

DYNAMIC CONTACT WITH NORMAL COMPLIANCE WEAR AND DISCONTINUOUS FRICTION COEFFICIENT*

KENNETH L. KUTTLER[†] AND MEIR SHILLOR[‡]

Abstract. We apply the recent theory of evolution inclusions for set-valued pseudomonotone maps, developed in Kuttler and Shillor [*Commun. Contemp. Math.*, 1 (1999), pp. 87–123] to the problem of dynamic frictional contact with normal compliance and wear. The friction coefficient is assumed to be slip rate dependent, and may be continuous, or discontinuous in the form of a graph with a vertical segment at the origin, representing the transition from the static to the dynamic value. The wear of the contacting surfaces is modeled by the Archard law. We prove the existence of a weak solution for the problem. We establish the uniqueness of the weak solution in the case when the friction coefficient is continuous. We also show that the problem with prescribed wear depends continuously on the wear.

Key words. dynamic frictional contact, set-valued inclusions, existence and uniqueness, discontinuous friction coefficient, normal compliance, wear

AMS subject classifications. 74M10, 74M15, 35R35, 35R05, 35R70

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1. Introduction. We use the theory of set-valued pseudomonotone maps, which we have developed in [18], to establish the existence of a weak solution of a dynamic frictional contact problem with wear, when the friction coefficient depends discontinuously on the slip velocity. The problem describes frictional dynamic contact between a deformable body, assumed to be viscoelastic, and a moving foundation and the resulting wear of the contact surface. This paper is a continuation of our investigation in [18], where the contact problem has been considered, however, with continuous coefficient of friction and without wear. The new features in the model are the description of friction with a discontinuous coefficient and inclusion of the wear of the contacting surfaces. We investigate the case when the friction coefficient jumps from a static value, when the contacting surfaces stick together, to the lower dynamic value at the onset of relative motion between them. Such a behavior is often assumed in engineering applications. The contact between the body and a moving rigid foundation is modeled with the normal compliance condition, and friction is modeled with the pressure dependent condition. We use the Archard law to describe the evolution of the wear. The problem is formulated as an abstract inclusion in a Banach space to which the results of [18] apply.

Dynamic frictional contact problems have been considered recently in [5, 6, 9, 19, 21, 26, 31], while quasi-static problems can be found in [2, 4, 7, 27, 30] and references to therein. See also [29] and the papers therein. It is a common assumption in engineering literature that the friction coefficient depends on the slip speed. However, there are only few and very recent mathematical publications which consider dynamic contact with a friction coefficient which depends on the slip velocity of the contacting surface [2, 10, 18, 19]. The last reference deals with a discontinuous slip-dependent

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[†]Department of Mathematics, Brigham Young University, Provo, UT 84602 (klkuttler@math.byu.edu).

[‡]Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309 (shillor@oakland.edu).

coefficient, but in that problem the contact was assumed to be maintained and there was no wear. A simple one-dimensional dynamic problem was analyzed in [10], where a criterion for the appearance of dynamic instabilities was discovered. Analysis and numerical simulations of thermoelastic frictional contact of a beam were performed in [17]. The quasi-static problem with slip or total slip (over the contact history) dependent coefficient of friction can be found in [2] and the dynamic thermoviscoelastic problem in [3]. Frictional contact problems with wear can be found in [6] and [31].

In section 2, we present preliminary material which includes the abstract existence theorem of [18] that underlies our results here. The classical model for the process, its abstract formulation, the assumptions on the problem data, and the statement of our main result, Theorem 3.2, are given in section 3. Section 4 is devoted to approximate problems, with a known wear function, whose unique solvability, stated in Theorem 4.1, follows from the existence theorem in section 2. A solution of the contact problem with known wear, when the friction coefficient μ is continuous, is obtained as a limit of these approximate solutions in section 5. Under a mild additional assumption on the problem data we show that the solution is unique. We investigate in section 6 the case of a discontinuous friction coefficient. It is found that many of the necessary estimates do not depend on the continuity of μ , and this fact is exploited in establishing the existence of a weak solution in the case when μ has a jump discontinuity at the origin, when slip motion is initiated. The result is stated in Theorem 3.2, in the case when the wear is known. Uniqueness remains an unsolved problem in this case. In section 7, we prove the continuous dependence of the solutions of the problem on the wear function w . The result is stated in Theorem 7.1, and it has some merit on its own. In section 8, we deal with the problem with wear, which is assumed to evolve according to a local version of the Archard law. We use the results up to this point to establish the existence of the weak solution to the problem with wear; however, the questions of uniqueness and stability of the solutions remain open.

2. Preliminaries. The existence results to be presented in this paper are based on our recent theorems [18] for differential inclusions of the form

$$(B(t)u(t))' + Au \ni f(t),$$

where A is a set-valued pseudomonotone map. Here, the prime denotes the time derivative which is understood in the sense of distributions. Let \mathcal{V} be a reflexive Banach space, over \mathbb{C} , and let \mathcal{V}' denote the space of conjugate linear maps. We start with (see, e.g., [22]) the following definition.

DEFINITION 2.1. *A map $A : \mathcal{V} \rightarrow \mathcal{P}(\mathcal{V}')$ is said to be pseudomonotone if*

1. *the set Au is nonempty, bounded, closed, and convex for all $u \in \mathcal{V}$;*
2. *if F is a finite-dimensional subspace of \mathcal{V} , $u \in F$, and if U is a weakly open set in \mathcal{V}' such that $Au \subseteq U$, then there exists $\delta > 0$ such that if $v \in B_\delta(u) \cap F$, then $Av \subseteq U$;*
3. *if $u_i \rightarrow u$ weakly in \mathcal{V} and $u_i^* \in Au_i$ is such that*

$$(2.1) \quad \limsup_{i \rightarrow \infty} \operatorname{Re} \langle u_i^*, u_i - u \rangle_{\mathcal{V}} \leq 0,$$

then, for each $v \in \mathcal{V}$, there exists $u^(v) \in Au$ such that*

$$(2.2) \quad \liminf_{i \rightarrow \infty} \operatorname{Re} \langle u_i^*, u_i - v \rangle_{\mathcal{V}} \geq \operatorname{Re} \langle u^*(v), u - v \rangle_{\mathcal{V}}.$$

Here $B_\delta(u)$ denotes the ball of radius δ centered at u .

Our theory of set-valued evolution equations was developed in general reflexive Banach spaces. Here we restrict ourselves to the spaces which we now describe. Let $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) be a domain, occupied by the deformable body, with Lipschitz boundary Γ . The surface is divided into three mutually disjoint parts Γ_D, Γ_N , and Γ_C such that $\Gamma_C \neq \emptyset$ is the potential contact surface. Next, we choose the space W as follows: if the body is clamped over Γ_D , with $\text{meas} \Gamma_D > 0$, then we set $W = \{\mathbf{u} \in (H^1(\Omega))^N : \mathbf{u} = 0 \text{ on } \Gamma_D\}$; if the body is not held fixed ($\text{meas} \Gamma_D = 0$), then $W = (H^1(\Omega))^N$. Now we let $q, p \geq 2$, set $D = W \cap (C^\infty(\bar{\Omega}))^N$, and define

$$(2.3) \quad \tilde{V}_p = \{\mathbf{u} \in W : \gamma \mathbf{u} \in (L^p(\Gamma_C))^N\},$$

with norm $\|\mathbf{u}\|_{\tilde{V}_p} = \|\mathbf{u}\|_W + \|\gamma \mathbf{u}\|_{(L^p(\Gamma_C))^N}$, where $\gamma : W \rightarrow (L^2(\Gamma_C))^N$ is the trace operator. \tilde{V}_p is a reflexive Banach space since it is isometric to a closed subspace of $W \times (L^p(\Gamma_C))^N$. We denote by V_p the closure of D in \tilde{V}_p . Then V_p is a reflexive Banach space, and for $p < q$

$$(2.4) \quad V_p \supseteq V_q, \quad V_q \text{ is dense in } V_p.$$

Since V_p is dense in $H = (L^2(\Omega))^N$, we identify H and H' and write $V_p \subseteq H = H' \subseteq V'_p$. Let

$$(2.5) \quad \mathcal{V}_p = \{\mathbf{u} \in L^2(0, T; V_p) : \|\mathbf{u}\|_{\mathcal{V}_p} < \infty\},$$

equipped with norm

$$(2.6) \quad \|\mathbf{u}\|_{\mathcal{V}_p} = \|\mathbf{u}\|_{L^2(0, T; W)} + \|\gamma \mathbf{u}\|_{L^p(0, T; (L^p(\Gamma_C))^N)}.$$

\mathcal{V}_p is a reflexive Banach space since it is isometric to a closed subspace of $L^2(0, T; W) \times L^p(0, T; (L^p(\Gamma_C))^N)$, and \mathcal{V}_q is dense in \mathcal{V}_p when $p < q$. Note that $\mathcal{V}'_p \subseteq L^p(0, T; V'_p)$ and the inclusion map is continuous.

Next, we define the Banach space X as follows:

$$(2.7) \quad X = \{\mathbf{u} \in \mathcal{V}_p : \mathbf{u}' \in \mathcal{V}'_p\}, \quad \|\mathbf{u}\|_X = \|\mathbf{u}\|_{\mathcal{V}_p} + \|\mathbf{u}'\|_{\mathcal{V}'_p}.$$

We shall use the following two results.

THEOREM 2.2 (see [20]). *Let $p \geq 1, q > 1, W \subseteq U \subseteq Y$ with compact inclusion map $i : W \rightarrow U$ and continuous inclusion map $i : U \rightarrow Y$ and let*

$$S_R = \{\mathbf{u} \in L^p(0, T; W) : \mathbf{u}' \in L^q(0, T; Y), \|\mathbf{u}\|_{L^p(0, T; W)} + \|\mathbf{u}'\|_{L^q(0, T; Y)} < R\}.$$

Then S_R is precompact in $L^p(0, T; U)$.

THEOREM 2.3 (see [28]). *Let W, U , and Y be as above and let*

$$S_{RT} = \{\mathbf{u} : \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0, T; Y)} \leq R, \quad t \in [0, T]\}$$

for $q > 1$. Then S_{RT} is precompact in $C(0, T; U)$.

We now describe the abstract setting we shall use. Let V and W be reflexive Banach spaces over \mathbb{C} and let $I = [a, b]$. We denote $\mathcal{W}_I \equiv L^2(I; W)$ and then $\mathcal{W}'_I = L^2(I; W')$. Also, when $I = [0, T]$, we write \mathcal{V} instead of \mathcal{V}_I .

We assume that the family of operators $B(t)$ satisfies $B(t) \in \mathcal{L}(W, W')$ and

$$(2.8) \quad \langle B(t)u, v \rangle = \overline{\langle B(t)v, u \rangle},$$

$$(2.9) \quad \langle B(t)u, u \rangle \geq 0,$$

$$(2.10) \quad B(t) = B(0) + \int_0^t B'(s) ds.$$

The operator L , associated with B , is defined as

$$(2.11) \quad D(L) \equiv \{u \in \mathcal{V} : (i^*Bu)' \in \mathcal{V}'\},$$

$$(2.12) \quad Lu \equiv (i^*Bu)' \quad \text{for } u \in D(L),$$

where i is the inclusion map of V into W . The following lemma results from the definitions.

LEMMA 2.4. L is a closed operator.

We define

$$X \equiv D(L), \quad \|u\|_X \equiv \|Lu\|_{\mathcal{V}'} + \|u\|_{\mathcal{V}}.$$

By Lemma 2.4, X is isometric to a closed subspace of a product of reflexive Banach spaces and thus X is also reflexive. Under these conditions the following theorem was proved in [18].

THEOREM 2.5 (see [18]). *Let $u, v \in X$; then the following hold.*

1. $t \rightarrow \langle B(t)u(t), v(t) \rangle_{W',W}$ equals an absolutely continuous function a.e. t , denoted by $\langle Bu, v \rangle(\cdot)$.
 2. $\text{Re}\langle Lu(t), u(t) \rangle = \frac{1}{2} [\langle Bu, u \rangle'(t) + \langle B'(t)u(t), u(t) \rangle]$ for a.e. t .
 3. $|\langle Bu, v \rangle(t)| \leq C \|u\|_X \|v\|_X$ for some $C > 0$ and for all $t \in [0, T]$.
 4. $t \rightarrow B(t)u(t)$ equals a function in $C(0, T; W')$, a.e. t , denoted by $Bu(\cdot)$.
 5. $\sup\{\|Bu(t)\|_{W'}, t \in [0, T]\} \leq C \|u\|_X$ for some $C > 0$.
- If $K : X \rightarrow X'$ is given by

$$\langle Ku, v \rangle_{X',X} \equiv \int_0^T \langle Lu(t), v(t) \rangle dt + \langle Bu, v \rangle(0),$$

then

6. K is linear, continuous, and weakly continuous.
7. $\text{Re}\langle Ku, u \rangle = \frac{1}{2} [\langle Bu, u \rangle(T) + \langle Bu, u \rangle(0)] + \frac{1}{2} \int_0^T \langle B'(t)u(t), u(t) \rangle dt$.

