ON GENERALIZED SUPERELLIPTIC RIEMANN SURFACES

RUBEN A. HIDALGO, SAÚL QUISPE AND TONY SHASKA

Abstract. A closed Riemann surface $\mathcal{X}$, of genus $g \geq 2$, is called a generalized superelliptic curve of level $n \geq 2$ if it admits an order n conformal automorphism τ so that $\mathcal{X}/\langle\tau\rangle$ has genus zero and τ is central in $\mathrm{Aut}(\mathcal{X})$; the cyclic group $H = \langle\tau\rangle$ is called a generalized superelliptic group of level n for $\mathcal{X}$. These Riemann surfaces are natural generalizations of hyperelliptic Riemann surfaces. We provide an algebraic curve description of these Riemann surfaces in terms of their groups of automorphisms. Also, we observe that the generalized superelliptic group H of level n is unique, with the exception of a very particular family of exceptional generalized superelliptic Riemann surfaces for n even. In particular, the uniqueness holds if either: (i) n is odd or (ii) the quotient $\mathcal{X}/H$ has all its cone points of order n. In the non-exceptional case, we use this uniqueness property to observe that the corresponding curves are definable over their fields of moduli if $\mathrm{Aut}(\mathcal{X})/H$ is neither trivial or cyclic.

1. Introduction

Let $\mathcal{X}$ denote a closed Riemann surface of genus $g \geq 2$. A natural question is to determine the group $\mathrm{Aut}(\mathcal{X})$ of conformal automorphism of $\mathcal{X}$, which is known to be a finite group of order $\leq 84(g-1)$, and to determine algebraic curves representing it over which one may realize a given subgroup of its automorphisms. Another related question is to determine if such Riemann surface can be defined over its field of moduli and to describe an algebraic curve description defined over a "minimal" field of definition of it. These questions have been studied for a long time and complete answers to them are not known, except for certain particular cases.

For an integer $n \geq 2$ we say that $\mathcal{X}$ is a *cyclic n-gonal Riemann surface* if it admits an order n conformal automorphism τ so that the quotient orbifold $\mathcal{O} = \mathcal{X}/\langle\tau\rangle$ has genus zero (so it can be identified with the Riemann sphere); τ is called a *n-gonal automorphism* and $H = \langle\tau\rangle \cong C_n$ a *n-gonal group* of $\mathcal{X}$. It is well known that $\mathcal{X}$ can be represented by an affine irreducible (which might have singularities) algebraic curve of the form (called a *cyclic n-gonal curve*)

$$y^n = \prod_{j=1}^{r} (x - a_j)^{l_j}, \tag{1}$$

where

(i) $a_1, \ldots, a_r \in \mathbb{C}$ are distinct,
(ii) $l_j \in \{1, \ldots, n-1\}$,
(iii) $\gcd(n, l_1, \ldots, l_r) = 1$.

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In this algebraic model, $\tau(x,y) = (x, \omega_n y)$, where $\omega_n = e^{2\pi i/n}$, and $\pi(x,y) = x$ is a regular branched cover with deck group H.

There exists an extensive literature about cyclic p-gonal Riemann surfaces when p is a prime integer (see, for instance, [2, 6–9, 25, 26, 28, 29, 34, 35]). Castelnuovo-Severi's inequality [1, 11] asserts that a cyclic p-gonal Riemann surface of genus $g > (p-1)^2$ has a unique p-gonal group (similar uniqueness results holds for n not necessarily a prime when all cone points of S/H have branch order n [25]). In [18] it is proved that a cyclic p-gonal Riemann surface of genus $2 \leq g < (p-1)(p-5)/10$ also has a unique p-gonal group (for instance, for $p \geq 11$ and $g = (p-1)/2$). The uniqueness property of the p-gonal group, in those cases, has permitted to determine the groups of automorphisms of such cyclic p-gonal Riemann surfaces and their equations [34, 35].

A particular class of cyclic n-gonal Riemann surfaces, called *superelliptic curves of level* n [5, 20, 27], are those for which, in the above algebraic description (1), all the exponents l_j are equal to 1 (and if $r \not\equiv 0 \mod (n)$, then $\gcd(n,r) = 1$) and τ is assumed to be central in $\mathrm{Aut}(\mathcal{X})$ (this is a generic condition and generalizes the hyperelliptic situation); τ is called a *superelliptic automorphism of level* n and $H = \langle \tau \rangle$ a *superelliptic group of level* n. In this case, all cone points of $\mathcal{X}/H$ have order n. A classification of those superelliptic curves of genus $g \leq 48$ according to their group of automorphisms was provided in [27, 28]. In [20] it has been shown that in most cases a superelliptic curve can be defined over its field of moduli and, for $g \leq 10$, it is described those which might not be definable over their field of moduli. In that paper is also noted that every superelliptic curve of level n admits a unique superelliptic group of level n (see also Corollary 3).

