *SIAM J. Numer. Anal.*, vol. 42, no. 5, pp. 1959–1977, 2005.

[10] W. Layton, F. Schieweck, and I. Yotov, "Coupling fluid flow with porous media flow," *SIAM J. Numer. Anal.*, vol. 40, no. 6, pp. 2195–2218, 2003.

[11] E. Burman and P. Hansbo, "A unified stabilized method for Stokes' and Darcy's equations," *J. Comput. Appl. Math.*, vol. 198, no. 1, pp. 35 – 51, 2007.

[12] M. Mu and J. Xu, "A two-grid method of a mixed Stokes–Darcy model for coupling fluid flow with porous media flow," *SIAM J. Numer. Anal.*, vol. 45, no. 5, pp. 1801–1813, 2007.

[13] M. Discacciati and A. Quarteroni, “Navier–Stokes/Darcy coupling: modeling, analysis, and numerical approximation,” *Rev. Mat. Compplut.*, vol. 22, no. 2, pp. 315–426, 2009.

[14] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang, and W. Zhao, “Finite element approximations for Stokes–Darcy flow with Beavers–Joseph interface conditions,” *SIAM J. Numer. Anal.*, vol. 47, no. 6, pp. 4239–4256, 2010.

[15] A. Cesmelioglu and S. Rhebergen, “A hybridizable discontinuous Galerkin method for the coupled Navier–Stokes and Darcy problem,” *Journal of Computational and Applied Mathematics*, vol. 422, 2023.

[16] T. Arbogast, J. Douglas, and U. Hornung, “Derivation of the double porosity model of single phase flow via homogenization theory,” *SIAM Journal on Mathematical Analysis*, vol. 21, no. 4, pp. 823–836, 1990.

[17] X. Hu and C. Douglas, “An implementation of a coupled dual-porosity-Stokes model with fenics,” *Journal of Computational and Applied Mathematics*, 2019.

[18] J. Warren and P. Root, “The behavior of naturally fractured reservoirs,” *SPE J.*, vol. 3, pp. 245–255, 1963.

[19] N. W. Haws, P. S. C. Rao, P. J. Simunekk, and I. C. Poyer, “Single-porosity and dual-porosity modeling of water flow and solute transport in subsurface-drained fields using effective field-scale parameters,” *Journal of Hydrology*, 2005.

[20] J. Hou, M. Qiu, X.He, C.Guo, M. .Wei, and B. Bai, “A dual-porosity-Stokes model and finite element method for coupling dual-porosity flow and free flow,” *SIAM Journal on Scientific Computing*, vol. 38, no. 5, pp. B710–B739, 2016.

[21] G. S. Beavers and D. D. Joseph, “Boundary conditions at a naturally impermeable wall,” *J. Fluid. Mech*, vol. 30, no. 1, pp. 197–207, 1967.

[22] P. Saffman, “On the boundary condition at the surface of a porous media,” *Stud. Appl. Math.*, vol. 50, pp. 292–315, 1971.

[23] M. A. A. Mahbub, F. Shi, N. J. Nasu, Y. Wang, and H. Zheng, “Mixed stabilized finite element method for the stationary Stokes-dual-permeability fluid flow model,” *Computer Methods in Applied Mechanics and Engineering*, vol. 358, p. 112616, 2020.

[24] M. A. A. Mahbub, X. He, N. J. Nasu, C. Qiu, and H. Zheng, “Coupled and decoupled stabilized mixed finite element methods for nonstationary dual-porosity-Stokes fluid flow model,” *International Journal for Numerical Methods in Engineering*, vol. 120, no. 6, pp. 803–833, 2019.

[25] M. Qiu, F. Qing, X. Yu, J. Hou, D. Li, and X. Zhao, “Finite element method for the stationary dual-porosity Navier-Stokes system with Beavers-Joseph interface conditions,” *Computers & Mathematics with Applications*, 2023.

[26] J. Wen, J. Su, Y. He, and Z. Wang, “A strongly conservative finite element method for the coupled Stokes and dual porosity model,” *J. Comput. Appl. Math.*, vol. 404, no. 113879, pp. 1–16, 2022.