The operator A in the theorem and below is assumed to satisfy

$$(2.13) \quad A : \mathcal{V} \rightarrow \mathcal{P}(\mathcal{V}') \text{ is bounded;}$$

$$(2.14) \quad \liminf_{\|u\|_{\mathcal{V}} \rightarrow \infty} \frac{\{2\text{Re}\langle u^*, u \rangle + \langle B'u, u \rangle + \langle Bu, u \rangle(T) : u^* \in Au\}}{\|u\|_{\mathcal{V}}} = \infty$$

for $u \in X$; and

$$(2.15) \quad A + K : X \rightarrow \mathcal{P}(X') \text{ is pseudomonotone.}$$

The following abstract theorem is the basis for the results in this paper.

THEOREM 2.6 (see [18]). *Let the spaces \mathcal{V} and \mathcal{W} be as defined above and let the operators $A : \mathcal{V} \rightarrow \mathcal{P}(\mathcal{V}')$ and $B(t)$ satisfy (2.13)–(2.15) and (2.10)–(2.12), respectively. If $f \in \mathcal{V}'$ and $u_0 \in W$, then there exists a solution $u \in \mathcal{V}$ to the initial value problem*

$$(i^*Bu)' + Au \ni f \text{ in } \mathcal{V}', \quad Bu(0) = Bu_0 \text{ in } W'.$$

Here, i is the inclusion map $i : V \rightarrow W$. The proof of the theorem can be found in [18].

3. The model. We describe the classical problem and the assumptions on the data, then we formulate it abstractly, and we state our main results in Theorems 3.2–3.3. We use the isothermal version of the problem in [6] (see also [21, 8]). We refer the reader there for a more detailed description of the model. We use the *normal compliance* contact condition (see, e.g., [6, 5, 15, 13, 21, 27]) to describe the contact, together with a condition for dry friction. Dynamic problems with this condition have been investigated in [15, 5, 9, 6]. We use the Archard law, as has been done in [6], to describe the wear of the contact surface (see also [27, 30, 31]).

A viscoelastic body occupies the reference configuration $\Omega \subset \mathbb{R}^N$, with boundary surface $\Gamma = \partial\Omega$, such that $\Gamma = \bar{\Gamma}_C \cup \bar{\Gamma}_D \cup \bar{\Gamma}_N$. It may come in contact with a deformable moving foundation on the part Γ_C . We set $\Omega_T = \Omega \times (0, T)$ for $0 < T$ and denote the displacements vector by $\mathbf{u} = (u_1, \dots, u_N)$ and the stress tensor by $\sigma = \sigma(\mathbf{u}, \mathbf{u}') = (\sigma_{ij})$, where here and below $i, j = 1, \dots, N$, and a comma separates the components of a vector or tensor from partial derivatives.

The equations of motion, in dimensionless form, are

$$(3.1) \quad \mathbf{u}'' - \text{Div } \sigma(\mathbf{u}, \mathbf{u}') = \mathbf{f}_B \quad \text{in } \Omega_T,$$

where \mathbf{f}_B represents the volume force acting on the body. Initially,

$$(3.2) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}'(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega,$$

where \mathbf{u}_0 and \mathbf{v}_0 are the prescribed displacement and velocity fields, respectively.

The body is held fixed on Γ_D (when $\text{meas } \Gamma_D \neq 0$) and tractions \mathbf{f}_N act on Γ_N , thus

$$(3.3) \quad \mathbf{u} = 0 \quad \text{on } \Gamma_D, \quad \sigma \mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N,$$

where \mathbf{n} is the unit outward normal to Ω on Γ .

Our interest lies in the process on the contact surface Γ_C . We denote the normal component of the displacements vector on Γ by $u_n = \mathbf{u} \cdot \mathbf{n}$, the tangential components by $\mathbf{u}_T = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$, the normal component of the traction by $\sigma_n = \sigma_{ij}n_jn_i$, and the tangential tractions by $\sigma_{Ti} = \sigma_{ij}n_j - \sigma_n n_i$.

We model the contact between the body and the foundation by the normal compliance condition. Let $g = g(\mathbf{x})$ be a nonnegative function on Γ_C , representing the gap between the body's surface (in the reference configuration) and the foundation, measured along the normal \mathbf{n} . We denote by $w = w(\mathbf{x}, t)$ the *wear function* which measures the wear of Γ_C at position \mathbf{x} and time t . It describes the change in the surface, in the (negative) direction of the normal, resulting from material removal because of friction. We assume that the contact pressure is given by

$$(3.4) \quad \sigma_n = -p(u_n - w - g),$$

where $p(\cdot)$ is a nonnegative monotone function which vanishes for negative argument values. Thus, the pressure on the contact surface depends on the interpenetration $u_n - w - g$, when positive. The choice $p(r) = (r)_+^{m_n}$ can be found in [13, 21].

We note that as the wear of the surface increases the normal displacement needed for contact increases, too. In the tangential direction we employ a dry friction condition that is compatible with (3.4) and which has a slip dependent and discontinuous friction coefficient. Let μ^* denote the *friction graph*,

$$(3.5) \quad \mu^*(r) = \begin{cases} [\mu_d, \mu_s] & \text{when } r = 0, \\ \mu_c(r) & \text{when } r > 0, \end{cases}$$

where $r = |\mathbf{u}'_T - \mathbf{v}_*|$ denotes the relative slip between the surface and the foundation. Here, \mathbf{v}_* is the tangential velocity of the foundation, and generally it depends on the location on the surface, thus it is assumed to lie in $L^\infty(0, T; L^\infty(\mathbb{R}^N))$. If the contact surface is flat, a portion of a plane, we may choose \mathbf{v}_* to be a function of time only, but when the contact surface is not flat, even if the velocity of the foundation is constant, the tangential velocity is not constant and depends on the position and on time.

In the slip state ($0 < r$) the coefficient μ is given by $\mu_c(r)$, and $\mu_d = \lim_{s \rightarrow 0} \mu_c(s)$ denotes the dynamic value at zero slip. In the absence of relative slip μ may have any value in the interval $[\mu_d, \mu_s]$. Thus, we do not insist that it has the static value μ_s , although it is likely when the body is in stick state for a while. We assume that $\mu_c(r)$, for $r \geq 0$, is a given positive Lipschitz function which satisfies the conditions below.

Next, we consider the friction condition. As is well known in applications, and explained well in [25, 32], when the contact pressure is low to moderate, the real contact area is a small fraction of the nominal contact area, and the frictional tangential traction is proportional to the contact pressure, given by μp . This is the usual Coulomb's condition which is often used both in engineering and mathematical publications. However, when the contact pressure is very high, such as in metal forming processes, the fraction of the real contact area approaches unity, and the frictional traction reaches saturation and the maximal frictional resistance becomes independent of the contact pressure. Thus, there is a transition from the Coulomb law to the so-called Tresca law; see, e.g., [32]. Such a transition is observed both in elastic and plastic materials. A simple way to model such behavior is to introduce the truncated contact pressure function

$$p_R = \begin{cases} p & \text{if } p \leq R, \\ R & \text{if } R \leq p. \end{cases}$$

Here, $R = \text{const.}$ is the pressure at which the friction traction levels off. We could have used, instead, a more general, and less transparent, function F such that $F = \mu p$ for small p , and asymptotically $F \rightarrow \mu R$ as $p \rightarrow \infty$.

Then the *friction bound* is defined as μp_R , and the friction law is

$$(3.6) \quad \mu(|\mathbf{u}'_T - \mathbf{v}_*|) \in \mu^*(|\mathbf{u}'_T - \mathbf{v}_*|) \quad \text{a.e. on } \Gamma_C,$$

$$(3.7) \quad |\sigma_T| \leq \mu_s p_R(u_n - w - g),$$

$$(3.8) \quad \sigma_T = -\frac{\mathbf{u}'_T - \mathbf{v}_*}{|\mathbf{u}'_T - \mathbf{v}_*|} \mu_c(|\mathbf{u}'_T - \mathbf{v}_*|) p_R(u_n - w - g) \quad \text{if } \mathbf{u}'_T - \mathbf{v}_* \neq 0.$$

Conditions (3.6)–(3.8) model friction as follows. The tangential part of the traction is bounded by $\mu_s p_R$. Sliding commences when $|\sigma_T|$ reaches the bound $\mu_s p_R$, and then the tangential force has a direction opposite to the relative tangential velocity. The actual value of μ is a selection out of the graph, (3.6).

The contact surface Γ_C is divided, at each time instant, into the *separation*, *slip*, and *stick* zones.

We assume that the wear of the surface is either a given function or else it is proportional to the friction force and to the sliding speed, as in the Archard law,

$$(3.9) \quad \frac{\partial w}{\partial t} = k_w \mu_c(|\mathbf{u}'_T - \mathbf{v}_*|) p_R(u_n - w - g) s_c(|\mathbf{u}'_T - \mathbf{v}_*|).$$

Here, k_w is a positive material constant, very small in practice. The function s_c is a regularization of $|\cdot|$ on \mathbb{R}^N which is uniformly bounded and such that $s_c(r) = 0$ for $r = 0$. Note that we used μ_c in (3.9) since s_c vanishes when there is no slip.

The new features in the model are the dependence, which often can be observed experimentally, of the friction coefficient on the magnitude of the slip, $|\mathbf{u}'_T - \mathbf{v}_*|$, with a jump from a static to a dynamic value at the onset of sliding, and the wear of the contact surface. The problem with Lipschitz μ and without wear was investigated in [18].

Finally, we assume that the material is viscoelastic with constitutive law

$$(3.10) \quad \sigma = \sigma(\mathbf{u}, \mathbf{u}') = A\mathbf{u} + C\mathbf{u}'$$

i.e., $\sigma_{ij} = A_{ijkl}u_{k,l} + C_{ijkl}u'_{k,l}$, where the elasticity tensor A has the components A_{ijkl} and the viscosity tensor C has the components C_{ijkl} .

The classical formulation of the problem of *dynamic frictional contact with normal compliance wear and discontinuous slip dependent friction coefficient* is as follows: Find $\{\mathbf{u}, w\}$ such that (3.1)–(3.10) hold.

We make the following assumptions on the problem data. The normal pressure function $p(\cdot)$ is increasing and satisfies

$$(3.11) \quad |p(r_1) - p(r_2)| \leq K(1 + r_1^{p-2} + r_2^{p-2})|r_1 - r_2|,$$

and either

$$(3.12) \quad 0 \leq p(r) \leq K \text{ and } p = 2; \quad p(r) = 0, \quad r < 0,$$

or

$$(3.13) \quad \delta^2 r^{p-1} - K \leq p(r) \leq K(1 + r^{p-1}), \quad r \geq 0; \quad p(r) = 0, \quad r < 0,$$

where $p \geq 2$ is a fixed exponent here and everywhere below, and δ and K are positive constants. Also, p is the exponent and $p(\cdot)$ is the normal compliance function. The choice made in [21] and [13] of $p(r) = r_+^{m_n}$, where $1 < m_T \leq m_n$, corresponds to $p - 1 = m_n$ and clearly (3.13) holds for suitable constants K and δ . The function s_c satisfies

$$(3.14) \quad s_c(r) \leq s_c^*, \quad |s_c(r_1) - s_c(r_2)| \leq \delta_c^* |r_1 - r_2|.$$

We assume that the coefficient of friction is a graph composed of the vertical segment $[\mu_d, \mu_s]$ and the function μ_c is bounded, positive, and Lipschitz continuous,

$$(3.15) \quad |\mu_c(r_1) - \mu_c(r_2)| \leq \text{Lip}_\mu |r_1 - r_2|, \quad \|\mu_c\|_{L^\infty} \leq c_\mu.$$

We assume that the elasticity and viscosity coefficients A and C lie in $L^\infty(\Omega)$ and satisfy the following symmetries for $B = A$ or C :

$$(3.16) \quad B_{ijkl} = B_{ijlk}, \quad B_{jikl} = B_{ijkl}, \quad B_{ijkl} = B_{klij},$$

and

$$(3.17) \quad B_{ijkl}\zeta_{ij}\zeta_{kl} \geq \lambda\zeta_{rs}\zeta_{rs},$$

for all symmetric matrices ζ , where $0 < \lambda$.

We now obtain a weak formulation of problem (3.1)–(3.10) since, generally, the friction law and the set inclusion (3.6) preclude the existence of classical solutions.

We begin by defining the viscosity and elasticity operators $M, A : V_p \rightarrow V'_p$ as

$$(3.18) \quad \langle M\mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} C_{ijkl} \mathbf{u}_{k,l} \mathbf{v}_{i,j} \, dx,$$

$$(3.19) \quad \langle A\mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} A_{ijkl} \mathbf{u}_{k,l} \mathbf{v}_{i,j} \, dx.$$

It follows from our assumptions and Korn's inequality [23] that both M and A satisfy

$$\langle B\mathbf{u}, \mathbf{u} \rangle \geq \delta^2 \|\mathbf{u}\|_W^2 - \lambda_0 \|\mathbf{u}\|_H^2, \quad \langle B\mathbf{u}, \mathbf{u} \rangle \geq 0, \quad \langle B\mathbf{u}, \mathbf{v} \rangle = \langle B\mathbf{v}, \mathbf{u} \rangle$$

for $B = M$ or A , for some $\delta > 0$, and for $\lambda_0 \geq 0$.

The *normal compliance* operator $(\mathbf{v}, w) \rightarrow P(\mathbf{u}, w)$, which maps $\mathcal{V}_q \times L^p(\Gamma_C)$ to \mathcal{V}'_q (for each $q \geq p$), is given by

$$(3.20) \quad \langle P(\mathbf{u}, w), \mathbf{z} \rangle = \int_0^T \int_{\Gamma_C} p(u_n - w - g) z_n \, d\Gamma dt,$$

where $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds$, for $\mathbf{u}_0 \in V_q$. Next, we define $\mathbf{f} \in \mathcal{W}'$ as

$$(3.21) \quad \langle \mathbf{f}, \mathbf{z} \rangle_{\mathcal{W}', \mathcal{W}} = \int_0^T \int_{\Omega} \mathbf{f}_B \mathbf{z} \, dx dt + \int_0^T \int_{\partial\Omega} \mathbf{f}_N \gamma \mathbf{z} \, d\Gamma dt$$

for all $\mathbf{z} \in \mathcal{W}$. Here \mathbf{f}_B represents a body force in $L^2(0, T; H)$ and \mathbf{f}_N is a traction force in $L^2(0, T; L^2(\partial\Omega)^N)$.

Let $\gamma_T^* : L^{p'}(0, T; L^{p'}(\Gamma_C)^N) \rightarrow \mathcal{V}'_p$ be defined as

$$\langle \gamma_T^* \xi, \mathbf{w} \rangle = \int_0^T \int_{\Gamma_C} \xi \cdot \mathbf{w}_T \, d\Gamma dt.$$

The abstract form of the problem for the displacement \mathbf{u} , the velocity \mathbf{v} , and the wear w , is the following.

Problem \mathcal{P} . Find $\mathbf{u}, \mathbf{v} \in \mathcal{V}_p, w \in L^p(0, T; L^p(\Gamma_C))$ such that

$$(3.22) \quad \mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P(\mathbf{u}, w) + \gamma_T^* \xi = \mathbf{f} \quad \text{in } \mathcal{V}'_p,$$

$$(3.23) \quad w' = k_w \mu_c (|\mathbf{v}_T - \mathbf{v}_*|) p_R (u_n - w - g) s_c (|\mathbf{v}_T - \mathbf{v}_*|),$$

$$(3.24) \quad w(0) = 0, \quad \mathbf{v}(0) = \mathbf{v}_0 \in H,$$

$$(3.25) \quad \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) \, ds, \quad \mathbf{u}_0 \in V_p,$$

the inclusion (3.6) holds and for all $\mathbf{w} \in \mathcal{V}_p$,

$$(3.26) \quad \langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu_c p_R (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma dt,$$

where $\mu_c = \mu_c(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p_R = p_R(u_n - w - g)$.

When the triplet $\{\mathbf{u}, \mathbf{v}, w\}$ solves the abstract problem (3.22)–(3.26), then \mathbf{u} and w are a weak solution of (3.1)–(3.10).

The main results in this paper are presented according to whether the wear is a given function or is determined by the differential equation (3.23). To begin with, we consider the following basic result, proved in section 5, in the case of a given wear function. We note that it includes all the published versions of the problem, such as [14, 21] or [15].

THEOREM 3.1. *Let $p \geq 2$ and let $w \in L^p(0, T; L^p(\Gamma_C))$, $w' \in L^p(0, T; L^p(\Gamma_C))$, $w' \geq 0$, $\mathbf{u}_0 \in V_p$, $\mathbf{v}_0 \in H$, $\mathbf{f} \in \mathcal{V}'_p$ and assume $\mu^*(r) = \mu_c(r)$, where μ_c is bounded and Lipschitz. Then there exists $\xi \in L^p(0, T; L^p(\Gamma_C)^N)$ and $\mathbf{v} \in L^2(0, T; W)$ such that*

$$(3.27) \quad (u_n - w - g)_+ \in L^\infty(0, T; L^p(\Gamma_C)),$$

$$(3.28) \quad \mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P(\mathbf{u}, w) + \gamma_T^* \xi = \mathbf{f} \text{ in } \mathcal{V}'_p,$$

$$(3.29) \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds,$$

and

$$(3.30) \quad \langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu_c p_R (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) d\Gamma dt,$$

where $\mu_c = \mu_c(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p_R = p_R(u_n - w - g)$.

If, in addition, $p \leq 4$, then the solution $\{\mathbf{u}, \mathbf{v}\}$ is unique.

Next, we consider the case of a set-valued friction coefficient and given wear function. The proof can be found in section 6.

THEOREM 3.2. *Let $p \geq 2$ and let $\mathbf{u}_0 \in V_p$, $\mathbf{v}_0 \in H$, $\mathbf{f} \in \mathcal{V}'_p$, and $w, w' \in L^p(0, T; L^p(\Gamma_C))$ with $w' \geq 0$. Then there exists a pair $\{\mathbf{v}, \xi\}$ such that*

$$(3.31) \quad \mathbf{v} \in L^2(0, T; W), \quad (u_n - w - g)_+ \in L^\infty(0, T; L^p(\Gamma_C)),$$

$$(3.32) \quad \mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P(\mathbf{u}, w) + \gamma_T^* \xi = \mathbf{f} \text{ in } \mathcal{V}'_p,$$

$$(3.33) \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds,$$

where ξ satisfies the inequality

$$(3.34) \quad \langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu_c p_R (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) d\Gamma dt,$$

and where $\mu_c = \mu_c(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p_R = p_R(u_n - w - g)$, for an element $(|\mathbf{v}_T - \mathbf{v}_*|, \mu(|\mathbf{v}_T - \mathbf{v}_*|))$ from the graph μ^* , a.e., and for all $\mathbf{w} \in \mathcal{V}_p$.

We note that this theorem guarantees only the existence of a solution. Indeed, it seems unreasonable to expect uniqueness when we have a graph in the problem; however, the question remains open.

Finally, we consider the case where the wear is a solution of the differential equation of Archard's law and $\mu^* = \mu_c$. This leads to the following theorem whose proof is in section 8.

THEOREM 3.3. *Assume (3.12) and that $\mu = \mu_c = \mu^*$ and s_c are bounded and Lipschitz continuous. Let $\mathbf{u}_0 \in V_2, \mathbf{v}_0 \in H$, and $\mathbf{f} \in \mathcal{V}'_2$. Then there exists a unique solution $\{\mathbf{u}, \mathbf{v}, w\}$ of problem (3.22)–(3.26), and it satisfies*

$$\mathbf{v} \in L^2(0, T; V_2), \quad \mathbf{v}' \in L^2(0, T; V'_2), \quad w, w' \in L^\infty(0, T; L^\infty(\Gamma_C)).$$

If we wish to take into account the possible dependence of μ and $p(\cdot)$ on the position x on the contact surface, all we need to do is to assume that both functions are measurable in x , in addition to the other assumptions above. This increase in generality is mainly technical and does not change any of the arguments and conclusions that follow. Therefore, we have omitted an explicit reference to it in the models.

Existence of weak solutions for the problem with friction graph and a wear function that is an unknown of the problem remains an important unresolved problem.

4. Approximate problems with given wear. In this section we consider regularized approximate problems in which w is a given function satisfying

$$w \in L^p(0, T; L^p(\Gamma_C)), \quad w' \in L^p(0, T; L^p(\Gamma_C)), \quad w' \geq 0,$$

and $\mu = \mu_c = \mu^*$ is a given Lipschitz continuous function of $|\mathbf{v}_T - \mathbf{v}_*|$.

First, let $\mathbf{u}_{0\varepsilon}$ be a sequence in D satisfying $\lim_{\varepsilon \rightarrow 0} \mathbf{u}_{0\varepsilon} = \mathbf{u}_0$ in V_p . We assume that $q = p^2(p-1)^{-1}$, thus

$$\frac{p-1}{q} + \frac{1}{p} + \frac{1}{q} = 1.$$

Next, let the operator J be defined by

$$(4.1) \quad \langle J\mathbf{u}, \mathbf{v} \rangle = \int_{\Gamma_C} \|\gamma\mathbf{u}\|^{q-2} \gamma\mathbf{u} \cdot \gamma\mathbf{v} \, d\Gamma.$$

We use J to regularize problem (3.22)–(3.26) and for each $\varepsilon > 0$ the approximate problem is the following.

Problem $\mathcal{P}(\varepsilon)$. Find $\mathbf{v}_\varepsilon \in \mathcal{V}_q$ such that

$$(4.2) \quad \mathbf{v}'_\varepsilon + M\mathbf{v}_\varepsilon + A\mathbf{u}_\varepsilon + \varepsilon J\mathbf{v}_\varepsilon + P(\mathbf{u}_\varepsilon, w) + Q(\mathbf{v}_\varepsilon, w) \ni \mathbf{f} \text{ in } \mathcal{V}'_q,$$

$$(4.3) \quad \mathbf{v}_\varepsilon(0) = \mathbf{v}_0 \in H,$$

$$(4.4) \quad \mathbf{u}_\varepsilon(t) = \mathbf{u}_{0\varepsilon} + \int_0^t \mathbf{v}_\varepsilon(s) \, ds.$$

Here, by $\mathbf{v}^* \in Q(\mathbf{v}, w) \subseteq \mathcal{V}'_p$ we mean that there exists $\mathbf{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^N)$ such that

$$(4.5) \quad \langle \mathbf{v}^*, \mathbf{w} \rangle = \int_0^T \int_{\Gamma_C} \mu(|\mathbf{v}_T - \mathbf{v}_*|) p_R(u_n - w - g) \mathbf{z} \cdot \mathbf{w}_T \, d\Gamma dt,$$

and \mathbf{z} satisfies

$$(4.6) \quad \int_0^T \int_{\Gamma_C} \mathbf{z} \cdot \mathbf{w}_T \, d\Gamma dt \leq \int_0^T \int_{\Gamma_C} (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma dt,$$

for all $\mathbf{w} \in \mathcal{V}_p$.

Below we omit the subscript ε for the sake of simplicity. We have the following result for the approximate problems.

THEOREM 4.1. *Assume that $p(\cdot)$ satisfies (3.13). Then for each $\varepsilon > 0$ there exists a solution $\mathbf{v}_\varepsilon \in \mathcal{V}_q$ of $\mathcal{P}(\varepsilon)$.*

The proof of the theorem is accomplished in a number of steps. We begin with the following assertion which follows directly from the definitions.

LEMMA 4.2. *The operators $J, Q, M, A,$ and $P(\cdot, w)$ are bounded maps from \mathcal{V}_q or $\mathcal{V}_q \times L^p(0, T; L^p(\Gamma_C))$ into \mathcal{V}'_q or $\mathcal{P}(\mathcal{V}'_q)$.*

Next, we change the dependent variable and set $\mathbf{y}e^{\lambda t} = \mathbf{v}$. Then, in terms of \mathbf{y} , the problem $\mathcal{P}(\varepsilon)$ consists of finding $\mathbf{y} \in \mathcal{V}_q$ such that

$$(4.7) \quad \begin{aligned} & \mathbf{y}' + \lambda \mathbf{y} + M\mathbf{y} + e^{-\lambda(\cdot)} A\mathbf{u} + \varepsilon e^{-\lambda(\cdot)} J(e^{\lambda(\cdot)} \mathbf{y}) \\ & + e^{-\lambda(\cdot)} P(\mathbf{u}, w) + e^{-\lambda(\cdot)} Q(e^{\lambda(\cdot)} \mathbf{y}, w) \ni e^{-\lambda(\cdot)} \mathbf{f} \quad \text{in } \mathcal{V}'_q, \end{aligned}$$

$$(4.8) \quad \mathbf{y}(0) = \mathbf{v}_0 \in H.$$

Let X be the space given in (2.7). The next lemma will be used to show the operator Q_λ given by

$$(4.9) \quad \mathbf{y} \rightarrow Q_\lambda(\mathbf{y}, w) \equiv e^{-\lambda(\cdot)} Q(e^{\lambda(\cdot)} \mathbf{y}, w)$$

is pseudomonotone.