[27] L.Cao, Y. He, J. Li, and M. A. A. Mahbub, “Decoupled modified characteristic finite element method with different subdomain time steps for nonstationary dual–porosity–Navier–Stokes model,” *Applied Numerical Mathematics*, vol. 166, pp. 238–271, 2021.

[28] L. Shan, J. Hou, W. Yan, and J. Chen, “Partitioned Time Stepping Method for a Dual-Porosity-Stokes Model,” *Journal of Scientific Computing*, vol. 79, pp. 389–413, 2019.

[29] L. Cao, Y. He, and J. Li, “A parallel robin–robin domain decomposition method based on modified characteristic FEMs for the time-dependent dual-porosity-n–Stokes model with the beavers–joseph interface condition,” *Journal of Scientific Computing*, vol. 90, 2021.

[30] L. Gao and J. Li, “A decoupled stabilized finite element method for the dual-porosity-Navier–Stokes fluid flow model arising in shale oil,” *Numerical Methods for Partial Differential Equations*, vol. 37, no. 3, pp. 2357–2374, 2021.

[31] L. Cao, Y. He, J. Li, and D. Yang, “Decoupled modified characteristic FEMs for fully evolutionary Navier–Stokes–Darcy model with the Beavers–Joseph interface condition,” *Journal of Computational and Applied Mathematics*, vol. 383, p. 113128, 2021.

[32] L. Cao, Y. He, and J. Li, “A parallel Robin–Robin domain decomposition method based on modified characteristic FEMs for the time-dependent dual-porosity-Navier–Stokes model with the Beavers–Joseph interface condition,” *J. Sci. Comput.*, vol. 90, 2022.

[33] Y. Li, D. Xue, Y. Rong, and Y. Qin, “A second order partitioned method with grad-div stabilization for the non-stationary dual-porosity-Stokes model,” *Comput. Math. Appl*, vol. 124, pp. 111–128, 2022.

[34] N. C. Nguyen, J. Peraire, and B. Cockburn, “An implicit high-order hybridizable discontinuous Galerkin method for linear convection-diffusion equations,” *J. Comput. Phys.*, vol. 228, no. 9, pp. 3232–3254, 2009.

[35] B. Cockburn, “Static condensation, hybridization, and the devising of the hdg methods,” *Lecture Notes in Computational Science and Engineering*, pp. 129–177, 2016.

[36] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, “Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems,” *SIAM J. Numer. Anal.*, vol. 47, no. 2, pp. 1319–1365, 2009.

[37] A. Cesmelioglu, S. Rhebergen, and G. N. Wells, “An embedded–hybridized discontinuous Galerkin method for the coupled Stokes–Darcy system,” *Journal of Computational and Applied Mathematics*, vol. 367, p. 112476, 2020.

[38] S. Rhebergen and G. N. Wells, “Analysis of a hybridized/interface stabilized finite element method for the Stokes equations,” *SIAM J. Numer. Anal.*, vol. 55, no. 4, pp. 1982–2003, 2017.

[39] D. N. Arnold and F. Brezzi, “Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates,” *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 19, no. 1, pp. 7–32, 1985.

[40] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, vol. 44 of *Springer Series in Computational Mathematics*. Springer–Verlag Berlin Heidelberg, 2013.

[41] S. Du and F.-J. Sayas, *An Invitation to the Theory of the Hybridizable Discontinuous Galerkin Method*. SpringerBriefs in Mathematics, Springer Cham, 2019.

[42] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*, vol. 159 of *Applied Mathematical Sciences*. Springer–Verlag New York, 2004.

[43] L. C. Evans, *Partial differential equations*. American Mathematical Society, 2010.

[44] R. A. Adams and J. F. Fournier., *Sobolev Spaces.* Pure and Applied Mathematics, Academic Press, 2003.