LEMMA 4.3. *If $\mathbf{v}^k \rightharpoonup \mathbf{v}$ in X , then $\gamma \mathbf{v}^k \rightarrow \gamma \mathbf{v}$ in $L^p(0, T; (L^p(\Gamma_C))^N)$.*

Proof. Since $p \geq 2$, it is straightforward to verify that if $\mathbf{v} \in X$, then $\mathbf{v}' \in L^q(0, T; V'_q)$ and $\mathbf{v}(t_1) - \mathbf{v}(t_2) = \int_{t_2}^{t_1} \mathbf{v}'(s) ds$. Let $W \subseteq U$ be such that the injection $W \rightarrow U$ is compact and $\gamma : U \rightarrow (L^2(\Gamma_C))^N$ is continuous. Since \mathcal{V}_q embeds continuously into $L^2(0, T; W)$, Theorem 2.2 implies that $\mathbf{v}^k \rightarrow \mathbf{v}$ in $L^2(0, T; U)$. It follows that $\gamma \mathbf{v}^k \rightarrow \gamma \mathbf{v}$ in $L^2(0, T; (L^2(\Gamma_C))^N)$. Now if the lemma is not true, then there exists a sequence $\{\mathbf{v}^k\} \subseteq X$ such that $\mathbf{v}^k \rightharpoonup \mathbf{v}$ in X but $\|\gamma \mathbf{v}^k - \gamma \mathbf{v}\|_{L^p(0, T; (L^p(\Gamma_C))^N)} \geq \eta > 0$ for some η . By taking a subsequence, we may assume $\widetilde{\gamma \mathbf{v}^k}(\mathbf{x}, t) \rightarrow \widetilde{\gamma \mathbf{v}}(\mathbf{x}, t)$ a.e. $(\mathbf{x}, t) \in \Gamma_C \times (0, T)$, since $\gamma \mathbf{v}^k \rightarrow \gamma \mathbf{v}$ in $L^2(0, T; (L^2(\Gamma_C))^N)$. Here, “ \sim ” means a product measurable representative. Since $\widetilde{\gamma \mathbf{v}^k}$ is bounded in $(L^q((0, T) \times \Gamma_C))^N$, the Fatou lemma guarantees that $\widetilde{\gamma \mathbf{v}}$ is also bounded in $L^q((0, T) \times \Gamma_C)$. Thus, the sequence $\{|\widetilde{\gamma \mathbf{v}^k} - \widetilde{\gamma \mathbf{v}}|^p\}$ is uniformly integrable, so it follows from the Vitali convergence theorem that

$$\lim_{k \rightarrow \infty} \int_{(0, T) \times \Gamma_C} |\widetilde{\gamma \mathbf{v}^k} - \widetilde{\gamma \mathbf{v}}|^p d\Gamma dt = 0.$$

This contradicts the assumption that $\|\gamma \mathbf{v}^k - \gamma \mathbf{v}\|_{L^p(0, T; (L^p(\Gamma_C))^N)} \geq \eta > 0$ and thus proves the lemma.

LEMMA 4.4. *If $\mathbf{y}^k \rightharpoonup \mathbf{y}$ in X , then*

$$(4.10) \quad p(u_n^k - w - g) \rightarrow p(u_n - w - g) \quad \text{in } L^{p'}(0, T; L^{p'}(\Gamma_C))$$

and

$$(4.11) \quad \mu(|\mathbf{v}_T^k - \mathbf{v}_*|) \rightarrow \mu(|\mathbf{v}_T - \mathbf{v}_*|) \quad \text{in } L^p(0, T; L^p(\Gamma_C)).$$

Proof. To simplify the notation we let $F = p(u_n - w - g)$, $F^k = p(u_n^k - w - g)$, $\mu = \mu(|\mathbf{v}_T - \mathbf{v}_*|)$, and $\mu^k = \mu(|\mathbf{v}_T^k - \mathbf{v}_*|)$. Now it follows from (3.13) that

$$|F^k - F| \leq K \left(1 + |u_n^k|^{p-2} + |u_n|^{p-2} \right) |u_n^k - u_n|.$$

We will show that $|u_n^k|^{p-2}|u_n^k - u_n| \rightarrow 0$ in $L^{p'}(0, T; L^{p'}(\Gamma_C))$ and observe that simpler arguments apply to the other two terms. We have

$$\begin{aligned} & \int_0^T \int_{\Gamma_C} |u_n^k|^{(p-2)p'} |u_n^k - u_n|^{p'} d\Gamma dt \\ & \leq \left(\int_0^T \int_{\Gamma_C} |u_n^k - u_n|^p d\Gamma dt \right)^{p'/p} \left(\int_0^T \int_{\Gamma_C} |u_n^k|^p d\Gamma dt \right)^{(p-p')/p} \\ & \leq c \left(\int_0^T \int_{\Gamma_C} |u_n^k - u_n|^p d\Gamma dt \right)^{p'/p}, \end{aligned}$$

which converges to zero by Lemma 4.3. Moreover,

$$\int_0^T \int_{\Gamma_C} |\mu^k - \mu|^p d\Gamma dt \leq C \operatorname{Lip}_\mu^p \int_0^T \int_{\Gamma_C} |\gamma \mathbf{v}^k - \gamma \mathbf{v}|^p d\Gamma dt,$$

which also converges to zero by Lemma 4.3. The other terms behave similarly.

LEMMA 4.5. *Let $\mathbf{y}^k \rightharpoonup \mathbf{y}$ in X and $\mathbf{z}^k \rightharpoonup \mathbf{z}$ in $L^\infty(0, T; L^\infty(\Gamma_C)^N)$. If $\mathbf{w} \in L^p(0, T; L^p(\Gamma_C)^N)$, then*

$$(4.12) \quad \int_0^T \int_{\Gamma_C} F^k \mu^k \mathbf{z}^k \cdot \mathbf{w}_T d\Gamma dt \rightarrow \int_0^T \int_{\Gamma_C} F \mu \mathbf{z} \cdot \mathbf{w}_T d\Gamma dt.$$

Proof. We argue by contradiction. If (4.12) does not hold, then there exist two sequences $\mathbf{y}^k \rightharpoonup \mathbf{y}$ in X and $\mathbf{z}^k \rightharpoonup \mathbf{z}$ in $L^\infty(0, T; L^\infty(\Gamma_C)^N)$ and $\mathbf{w} \in L^p(0, T; L^p(\Gamma_C)^N)$ such that

$$\left| \int_0^T \int_{\Gamma_C} F^k \mu^k \mathbf{z}^k \cdot \mathbf{w}_T d\Gamma dt - \int_0^T \int_{\Gamma_C} F \mu \mathbf{z} \cdot \mathbf{w}_T d\Gamma dt \right| \geq 2\hat{\varepsilon}.$$

Since $L^\infty(0, T; L^\infty(\Gamma_C)^N)$ is dense in $L^p(0, T; L^p(\Gamma_C)^N)$, we find that, for $\mathbf{w} \in L^\infty(0, T; L^\infty(\Gamma_C)^N)$,

$$(4.13) \quad \left| \int_0^T \int_{\Gamma_C} F^k \mu^k \mathbf{z}^k \cdot \mathbf{w}_T d\Gamma dt - \int_0^T \int_{\Gamma_C} F \mu \mathbf{z} \cdot \mathbf{w}_T d\Gamma dt \right| \geq \hat{\varepsilon}.$$

However, by Lemma 4.4, $\mu(|\mathbf{v}_T^k - \mathbf{v}_*|)p(u_n^k - w - g) \rightarrow \mu(|\mathbf{v}_T - \mathbf{v}_*|)p(u_n - w - g)$ in $L^1(0, T; L^1(\Gamma_C))$. Therefore, (4.13) cannot hold for all k , which proves the lemma.

LEMMA 4.6. Q_λ is a bounded pseudomonotone operator.

Proof. We have already observed that Q_λ is bounded, and it is straightforward to show that $Q_\lambda(\mathbf{y})$ is convex. Suppose that $Q_\lambda(\mathbf{y}) \subseteq U$, where U is a weakly open set in X' , that $\mathbf{y}_k^* \in Q_\lambda(\mathbf{y}) \setminus U$, and that $\mathbf{y}^k \rightharpoonup \mathbf{y}$ in X , where $\mathbf{y}_k^* \in Q_\lambda(\mathbf{y}^k)$. Let $U_\lambda \equiv e^{\lambda(\cdot)}U$; then U_λ is weakly open in X' containing $Q(\mathbf{v})$, $\mathbf{v}^k \rightharpoonup \mathbf{v}$ in X , and $\mathbf{v}_k^* \equiv e^{\lambda(\cdot)}\mathbf{y}_k^* \in Q(\mathbf{v}^k) \setminus U_\lambda$. Next, let $\{\mathbf{z}^k\}$ be a sequence in $L^\infty(0, T; L^\infty(\Gamma_C)^N)$ as in the definition of Q such that, possibly for a subsequence, $\mathbf{z}^k \rightharpoonup \mathbf{z}$ in $L^\infty(0, T; L^\infty(\Gamma_C)^N)$.

From Lemma 4.3,

$$\begin{aligned} \int_0^T \int_{\Gamma_C} \mathbf{z} \cdot \mathbf{w}_T \, d\Gamma dt &= \lim_{k \rightarrow \infty} \int_0^T \int_{\Gamma_C} \mathbf{z}^k \cdot \mathbf{w}_T \, d\Gamma dt \\ &\leq \lim_{k \rightarrow \infty} \int_0^T \int_{\Gamma_C} (|\mathbf{v}_T^k - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T^k - \mathbf{v}_*|) \, d\Gamma dt \\ &\leq \int_0^T \int_{\Gamma_C} (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma dt. \end{aligned}$$

Now, using the notation $p_R^k = p_R(u_n^k - w - g)$ and $p_R = p_R(u_n - w - g)$, we have

$$\langle \mathbf{v}_k^*, \mathbf{w} \rangle \equiv \int_0^T \int_{\Gamma_C} \mu (|\mathbf{v}_T^k - \mathbf{v}_*|) p_R^k \mathbf{z}^k \cdot \mathbf{w}_T \, d\Gamma dt,$$

and so from Lemma 4.5 we know that $\mathbf{v}_k^* \rightharpoonup \mathbf{v}^*$, where

$$(4.14) \quad \langle \mathbf{v}^*, \mathbf{w} \rangle \equiv \int_0^T \int_{\Gamma_C} \mu (|\mathbf{v}_T - \mathbf{v}_*|) p_R \mathbf{z} \cdot \mathbf{w}_T \, d\Gamma dt.$$

Thus, $\mathbf{v}^* \in Q(\mathbf{v}) \subseteq U_\lambda$ by the definition of Q . This contradicts the assumption that $\mathbf{v}_k^* \notin U_\lambda$ for all k , and hence $Q(\mathbf{v}^k) \subseteq U_\lambda$ for all large k . This argument also shows that $Q_\lambda(\mathbf{y})$ is closed. It remains to verify conditions (2.1) and (2.2).

To that end let $\mathbf{y}^k \rightharpoonup \mathbf{y}$ and $\mathbf{y}_k^* \in Q_\lambda(\mathbf{y}^k)$. We show that if $\mathbf{w} \in X$, then

$$\liminf_{k \rightarrow \infty} \langle \mathbf{y}_k^*, \mathbf{y}^k - \mathbf{w} \rangle \geq \langle \mathbf{y}^*(\mathbf{w}), \mathbf{y} - \mathbf{w} \rangle, \quad \mathbf{y}^*(\mathbf{w}) \in Q_\lambda(\mathbf{y}).$$

We choose a subsequence \mathbf{y}^k (depending on \mathbf{w}) such that

$$\lim_{k \rightarrow \infty} \langle \mathbf{y}_k^*, \mathbf{y}^k - \mathbf{w} \rangle = \liminf_{k \rightarrow \infty} \langle \mathbf{y}_k^*, \mathbf{y}^k - \mathbf{w} \rangle.$$

For $\mathbf{v}_k^* = e^{\lambda(\cdot)} \mathbf{y}_k^* \in Q(\mathbf{v}^k)$ we let $\mathbf{z}^k \in L^\infty(0, T; L^\infty(\Gamma_C)^N)$ be as in the definition of Q . We take a further subsequence, if necessary, such that $\mathbf{z}^k \rightharpoonup \mathbf{z}$ in $L^\infty(0, T; L^\infty(\Gamma_C)^N)$. Then \mathbf{z} satisfies (4.6) by Lemma 4.3. It follows from Lemma 4.5 that if we define $\mathbf{y}^*(\mathbf{w})$ by

$$\langle \mathbf{y}^*(\mathbf{w}), \mathbf{b} \rangle = \int_0^T \int_{\Gamma_C} e^{-\lambda t} p_R (u_n - w - g) \mu (|\mathbf{v}_T - \mathbf{v}_*|) \mathbf{z} \cdot \mathbf{b}_T \, d\Gamma dt,$$

then

$$\begin{aligned} \liminf_{k \rightarrow \infty} \langle \mathbf{y}_k^*, \mathbf{y}^k - \mathbf{w} \rangle &= \lim_{k \rightarrow \infty} \langle \mathbf{y}_k^*, \mathbf{y}^k - \mathbf{w} \rangle \\ &= \lim_{k \rightarrow \infty} \int_0^T \int_{\Gamma_C} e^{-\lambda t} \mu^k p_R^k \mathbf{z}^k \cdot (\mathbf{y}_T^k - \mathbf{w}_T) \, d\Gamma dt \\ &= \int_0^T \int_{\Gamma_C} e^{-\lambda t} \mu p_R \mathbf{z} \cdot (\mathbf{y}_T - \mathbf{w}_T) \, d\Gamma dt = \langle \mathbf{y}^*(\mathbf{w}), \mathbf{y} - \mathbf{w} \rangle. \end{aligned}$$

This proves the lemma.

LEMMA 4.7. *If $\mathbf{v}^k \rightharpoonup \mathbf{v}$ in X , then $P(\mathbf{u}^k, w) \rightarrow P(\mathbf{u}, w)$ in \mathcal{V}'_q .*

Proof. Let $\mathbf{w} \in \mathcal{V}_q$. Then we have from the definition of P and (3.13) that

$$\begin{aligned} & |\langle P(\mathbf{u}^k, w) - P(\mathbf{u}, w), \mathbf{w} \rangle| \\ & \leq K \int_0^T \int_{\Gamma_C} (1 + |u_n^k|^{p-2} + |u_n|^{p-2}) |u_n^k - u_n| |w_n| \, d\Gamma \, dt \\ & \leq K \int_0^T \left(\int_{\Gamma_C} (1 + |u_n^k|^p + |u_n|^p) \, d\Gamma \right)^{\frac{p-2}{p}} \left(\int_{\Gamma_C} |u_n^k - u_n|^p \, d\Gamma \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{\Gamma_C} |w_n|^p \, d\Gamma \right)^{\frac{1}{p}} \, dt \\ & \leq K \|u_n^k - u_n\|_{L^p(0,T;L^p(\Gamma_C))^N} \|\mathbf{w}\|_{\mathcal{V}_q}. \end{aligned}$$

Thus, $\|P(\mathbf{u}^k, w) - P(\mathbf{u}, w)\|_{\mathcal{V}_q'} \leq K \|\gamma \mathbf{u}^k - \gamma \mathbf{u}\|_{L^p(0,T;L^p(\Gamma_C))^N}$, and the result follows from Lemma 4.3.

Now for each $\lambda \geq 0$ the map $\mathbf{y} \rightarrow e^{-\lambda(\cdot)} \mathbf{A} \mathbf{u}$ is monotone; in fact,

$$\begin{aligned} (4.15) \quad \langle e^{-\lambda(\cdot)} \mathbf{A}(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{y}_1 - \mathbf{y}_2 \rangle &= \frac{1}{2} \int_0^T e^{-2\lambda t} \frac{d}{dt} \langle \mathbf{A}(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \, dt \\ &= \frac{1}{2} e^{-2\lambda T} \langle \mathbf{A}(\mathbf{u}_1(T) - \mathbf{u}_2(T)), \mathbf{u}_1(T) - \mathbf{u}_2(T) \rangle \\ & \quad + \lambda \int_0^T \langle \mathbf{A}(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle e^{-2\lambda t} \, dt. \end{aligned}$$

Also, the map $\mathbf{y} \rightarrow \varepsilon e^{-\lambda(\cdot)} J(e^{\lambda(\cdot)} \mathbf{y})$ is monotone. Next, $\mathbf{y}^k \rightarrow \mathbf{y}$ in X if and only if $\mathbf{v}^k \rightarrow \mathbf{v}$ in X , and Lemma 4.7 implies that the operator $\mathbf{y} \rightarrow e^{-\lambda(\cdot)} P(\mathbf{u}, w)$ is completely continuous; and if we let

$$(4.16) \quad \begin{aligned} \mathcal{A}_\lambda \mathbf{y} &= \lambda \mathbf{y} + M \mathbf{y} + e^{-\lambda(\cdot)} \mathbf{A}(\mathbf{u}) + \varepsilon e^{-\lambda(\cdot)} J(e^{\lambda(\cdot)} \mathbf{y}) \\ & \quad + e^{-\lambda(\cdot)} Q(e^{\lambda(\cdot)} \mathbf{y}) + e^{-\lambda(\cdot)} P(\mathbf{u}, w), \end{aligned}$$

then \mathcal{A}_λ is a sum of bounded pseudomonotone operators. Consequently, $\mathcal{A}_\lambda : X \rightarrow \mathcal{P}(X')$ is pseudomonotone [22], verifying condition (2.15) for \mathcal{A}_λ . We now check the coercivity of \mathcal{A}_λ (2.14). To this end, we consider the various terms of $\langle \mathcal{A}_\lambda \mathbf{y}, \mathbf{y} \rangle$. Let $\mathbf{y}^* \in Q_\lambda(\mathbf{y})$, which implies that $\mathbf{y}^* \in e^{-\lambda(\cdot)} Q(e^{\lambda(\cdot)} \mathbf{v})$ and so $\mathbf{y}^* = e^{-\lambda(\cdot)} \mathbf{v}^*$, where $\mathbf{v}^* \in Q(e^{\lambda(\cdot)} \mathbf{v})$. Therefore,

$$\langle \mathbf{y}^*, \mathbf{y} \rangle = \langle e^{-\lambda(\cdot)} \mathbf{v}^*, e^{\lambda(\cdot)} \mathbf{v} \rangle = \langle \mathbf{v}^*, \mathbf{v} \rangle = \int_0^T \int_{\Gamma_C} \mu p_R \mathbf{z} \cdot \mathbf{v}_T \, d\Gamma \, dt,$$

where $p_R = p_R(u_n - w - g)$ and $\mathbf{z} \in L^\infty(0, T; L^\infty(\Gamma_C)^N)$ satisfies

$$(4.17) \quad \int_0^T \int_{\Gamma_C} \mathbf{z} \cdot \mathbf{w}_T \, d\Gamma \, dt \leq \int_0^T \int_{\Gamma_C} (|e^{\lambda(\cdot)} \mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |e^{\lambda(\cdot)} \mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma \, dt,$$

and $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t e^{\lambda s} \mathbf{v}(s) \, ds$. Thus,

$$(4.18) \quad \begin{aligned} \langle \mathbf{y}^*, \mathbf{y} \rangle &= \int_0^T \int_{\Gamma_C} e^{-\lambda t} \mu p_R \mathbf{z} \cdot (e^{\lambda t} \mathbf{v}_T - \mathbf{v}_*) \, d\Gamma \, dt \\ & \quad + \int_0^T \int_{\Gamma_C} e^{-\lambda t} \mu p_R \mathbf{z} \cdot \mathbf{v}_* \, d\Gamma \, dt. \end{aligned}$$

Now the first integral is nonnegative by a routine argument involving (4.17), and since $\mathbf{v}_* \in L^\infty(0, T; L^p(\Gamma_C)^N)$ we have that the second integral is bounded below by

$$\begin{aligned}
 & -c - c \int_0^T \left(\int_{\Gamma_C} (u_n - w - g)_+^p d\Gamma \right)^{1/p} dt \\
 (4.19) \quad & \geq -c_\eta - \eta \int_0^T \int_0^t |v_n(s) - w_t(s)|_{L^p(\Gamma_C)}^p ds dt
 \end{aligned}$$

for $\eta > 0$. Next, we examine the term $\langle e^{-\lambda(\cdot)} P(\mathbf{u}, w), \mathbf{y} \rangle$. Let $h(r, \mathbf{x}) = \int_{g(\mathbf{x})}^r p(s - g(\mathbf{x})) ds$ and define $H : L^2(\Gamma_C) \rightarrow [0, \infty)$ by

$$(4.20) \quad H(u) = \int_{\Gamma_C} h(u, \mathbf{x}) d\Gamma.$$

Then

$$\begin{aligned}
 & \frac{d}{dt} H(u_n - w) = \langle DH(u_n - w), v_n - w' \rangle \\
 (4.21) \quad & = \int_{\Gamma_C} p(u_n - w - g) (v_n - w') d\Gamma = \langle P(\mathbf{u}, w), \mathbf{v} \rangle - \int_{\Gamma_C} p(u_n - w - g) w' d\Gamma.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \langle e^{-\lambda(\cdot)} P(\mathbf{u}, w), \mathbf{y} \rangle = \int_0^T e^{-2\lambda t} \langle P(\mathbf{u}, w), \mathbf{v} \rangle dt \\
 (4.22) \quad & = \int_0^T e^{-2\lambda t} \frac{d}{dt} H(u_n - w) dt + \int_0^T \int_{\Gamma_C} p(u_n - w - g) w' d\Gamma \\
 & \geq H(u_n(T) - w(T)) e^{-2\lambda T} - H(u_{0\varepsilon n}) + 2\lambda \int_0^T H(\mathbf{u}) e^{-2\lambda t} dt,
 \end{aligned}$$

due to the assumptions that $w' \geq 0$ and $p(\cdot) \geq 0$. Similarly,

$$\begin{aligned}
 & \langle e^{-\lambda(\cdot)} A\mathbf{u}, \mathbf{y} \rangle = \frac{1}{2} \langle A\mathbf{u}(T), \mathbf{u}(T) \rangle e^{-2\lambda T} \\
 (4.23) \quad & - \frac{1}{2} \langle A\mathbf{u}_{0\varepsilon}, \mathbf{u}_{0\varepsilon} \rangle + \lambda \int_0^T \langle A\mathbf{u}, \mathbf{u} \rangle e^{-2\lambda t} dt.
 \end{aligned}$$

It follows from (4.19), (4.22), and (4.23) that

$$\begin{aligned}
 \langle \mathcal{A}_\lambda \mathbf{y}, \mathbf{y} \rangle & \geq \delta^2 \|\mathbf{y}\|_{L^2(0, T; W)}^2 + \varepsilon e^{-2\lambda T} \|\gamma \mathbf{y}\|_{L^q(0, T; (L^q(\Gamma_C))^N)}^q \\
 & - c_\eta - \eta \int_0^T \int_0^t |v_n(s) - w'(s)|_{L^p(\Gamma_C)}^p ds dt - H(u_{0\varepsilon n}).
 \end{aligned}$$

We conclude that \mathcal{A}_λ is coercive when η is sufficiently small and by Lemma 4.2 that $\mathcal{A}_\lambda : \mathcal{V}_q \rightarrow \mathcal{V}'_q$ is bounded. All the assumptions of Theorem 2.6 are satisfied now, and the proof of Theorem 4.1 is complete.

We use this result in the following section. However, we note that the theorem has merit of its own.

5. Existence and uniqueness. We obtain a solution for problem \mathcal{P} , when w is a known function, by deriving estimates on the solutions of $\mathcal{P}(\varepsilon)$ and passing to the limit $\varepsilon \rightarrow 0$, thus proving Theorem 3.1. We are still assuming that μ is Lipschitz continuous.