[45] V. Girault and B. Rivière, “DG approximation of coupled Navier–Stokes and Darcy equations by Beaver–Joseph–Saffman interface condition,” *SIAM J. Numer. Anal.*, vol. 47, no. 3, pp. 2052–2089, 2009.

[46] P. G. Ciarlet, *Linear and nonlinear functional analysis with applications*, vol. 130. SIAM, 2013.

[47] *Investigation of the Shape Factor Used in the Dual-Porosity Reservoir Simulator*, vol. All Days of *SPE Asia Pacific Oil and Gas Conference and Exhibition*, 09 1989.

[48] D. A. Di Pietro and A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, vol. 69 of *Mathématiques et Applications*. Springer–Verlag Berlin Heidelberg, 2012.

[49] T. Warburton and J. S. Hesthaven, "On the constants in *hp*-finite element trace inverse inequalities," *Comput. Methods Appl. Mech. Engrg.*, vol. 192, no. 25, pp. 2765–2773, 2003.

[50] P. Hansbo and M. G. Larson, "Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method," *Comput. Methods Appl. Mech. Engrg.*, vol. 191, pp. 1895–1908, 2002.

[51] J. S. Howell and N. J. Walkington, "Inf-sup conditions for twofold saddle point problems," *Numer. Math.*, vol. 118, pp. 663–693, 2011.

[52] S. Rhebergen and G. N. Wells, "Preconditioning of a hybridized discontinuous Galerkin finite element method for the Stokes equations," *J. Sci. Comput.*, vol. 77, no. 3, pp. 1936–1952, 2018.

[53] J. Schöberl, "An advancing front 2D/3D-mesh generator based on abstract rules," *J. Comput. Visual Sci.*, vol. 1, no. 1, pp. 41–52, 1997.

[54] J. Schöberl, "C++11 implementation of finite elements in NGSolve," Tech. Rep. ASC Report 30/2014, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2014.

[55] A. Linke and C. Merdon, "On velocity errors due to irrotational forces in the Navier–Stokes momentum balance," *J. Comput. Phys.*, vol. 313, pp. 654–661, 2016.

[56] A. Linke and C. Merdon, "Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations," *Comput. Methods Appl. Mech. Engrg.*, vol. 311, pp. 304–326, 2016.

[57] *Effective Open Hole Horizontal Completion System for Multistage Fracturing and Stimulation*, vol. All Days of *SPE Unconventional Resources Conference / Gas Technology Symposium*, 06 2008.

[58] M. Discacciati and R. Oyarzúa, "A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem," *Numer. Math.*, vol. 135, pp. 571–606, 2017.

[59] J. Camaño, G. N. Gatica, R. Oyarzúa, and G.Tierra, "An augmented mixed finite element method for the Navier–Stokes equations with variable viscosity," *SIAM Journal on Numerical Analysis*, vol. 54, 2016.

[60] S. Rhebergen and G. N. Wells, "A hybridizable discontinuous Galerkin method for the Navier–Stokes equations with pointwise divergence-free velocity field," *J. Sci. Comput.*, vol. 76, no. 3, pp. 1484–1501, 2018.

[61] S. Caucao, M. Discacciati, G. N. Gatica, and R. Oyarzúa, “A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem,” *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 54, no. 5, pp. 1689–1723, 2020.

[62] A. Cesmelioglu, B. Cockburn, and W. Qiu, “Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier–Stokes equations,” *Math. Comp.*, vol. 86, pp. 1643–1670, 2017.

[63] *Horizontal Injection and Production Wells for EOR or Waterflooding*, vol. All Days of *SPE Permian Basin Oil and Gas Recovery Conference*, 1992.

[64] P. Chidyagwai, “A second order multirate scheme for the evolutionary Stokes–Darcy model.,” *Applied Numerical Mathematics*, vol. 170, pp. 364–383, 2021.

[65] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*. Paris: Masson et Cie, 1967.

[66] I. Babuška, *The Finite Element Method for Elliptic Equations with Discontinuous Coefficients*. Berlin: Springer, 1973.

[67] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15 of *Springer Series in Computational Mathematics*. Springer–Verlag New York Inc., 1991.