The proof of Theorem 3.1 is accomplished in a number of steps. We denote by c a generic positive constant which is independent of ε . Multiplying both sides of (4.2) by $\mathbf{v}\chi_{[0,t]}$ and using the above formulas along with the assumption that $w' \geq 0$, and performing routine manipulations, we obtain the following estimates for $\mathbf{v}^* \in Q\mathbf{v}$:

$$\begin{aligned} & \frac{1}{2}|\mathbf{v}(t)|_H^2 - \frac{1}{2}|\mathbf{v}_0|_H^2 + \delta^2 \int_0^t \|\mathbf{v}\|_W^2 ds + \frac{1}{2}\langle A\mathbf{u}(t), \mathbf{u}(t) \rangle \\ & + \varepsilon \int_0^t \int_{\Gamma_C} |\gamma \mathbf{v}|^q d\Gamma ds + \langle \mathbf{v}^*, \mathbf{v}\chi_{[0,t]} \rangle + H(u_n(t) - w(t)) - H(u_{0\varepsilon n}) \\ (5.1) \quad & \leq \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s) \rangle ds + \frac{1}{2}\langle A\mathbf{u}_{0\varepsilon}, \mathbf{u}_{0\varepsilon} \rangle. \end{aligned}$$

Now, when $\lambda = 0$ in (4.18), we obtain

$$\langle \mathbf{v}^*, \mathbf{v}\chi_{[0,t]} \rangle \geq -c - c \int_0^t \int_{\Gamma_C} (u_n - w - g)_+^p d\Gamma,$$

thus

$$\begin{aligned} & \frac{1}{2}|\mathbf{v}(t)|_H^2 + \delta^2 \int_0^t \|\mathbf{v}\|_W^2 ds + \frac{1}{2}\langle A\mathbf{u}(t), \mathbf{u}(t) \rangle + \varepsilon \int_0^t \int_{\Gamma_C} |\gamma \mathbf{v}|^q d\Gamma ds \\ & + \int_{\Gamma_C} h(u_n(t, \mathbf{x}) - w(t, \mathbf{x}), \mathbf{x}) d\Gamma \leq c + \frac{1}{2}|\mathbf{v}_0|_H^2 + \frac{1}{2}\langle A\mathbf{u}_{0\varepsilon}, \mathbf{u}_{0\varepsilon} \rangle + H(u_{0\varepsilon n}) \\ (5.2) \quad & + \frac{1}{2\delta^2} \int_0^t \|\mathbf{f}(s)\|_W^2 ds + \frac{\delta^2}{2} \int_0^t \|\mathbf{v}(s)\|_W^2 ds + c \int_0^t \int_{\Gamma_C} (u_n - w - g)_+^p d\Gamma ds. \end{aligned}$$

The assumptions on $p(\cdot)$ given in (3.13) imply that if $r \geq g(\mathbf{x})$, then

$$(5.3) \quad h(r, \mathbf{x}) \geq \int_{g(\mathbf{x})}^r (\delta^2(s - g)_+^{p-1} - c) ds = \frac{\delta^2}{p}(r - g(\mathbf{x}))_+^p - c(r - g(\mathbf{x}))_+.$$

Now, since $p(r) = 0$ for $r \leq 0$, (5.3) holds also when $r < g(\mathbf{x})$. Then (5.2) yields

$$\begin{aligned} & |\mathbf{v}(t)|_H^2 + \delta^2 \int_0^t \|\mathbf{v}\|_W^2 ds + \langle A\mathbf{u}(t), \mathbf{u}(t) \rangle + 2\varepsilon \int_0^t \int_{\Gamma_C} |\gamma \mathbf{v}|^q d\Gamma ds \\ & + \frac{2\delta^2}{p} \int_{\Gamma_C} (u_n(t) - w(t) - g)_+^p d\Gamma - 2c \int_{\Gamma_C} (u_n(t) - w(t) - g)_+ d\Gamma \\ & \leq c + |\mathbf{v}_0|_H^2 + \langle A\mathbf{u}_{0\varepsilon}, \mathbf{u}_{0\varepsilon} \rangle + 2H(\mathbf{u}_{0\varepsilon n}) + \frac{1}{\delta^2} \int_0^t \|\mathbf{f}(s)\|_W^2 ds \\ (5.4) \quad & + c \int_0^t \int_{\Gamma_C} (u_n - w - g)_+^p d\Gamma ds. \end{aligned}$$

Applying the Hölder inequality to the sixth term on the right-hand side we obtain

$$\begin{aligned}
 |\mathbf{v}(t)|_H^2 + \delta^2 \int_0^t \|\mathbf{v}\|_W^2 ds + \langle A\mathbf{u}(t), \mathbf{u}(t) \rangle + 2\varepsilon \int_0^t \int_{\Gamma_C} |\gamma\mathbf{v}|^q d\Gamma ds \\
 + \frac{\delta^2}{p} \int_{\Gamma_C} (u_n(t) - w - g)_+^p d\Gamma \leq c + |\mathbf{v}_0|^2 + \langle A\mathbf{u}_{0\varepsilon}, \mathbf{u}_{0\varepsilon} \rangle + 2H(\mathbf{u}_{0\varepsilon}) \\
 (5.5) \quad + \frac{1}{\delta^2} \int_0^t \|\mathbf{f}(s)\|_{W'}^2 ds + c \int_0^t \int_{\Gamma_C} (u_n - w - g)_+^p d\Gamma ds.
 \end{aligned}$$

Now using the Gronwall inequality yields

$$\begin{aligned}
 |\mathbf{v}(t)|_H^2 + \int_0^t \|\mathbf{v}\|_W^2 ds + \langle A\mathbf{u}(t), \mathbf{u}(t) \rangle + \varepsilon \int_0^t \int_{\Gamma_C} |\gamma\mathbf{v}|^q d\Gamma ds \\
 (5.6) \quad + \int_{\Gamma_C} (u_n(t) - w - g)_+^p d\Gamma \leq c,
 \end{aligned}$$

where c does not depend on ε , q (for $q > p$) or w . If $\mathbf{w} \in \mathcal{V}_q$, then (5.6) and the definition of J imply

$$\begin{aligned}
 |\langle \varepsilon J\mathbf{v}, \mathbf{w} \rangle| &\leq \varepsilon \langle J\mathbf{v}, \mathbf{v} \rangle^{(1/q')} \langle J\mathbf{w}, \mathbf{w} \rangle^{(1/q)} \\
 (5.7) \quad &\leq (\varepsilon \langle J\mathbf{v}, \mathbf{v} \rangle)^{(1/q')} \varepsilon^{(1/q)} \|\mathbf{w}\|_{\mathcal{V}_q} \leq c\varepsilon^{(1/q)} \|\mathbf{w}\|_{\mathcal{V}_q}.
 \end{aligned}$$

Thus, when \mathbf{v}_ε is a solution of problem $\mathcal{P}(\varepsilon)$ we have

$$(5.8) \quad \varepsilon J\mathbf{v}_\varepsilon \rightarrow 0 \quad \text{in } \mathcal{V}'_q.$$

From (5.6) and the growth conditions for $p(\cdot)$ we find that $Q(\mathbf{v}_\varepsilon, w)$ and $P(\mathbf{u}_\varepsilon, w)$ are bounded in $\mathcal{V}'_p \subseteq \mathcal{V}'_q$. Using Theorems 2.2 and 2.3 we find that there exists a subsequence, still denoted by $\varepsilon \rightarrow 0$, such that

$$(5.9) \quad \mathbf{v}_\varepsilon \rightarrow \mathbf{v} \quad \text{weakly in } L^2(0, T; W),$$

$$(5.10) \quad \mathbf{v}'_\varepsilon \rightarrow \mathbf{v}' \quad \text{in } \mathcal{V}'_q,$$

$$(5.11) \quad \mathbf{u}_\varepsilon \rightarrow \mathbf{u} \quad \text{in } C(0, T; U),$$

$$(5.12) \quad \mathbf{v}_\varepsilon \rightarrow \mathbf{v} \quad \text{in } L^2(0, T; U),$$

$$(5.13) \quad M\mathbf{v}_\varepsilon \rightarrow M\mathbf{v} \quad \text{weakly in } L^2(0, T; W'),$$

$$(5.14) \quad A\mathbf{u}_\varepsilon \rightarrow A\mathbf{u} \quad \text{weakly in } L^2(0, T; W').$$

Here U denotes a space containing W with compact identity map and such that the trace map $\gamma : U \rightarrow L^2(\Gamma_C)^N$ is continuous. Letting \mathbf{z}_ε be as in (4.5) and (4.6), (5.11) and (5.12) imply that, for a subsequence,

$$(5.15) \quad \widetilde{\gamma\mathbf{u}}_\varepsilon(\mathbf{x}, t) \rightarrow \widetilde{\gamma\mathbf{u}}(\mathbf{x}, t) \quad \text{a.e. in } \Gamma_C \times (0, T),$$

$$(5.16) \quad \widetilde{\gamma\mathbf{v}}_\varepsilon(\mathbf{x}, t) \rightarrow \widetilde{\gamma\mathbf{v}}(\mathbf{x}, t) \quad \text{a.e. in } \Gamma_C \times (0, T),$$

$$\begin{aligned}
 \mu(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) p_R(u_{\varepsilon n} - w - g) \mathbf{z}_\varepsilon \rightharpoonup \xi \\
 (5.17) \quad \text{weakly in } L^{p'}(0, T; L^{p'}(\Gamma_C)^N).
 \end{aligned}$$

LEMMA 5.1. $P(\mathbf{u}_\varepsilon, w) \rightarrow P(\mathbf{u}, w)$ in \mathcal{V}'_q and $P(\mathbf{u}_\varepsilon, w) \rightharpoonup P(\mathbf{u}, w)$ weakly in \mathcal{V}'_p .

Proof. Let $\mathbf{w} \in \mathcal{V}_q$; then, by (3.13),

$$\begin{aligned} |\langle P(\mathbf{u}_\varepsilon, w) - P(\mathbf{u}, w), \mathbf{w} \rangle| &\leq \int_0^T \int_{\Gamma_C} K(1 + (u_{\varepsilon n} - w - g)_+^{p-2} + (u_n - w - g)_+^{p-2}) \\ &\quad \times |(u_{\varepsilon n} - w - g)_+ - (u_n - w - g)_+| |w_n| d\Gamma dt \\ &\leq c \int_0^T \left(\int_{\Gamma_C} (1 + (u_{\varepsilon n} - w - g)_+^p + (u_n - w - g)_+^p) d\Gamma \right)^{\frac{p-2}{p}} \\ &\quad \times \left(\int_{\Gamma_C} |(u_{\varepsilon n} - w - g)_+ - (u_n - w - g)_+|^r d\Gamma \right)^{\frac{1}{r}} \cdot \left(\int_{\Gamma_C} |w_n|^q d\Gamma \right)^{\frac{1}{q}} dt, \end{aligned}$$

where $r = pq(2q - p)^{-1}$. It follows from (5.6) that

$$(5.18) \quad \begin{aligned} &|\langle P(\mathbf{u}_\varepsilon, w) - P(\mathbf{u}, w), \mathbf{w} \rangle| \\ &\leq c \left(\int_0^T \int_{\Gamma_C} |(u_{\varepsilon n} - w - g)_+ - (u_n - w - g)_+|^r d\Gamma dt \right)^{\frac{1}{r}} \|\mathbf{w}\|_{\mathcal{V}_q}. \end{aligned}$$

Now note that $r < p$ and so estimate (5.6) implies the functions $|(u_\varepsilon - w - g)_+ - (u_n - w - g)_+|^r$ are uniformly integrable. Then (5.15) and the Vitali convergence theorem imply

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma_C} |(u_{\varepsilon n} - w - g)_+ - (u_n - w - g)_+|^r d\Gamma dt = 0.$$

Now

$$\begin{aligned} &\|P(\mathbf{u}_\varepsilon, w) - P(\mathbf{u}, w)\|_{\mathcal{V}'_q} \\ &\leq c \left(\int_0^T \int_{\Gamma_C} |(u_{\varepsilon n} - w - g)_+ - (u_n - w - g)_+|^r d\Gamma dt \right)^{\frac{1}{r}}, \end{aligned}$$

and hence $P(\mathbf{u}_\varepsilon, w) \rightarrow P(\mathbf{u}, w)$ in \mathcal{V}'_q .

To obtain the other assertion, we note that $P(\mathbf{u}_\varepsilon, w)$ is bounded in \mathcal{V}'_p , and therefore it has a convergent subsequence such that $P(\mathbf{u}_\varepsilon, w) \rightharpoonup \ell$ weakly in \mathcal{V}'_p . However, \mathcal{V}_q is dense in \mathcal{V}_p and so $\ell = P(\mathbf{u}, w)$. Since this holds for every weakly convergent subsequence, it follows that $P(\mathbf{u}_\varepsilon, w) \rightharpoonup P(\mathbf{u}, w)$.

LEMMA 5.2. *For each $\mathbf{w} \in \mathcal{V}_p$,*

$$(5.19) \quad \langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu p_R (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) d\Gamma dt,$$

where $\mu(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p_R = p_R(u_n - w - g)$.

Proof. To simplify the notation we let $F = p_R(u_n - w - g)$, $F_\varepsilon = p_R(u_{\varepsilon n} - w - g)$, $\mu = \mu(|\mathbf{v}_T - \mathbf{v}_*|)$, and $\mu_\varepsilon = \mu(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|)$. First suppose that $\mathbf{w} \in \mathcal{V}_q$. It follows from the assumptions on \mathbf{z}_ε that $\mathbf{z}_\varepsilon \cdot \mathbf{w}_T \leq (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|)$ for a.e. t and a.e. \mathbf{x} . Therefore,

$$(5.20) \quad \begin{aligned} \langle \gamma_T^* \xi, \mathbf{w} \rangle &= \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma_C} F_\varepsilon \mu_\varepsilon \mathbf{z}_\varepsilon \cdot \mathbf{w}_T d\Gamma dt \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma_C} F_\varepsilon \mu_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) d\Gamma dt. \end{aligned}$$

Now the integrand converges pointwise to $F\mu(|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|)$ and is bounded in absolute value by $c(1 + (u_{\varepsilon n} - w - g)_+^{p-1})|\mathbf{w}_T|$. These functions are bounded in $L^r((0, T) \times \Gamma_C)$, independently of ε , where $r \equiv pq/(pq + p - q)$. Indeed, $(p-1)r q/(q-r) = p$, and thus

$$\begin{aligned} (u_{\varepsilon n} - w - g)_+^{(p-1)r} |\mathbf{w}_T|^r &\leq (u_{\varepsilon n} - w - g)_+^{\frac{(p-1)r q}{q-r}} + |\mathbf{w}_T|^q \\ &= (u_{\varepsilon n} - w - g)_+^p + |\mathbf{w}_T|^q, \end{aligned}$$

which is bounded in L^1 , independent of ε . Therefore, using the Vitali convergence theorem in (5.20), we may pass to the limit and obtain (5.19) for all $\mathbf{w} \in \mathcal{V}_q$, and since \mathcal{V}_q is dense in \mathcal{V}_p this inequality holds for all $\mathbf{w} \in \mathcal{V}_p$. This proves the lemma.

Next, from (4.2), (5.13), (5.14), (5.9), and Lemma 5.1 we obtain

$$\mathbf{v}' + M\mathbf{v} + A\mathbf{u} + \gamma_T^* \xi + P(\mathbf{u}, w) = \mathbf{f} \text{ in } \mathcal{V}'_q.$$

Since $\gamma_T^* \xi$, $A\mathbf{u}$, $M\mathbf{v}$ and \mathbf{f} are all in \mathcal{V}'_p , so is \mathbf{v}' . This proves the existence part of the theorem.

Proof of uniqueness. Suppose \mathbf{v}_1 and \mathbf{v}_2 are two solutions of \mathcal{P} . Let, for $i = 1, 2$, $\mathbf{u}_i(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_i(s) ds$. It follows that

$$\begin{aligned} &\frac{1}{2} |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \int_0^t \langle M\mathbf{v}_1 - M\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle ds \\ &\quad + \int_0^t \langle A(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{v}_1 - \mathbf{v}_2 \rangle ds + \int_0^t \langle \gamma_T^* \xi_1 - \gamma_T^* \xi_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle ds \\ (5.21) \quad &\quad + \int_0^t \langle P(\mathbf{u}_1, w) - P(\mathbf{u}_2, w), \mathbf{v}_1 - \mathbf{v}_2 \rangle ds = 0. \end{aligned}$$

Thus, if we denote by c a positive generic constant, we have

$$\begin{aligned} &\frac{1}{2} |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \frac{1}{2} \langle A(\mathbf{u}_1(t) - \mathbf{u}_2(t)), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle \\ &\quad + \int_0^t \langle P(\mathbf{u}_1, w) - P(\mathbf{u}_2, w), \mathbf{v}_1 - \mathbf{v}_2 \rangle ds + \delta^2 \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 ds \\ (5.22) \quad &\quad + \int_0^t \langle \gamma_T^* \xi_1 - \gamma_T^* \xi_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle ds \leq c \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 ds. \end{aligned}$$

Let $F^i = p_R(u_{in} - w - g)$, $\mu^i = \mu(|\mathbf{v}_{iT} - \mathbf{v}_*|)$, for $i = 1, 2$; then using condition (5.20) we observe

$$\begin{aligned} &\int_0^t \langle \gamma_T^* \xi_1 - \gamma_T^* \xi_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle ds \\ &\quad \geq \int_0^t \int_{\Gamma_C} (F^1 \mu^1 - F^2 \mu^2) (|\mathbf{v}_{1T} - \mathbf{v}_*| - |\mathbf{v}_{2T} - \mathbf{v}_*|) d\Gamma ds. \end{aligned}$$

Consequently, the last term on the left-hand side in (5.22) dominates

$$(5.23) \quad -c \int_0^t \int_{\Gamma_C} F^2 |\mathbf{v}_{1T} - \mathbf{v}_{2T}|^2 d\Gamma ds - c \int_0^t \int_{\Gamma_C} |F^1 - F^2| |\mathbf{v}_{1T} - \mathbf{v}_{2T}| d\Gamma ds.$$

The third term in (5.22) is greater than or equal to

$$(5.24) \quad - \int_0^t \int_{\Gamma_C} |p(u_{1n} - w - g) - p(u_{2n} - w - g)| |v_{1n} - v_{2n}| d\Gamma ds.$$

From the assumptions on $p(\cdot)$ and from (5.22) we obtain

$$\begin{aligned} & |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \langle A(\mathbf{u}_1(t) - \mathbf{u}_2(t)), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle + \delta^2 \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 ds \\ & \leq c \int_0^t \int_{\Gamma_C} (1 + |(u_{1n} - w - g)_+|^2 + |(u_{2n} - w - g)_+|^2) \\ & \quad \times |u_{1n} - u_{2n}| |v_{1n} - v_{2n}| d\Gamma ds \\ & \quad + c \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 ds + c \int_0^t \int_{\Gamma_C} |\mathbf{v}_{1T} - \mathbf{v}_{2T}|^2 d\Gamma ds. \end{aligned}$$

Since $(u_n - w - g)_+ \in L^\infty(0, T; L^p(\Gamma_C))$, we obtain, with another c which depends on \mathbf{u}_1 and \mathbf{u}_2 ,

$$\begin{aligned} & |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \delta^2 \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 ds \leq c \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_U^2 dt \\ & \quad + c \int_0^t \left(\int_{\Gamma_C} |u_{1n} - u_{2n}|^4 d\Gamma \right)^{\frac{1}{4}} \left(\int_{\Gamma_C} |v_{1n} - v_{2n}|^4 d\Gamma \right)^{\frac{1}{4}} ds \\ & \quad + c \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 ds \\ & \quad \leq c \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_W \|\mathbf{v}_1 - \mathbf{v}_2\|_W ds + c \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 ds \\ & \quad + K \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_U^2 dt, \end{aligned}$$

where we used the fact that the trace map $W \rightarrow L^4(\partial\Omega)$ is continuous. It follows from the compactness of the embedding $U \rightarrow W$ that

$$\begin{aligned} & |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \frac{\delta^2}{2} \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 ds \\ & \leq c_{\delta T} \int_0^t \int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 dr ds + K_\varepsilon \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 ds \\ & \quad + \varepsilon \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 dt. \end{aligned}$$

Choosing $\varepsilon = \frac{\delta^2}{4}$ and adjusting the constants yields

$$\begin{aligned} & |\mathbf{v}_1(t) - \mathbf{v}_2(t)|_H^2 + \frac{\delta^2}{4} \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 ds \\ & \leq c_{\delta T} \int_0^t \left(\int_0^s \|\mathbf{v}_1 - \mathbf{v}_2\|_W^2 dr + |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_H^2 \right) ds. \end{aligned}$$

By the Gronwall inequality we obtain $\mathbf{v}_1 = \mathbf{v}_2$. This concludes the proof of Theorem 3.1 in the case that p satisfies (3.13).

In the case when $p(\cdot)$ satisfies (3.12) the proof is much easier, not requiring the consideration of the approximate problems where εJ was added in.

THEOREM 5.3. *Let $p \geq 2$ and let $w \in L^p(0, T; L^p(\Gamma_C))$, $w' \in L^p(0, T; L^p(\Gamma_C))$, $w' \geq 0$, $\mathbf{u}_0 \in V_p$, $\mathbf{v}_0 \in H$, $\mathbf{f} \in \mathcal{V}'_p$ and assume $\mu^*(r) = \mu_c(r)$, where μ_c is bounded and Lipschitz. Then there exists $\xi \in L^{p'}(0, T; L^{p'}(\Gamma_C)^N)$ and $\mathbf{v} \in L^2(0, T; W)$ such that*

$$(5.25) \quad (u_n - w - g)_+ \in L^\infty(0, T; L^p(\Gamma_C)),$$

$$(5.26) \quad \mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P(\mathbf{u}, w) + \gamma_T^* \xi = \mathbf{f} \text{ in } \mathcal{V}'_p,$$

$$(5.27) \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) \, ds,$$

and

$$(5.28) \quad \langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu p_R(|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma \, dt,$$

where $\mu = \mu(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p_R = p_R(u_n - w - g)$.

Moreover, if the function $p(\cdot)$ satisfies (3.12), the solution $\{\mathbf{u}, \mathbf{v}\}$ is unique.

We note that (3.28) and the fact that $\mathbf{v}_1 = \mathbf{v}_2$ imply $\gamma_T^* \xi_1 = \gamma_T^* \xi_2$; however, we do not know if $\xi_1 = \xi_2$.

6. Discontinuous friction coefficient. In this section we consider the case when the coefficient of friction is a discontinuous function of the slip speed and establish Theorem 3.2. This is the case often described in elementary courses where it is stated that the coefficient of sliding friction is smaller than the coefficient of static friction. Therefore, we assume that the function μ has a jump discontinuity at zero, becoming smaller when slip takes place, and is represented by the friction graph μ^* (3.5).

To investigate this case when p satisfies (3.13), we regularize the graph μ^* by defining $\mu_\varepsilon(r) = \mu_d$ for all $r \leq 0$ and

$$\mu_\varepsilon(r) = \mu_c(r) - h'_\varepsilon(r) + \eta,$$

where $2\eta = \mu_s - \mu_d$ and $h_\varepsilon(r) \equiv (\eta^2 r^2 + \varepsilon)^{1/2}$, for $0 < \varepsilon$ small. Thus, η is half the size of the jump at 0 between μ_d and μ_s . From this definition, it follows that

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(r) = \begin{cases} \mu_c(r) & \text{if } r > 0, \\ \mu_c(r) + 2\eta = \mu_s & \text{if } r < 0, \\ \mu_d + \eta & \text{if } r = 0 \end{cases}$$

which is a function whose graph has a jump of height $2\eta = \mu_s - \mu_d$ at $r = 0$.

Let \mathbf{v}_ε be the solution of the approximate problem (4.2)–(4.6) in which μ is replaced with μ_ε . Then, estimate (5.6) holds for \mathbf{v}_ε and, consequently, there exists a subsequence such that (5.9)–(5.17) hold. Passing to a further subsequence if necessary, we may assume there exists $\psi \in L^\infty(0, T; L^\infty(\Gamma_C))$ such that

$$h'_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) \rightarrow \psi \text{ weak } * \text{ in } L^\infty(0, T; L^\infty(\Gamma_C)).$$

We note that Lemma 5.1 still holds. As above, we let $F = p_R(u_n - w - g)$ and

$F_\varepsilon = p_R(u_{\varepsilon n} - w - g)$. Let $\mathbf{w} \in \mathcal{V}_q$,

$$\begin{aligned} \langle \gamma_T^* \xi, \mathbf{w} \rangle &= \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma_C} F_\varepsilon \mu_\varepsilon \mathbf{z}_\varepsilon \cdot \mathbf{w}_T \, d\Gamma dt \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma_C} F_\varepsilon \mu_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) \, d\Gamma dt \\ &= \liminf_{\varepsilon \rightarrow 0} \left[\int_0^T \int_{\Gamma_C} F_\varepsilon (\mu_c (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) - \psi + \eta) \right. \\ &\quad \times (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) \, d\Gamma dt \\ &\quad + \int_0^T \int_{\Gamma_C} F_\varepsilon (\psi - h'_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|)) \\ &\quad \left. \times (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) \, d\Gamma dt \right]. \end{aligned}$$

As in the proof of Lemma 5.2, the first integral on the right-hand side converges to

$$\int_0^T \int_{\Gamma_C} (\mu_c (|\mathbf{v}_T - \mathbf{v}_*|) - \psi + \eta) p_R (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma dt,$$

where $p_R = p_R(u_n - w - g)$. We need to show that the second integral converges to zero. This follows from the observation that, since p_R is bounded,

$$|p_R(u_{\varepsilon n} - w - g)(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|)|$$

is bounded in $L^2((0, T) \times \Gamma_C)$, independently of ε , and converges pointwise to $|F (|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|)|$, which lies in $L^2((0, T) \times \Gamma_C)$. Thus, the sequence is uniformly integrable, and by the Vitali convergence theorem it converges strongly in $L^1((0, T) \times \Gamma_C)$. Since $\psi - h'_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|)$ converges weak* in L^∞ to zero, the second integral converges to zero as desired. Next, we consider ψ .

First, note that, from the convexity of h_ε ,

$$h'_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|) z \leq h_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*| + z) - h_\varepsilon (|\mathbf{v}_{\varepsilon T} - \mathbf{v}_*|),$$

thus for arbitrary $z \in L^1(0, T; L^1(\Gamma_C))$

$$\int_0^T \int_{\Gamma_C} \psi z \, d\Gamma dt \leq \int_0^T \int_{\Gamma_C} |\eta (|\mathbf{v}_T - \mathbf{v}_*| + z)| - |\eta (|\mathbf{v}_T - \mathbf{v}_*|)| \, d\Gamma dt$$

which implies that, for a.e. t ,

$$\psi z \leq |\eta (|\mathbf{v}_T - \mathbf{v}_*| + z)| - |\eta (|\mathbf{v}_T - \mathbf{v}_*|)|$$

for a.e. \mathbf{x} . Letting $\theta(r) \equiv |\eta r|$, it follows that, for a.e. \mathbf{x}, t ,

$$\psi(t, \mathbf{x}) \in \partial\theta (|\mathbf{v}_T - \mathbf{v}_*|(t, \mathbf{x})).$$

Therefore, for a.e. t, \mathbf{x} ,

$$\psi(t, \mathbf{x}) \in [-\eta, \eta].$$

More particularly, if $|\mathbf{v}_T - \mathbf{v}_*| > 0$, $\psi = \eta$, while if $|\mathbf{v}_T - \mathbf{v}_*| = 0$, the above holds. Therefore, the pair

$$(|\mathbf{v}_T - \mathbf{v}_*|, \mu_c(|\mathbf{v}_T - \mathbf{v}_*|) - \psi + \eta)$$

is an element of the graph of μ^* , a.e. The proof of Theorem 3.2 is now complete in the case where $p(\cdot)$ satisfies (3.13). Theorem 3.2 holds in the case where $p(\cdot)$ satisfies (3.12) from arguments similar to the above but without the necessity of dealing with the limit as $\varepsilon \rightarrow 0$ in the solutions of the approximate problems in which $\varepsilon J\mathbf{v}_\varepsilon$ was added.

The uniqueness of the solution remains an open question.

7. Dependence on w . In this section we investigate the dependence of the solution of (3.27)–(3.30) on w in the situation of (3.12) and $\mu^* = \mu = \mu_c$. Therefore, in this section we do not need to employ the truncation p_R . We need to identify the dependence of $\gamma_T^* \xi$ on w and for this reason we write $\gamma_T^* \xi_w$ and rewrite (3.27)–(3.29) as follows:

$$(7.1) \quad \mathbf{v}' + M\mathbf{v} + A\mathbf{u} + \gamma_T^* \xi_w + P(\mathbf{u}, w) = \mathbf{f} \text{ in } \mathcal{V}'_2,$$

$$(7.2) \quad \mathbf{v}(0) = \mathbf{v}_0, \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) \, ds,$$

and

$$(7.3) \quad \langle \gamma_T^* \xi_w, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu p(|\mathbf{v}_T - \mathbf{v}_* + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_*|) \, d\Gamma dt,$$

where $\mu = \mu(|\mathbf{v}_T - \mathbf{v}_*|)$ and $p = p(u_n - w - g)$.

Now let w_i , for $i = 1, 2$, be two wear functions as above and let \mathbf{v}^i denote the corresponding solutions of problem (7.1)–(7.3). We need the following estimates. From (7.3) we obtain

$$\begin{aligned} & \int_0^t \langle \gamma_T^* \xi_{w_1} - \gamma_T^* \xi_{w_2}, \mathbf{v}^1 - \mathbf{v}^2 \rangle \, ds \\ & \geq - \int_0^t \int_{\Gamma_C} F^1 \mu^1 (|\mathbf{v}_T^2 - \mathbf{v}_*| - |\mathbf{v}_T^1 - \mathbf{v}_*|) \, d\Gamma \, ds \\ & \quad - \int_0^t \int_{\Gamma_C} F^2 \mu^2 (|\mathbf{v}_T^1 - \mathbf{v}_*| - |\mathbf{v}_T^2 - \mathbf{v}_*|) \, d\Gamma \, ds \\ & = \int_0^t \int_{\Gamma_C} (F^2 \mu^2 - F^1 \mu^1) (|\mathbf{v}_T^2 - \mathbf{v}_*| - |\mathbf{v}_T^1 - \mathbf{v}_*|) \, d\Gamma \, ds, \end{aligned}$$

where $F^i = p(u_n^i - w_i - g)$ and $\mu^i = \mu(|\mathbf{v}_T^i - \mathbf{v}_*|)$, for $i = 1, 2$. Let c be a positive constant which depends on Lip_μ , Lip_p , $p(\cdot)$, and the bounds on μ and $p(\cdot)$; then

$$(7.4) \quad \begin{aligned} & \int_0^t \langle \gamma_T^* \xi_{w_1} - \gamma_T^* \xi_{w_2}, \mathbf{v}^1 - \mathbf{v}^2 \rangle \, ds \geq -c \int_0^t \int_{\Gamma_C} |\mathbf{v}_T^1 - \mathbf{v}_T^2|^2 \, d\Gamma \, ds \\ & - c \int_0^t \int_{\Gamma_C} |\mathbf{v}_T^2 - \mathbf{v}_T^1| |w_1 - w_2| \, d\Gamma \, ds - c \int_0^t \int_{\Gamma_C} |\mathbf{v}_T^1 - \mathbf{v}_T^2| |u_n^1 - u_n^2| \, d\Gamma \, ds. \end{aligned}$$

Next, we consider the term $\int_0^t \langle P(\mathbf{u}^1, w_1) - P(\mathbf{u}^2, w_2), \mathbf{v}^1 - \mathbf{v}^2 \rangle ds$. From (3.11) and (3.20), the definition of $P(\mathbf{u}, w)$, we obtain that this expression is no smaller than

$$\begin{aligned} & - \int_0^t \int_{\Gamma_C} (p(u_n^1 - w_1 - g) - p(u_n^2 - w_2 - g)) (v_n^1 - v_n^2) d\Gamma ds \\ (7.5) \quad & \geq -c \int_0^t \int_{\Gamma_C} (|u_n^1 - u_n^2| + |w_1 - w_2|) |v_n^1 - v_n^2| d\Gamma ds. \end{aligned}$$

Now let U be a space in which V_2 embeds compactly and for which the trace map from U to $L^2(\partial\Omega)$ is continuous. Then, after adjusting the constant c and denoting by H_C the Hilbert space $L^2(\Gamma_C)$, we obtain from (7.4) and (7.5)

$$\begin{aligned} & \int_0^t \langle \gamma_T^* \xi_{w_1} - \gamma_T^* \xi_{w_2}, \mathbf{v}^1 - \mathbf{v}^2 \rangle ds + \int_0^t \langle P(\mathbf{u}^1, w_1) - P(\mathbf{u}^2, w_2), \mathbf{v}^1 - \mathbf{v}^2 \rangle ds \\ (7.6) \quad & \geq -c \int_0^t \|\mathbf{v}^1 - \mathbf{v}^2\|_U^2 ds - c \int_0^t |w_1 - w_2|_{H_C}^2 ds. \end{aligned}$$

It follows from (7.6) and (7.1) that

$$\begin{aligned} & |\mathbf{v}^1(t) - \mathbf{v}^2(t)|_H^2 + \delta^2 \int_0^t \|\mathbf{v}^1(s) - \mathbf{v}^2(s)\|_{V_2}^2 ds \\ & \quad + \frac{1}{2} \langle A(\mathbf{u}^1(t) - \mathbf{u}^2(t)), \mathbf{u}^1(t) - \mathbf{u}^2(t) \rangle \\ & \leq c \int_0^t \|\mathbf{v}^1 - \mathbf{v}^2\|_U^2 ds + c \int_0^t |w_1 - w_2|_{H_C}^2 ds \\ & \quad + \delta^2 \int_0^t |\mathbf{v}^1(s) - \mathbf{v}^2(s)|_H^2 ds. \end{aligned}$$

By the compactness of the embedding $V_2 \rightarrow U$ we have $\|\mathbf{z}\|_U^2 \leq \frac{\delta^2}{2} \|\mathbf{z}\|_{V_2}^2 + c_\delta \|\mathbf{z}\|_H^2$; hence,

$$\begin{aligned} & |\mathbf{v}^1(t) - \mathbf{v}^2(t)|_H^2 + \frac{\delta^2}{2} \int_0^t \|\mathbf{v}^1(s) - \mathbf{v}^2(s)\|_{V_2}^2 ds \\ & \leq c_\delta \int_0^t |\mathbf{v}^1(s) - \mathbf{v}^2(s)|_H^2 ds + c \int_0^t |w_1 - w_2|_{H_C}^2 ds. \end{aligned}$$

It follows from the Gronwall inequality that

$$\begin{aligned} & |\mathbf{v}^1(t) - \mathbf{v}^2(t)|_H^2 + \int_0^t \|\mathbf{v}^1(s) - \mathbf{v}^2(s)\|_{V_2}^2 ds \\ (7.7) \quad & \leq c(\delta, T) \int_0^t |w_1 - w_2|_{H_C}^2 ds, \end{aligned}$$

where the constant c depends on the indicated quantities and the bounds and Lipschitz constants of p and μ but not on the choice of w_i . We conclude with the following theorem.

THEOREM 7.1. *The solutions \mathbf{v} of problem (3.27)–(3.30) depend continuously on w .*

8. Archard law. We now consider Theorem 3.3. We use a fixed point argument to prove Theorem 3.3, which guarantees the existence and uniqueness of the weak solution. Since $p(\cdot)$ is assumed to be bounded, we do not need to employ the truncation p_R .

The Archard law of wear, in its differential form (3.9), may be written as

$$w' = \Psi(\mathbf{v}_T) p(u_n - w - g),$$

where $\Psi(\mathbf{v}_T) \equiv k_w \mu(|\mathbf{v}_T - \mathbf{v}_*|) s_c(|\mathbf{v}_T - \mathbf{v}_*|)$. It follows from our assumptions that Ψ is bounded, nonnegative, and Lipschitz continuous. Let $\mathbf{v}^i \in \mathcal{V}_2$ and w_i , for $i = 1, 2$, be the solutions of the problem

$$(8.1) \quad w_i, w_i' \in L^2(0, T; H_C),$$

$$(8.2) \quad w_i' = \Psi(\mathbf{v}_T^i) p(u_n^i - w_i - g),$$

$$(8.3) \quad w_i(\cdot, 0) = 0.$$

Since the function Ψ is bounded, we actually have

$$w, w' \in L^\infty(0, T; L^\infty(\Gamma_C)),$$

and so these functions may be considered as known wear functions in the preceding theory. Thus,

$$\begin{aligned} & \frac{1}{2} |w_1(t) - w_2(t)|_{H_C}^2 \\ & \leq c(\Psi, R) \int_0^t (|u_n^1 - u_n^2|_{H_C} + |w_1 - w_2|_{H_C}) (|w_1 - w_2|_{H_C}) \, ds \\ & \quad + c(\text{Lip}_\Psi, R, p) \int_0^t |\mathbf{v}_T^1 - \mathbf{v}_T^2|_{H_C^N} |w_1 - w_2|_{H_C} \, ds, \end{aligned}$$

where $H_C = L^2(\Gamma_C)$. It follows that

$$\begin{aligned} & |w_1(t) - w_2(t)|_{H_C}^2 \\ & \leq c(\Psi, R, p, \text{Lip}_\Psi, T) \left(\int_0^t |w_1 - w_2|_{H_C}^2 \, ds + \int_0^t \|\mathbf{v}^2 - \mathbf{v}^1\|_U^2 \, ds \right), \end{aligned}$$

where U is an intermediate space. Thus, by the Gronwall inequality,

$$(8.4) \quad |w_1(t) - w_2(t)|_{H_C}^2 \leq c(\Psi, R, p, \text{Lip}_\Psi, T) \int_0^t \|\mathbf{v}^1 - \mathbf{v}^2\|_U^2 \, ds.$$

Now we construct the following mapping. Starting with $\mathbf{v} \in \mathcal{V}_2$, we denote by $w(\mathbf{v})$ the solution of problem (8.1)–(8.3), with i omitted. Then we use $w(\mathbf{v})$ as the wear function in the system (7.1)–(7.3). In this manner we define a mapping, $\Lambda : \mathcal{V}_2 \rightarrow \mathcal{V}_2$, where $\mathbf{z} = \Lambda \mathbf{v}$, and \mathbf{z} is the solution of (7.1)–(7.3) with the given wear function $w(\mathbf{v})$. Now, from (7.7) and (8.4) we obtain

$$\int_0^t \|\Lambda \mathbf{v}^1 - \Lambda \mathbf{v}^2\|_{V_2}^2 \, ds \leq c(\delta, T, \Psi, R, p, \text{Lip}_\Psi) \int_0^t \int_0^s \|\mathbf{v}^1 - \mathbf{v}^2\|_{V_2}^2 \, dr \, ds.$$

By iterating this inequality m times we find that every Λ^m is a contraction mapping on \mathcal{V}_2 for all sufficiently large m . Consequently, Λ has a unique fixed point, which is the unique solution of problem \mathcal{P} . This establishes Theorem 3.3.

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