A natural generalization of superelliptic curves is by not requiring the cone points of $\mathcal{X}/H$ to be all of order n. A *generalized superelliptic curve of level* n is a closed Riemann surfaces $\mathcal{X}$ admitting an order n automorphism τ which is central in $G = \mathrm{Aut}(\mathcal{X})$ and $\mathcal{X}/\langle \tau \rangle$ has genus zero; we call τ a *generalized superelliptic automorphism of level* n and the cyclic group $H = \langle \tau \rangle$ is a *generalized superelliptic group of level* n. The condition for τ to be central imposes some conditions of the exponents l_j in the algebraic curve (1) in terms of the reduced group of conformal automorphisms $\overline{G} = G/H$ relative to H (see Lemma 1). By Singerman's list of finitely maximal signatures [31], generically, cyclic n-gonal Riemann surface are generalized superelliptic curve of level n.

Following similar arguments as done by Horiuchi for the hyperelliptic situation in [21] (because of Lemma 1), in Theorem 2 we provide a description of generalized superelliptic Riemann surfaces $\mathcal{X}$ in terms of $\overline{G}$. In Theorem 3 we study the uniqueness of the generalized superelliptic group of level n. It says that except for a very particular family of exceptional generalized superelliptic Riemann surfaces of level n even, there is only one generalized superelliptic group of level n. As a consequence we have uniqueness of the generalized superelliptic group H of level n in the following cases: (i) n is odd, (ii) the quotient $\mathcal{X}/H$ has all its cone points of order n (for instance, when $\mathcal{X}$ is a superelliptic curve of level n). For the exceptional cases, the groups of conformal automorphisms can be described and it can be see that they are defined over their fields of moduli.

Let us assume now that $\mathcal{X}$ is a non-exceptional generalized superelliptic curve of level n; so its has a unique generalized superelliptic group H of level n. The uniqueness of the generalized superelliptic group H in G permits to observe (see

Theorem 8) that in the case that $\overline{G}$ is different from trivial or cyclic group the generalized superelliptic curve $\mathcal{X}$ is definable over its field of moduli (the field of definition of the point in moduli space defining the conformal class of $\mathcal{X}$). In the case that $\overline{G}$ is cyclic a similar answer is provided under a mild condition on the signature of $\mathcal{X}/G$ (see Theorem 9). At this point we should mention the paper [26] where the authors study the Galois descent obstruction for hyperelliptic curves of arbitrary genus whose reduced automorphism groups are cyclic (in their paper it is also considered the case of fields of positive characteristic); they an explicit and effectively computable description of this obstruction and obtain an arithmetic criterion for the existence of a hyperelliptic descent.

Notation 1. Throughout this paper we denote by C_n the cyclic group of order n, by D_m the dihedral group of order $2m$, by A_4 and A_5 the alternating groups of orders 12 and 60, respectively, and by S_4 the alternating group in four letters. Also, when writing an algebraic curve $y^n = \prod_{j=1}^{s}(x-b_j)^{l_j}$, if $l_i = 0$, then we delete the corresponding factor $(x-b_i)$ from it and the corresponding $\frac{n}{n_i}$ from the signature.

2. Preliminaries

2.1. **Co-compact Fuchsian groups.** A co-compact Fuchsian group is a discrete subgroup K of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$, so a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, so that the quotient orbifold $\mathbb{H}/K$ is compact. The algebraic structure of a co-compact Fuchsian group K is determined by its signature

$$(\gamma; n_1, \ldots, n_r), \tag{2}$$

where the quotient orbifold $\mathbb{H}/K$ has genus γ and r cone points having branch orders $n_1, \ldots, n_r$. The algebraic structure of K is given as

$$K = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma, c_1, \ldots, c_r : c_1^{n_1} = \cdots = c_r^{n_r} = 1, c_1 \cdots c_r [a_1, b_1] \cdots [a_\gamma, b_\gamma] = 1 \rangle \tag{3}$$

where $[a,b] = aba^{-1}b^{-1}$.

The hyperbolic area of the orbifold $\mathbb{H}/K$ is equal to

$$\mu(K) = 2\pi \left(2\gamma - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{n_j} \right) \right). \tag{4}$$

If the co-compact Fuchsian group K has no torsion, the quotient orbifold $\mathbb{H}/K$ is a closed Riemann surface of genus $g \geq 2$ and its signature is $(\gamma; -)$. Conversely, by the uniformization theorem, every closed Riemann surface of genus $g \geq 2$ can be represented as a quotient $\mathbb{H}/K$, where K is a torsion free Fuchsian group.

Let $R(K)$ be the set of all isomorphisms $\rho : K \to \mathrm{PSL}(2,\mathbb{R})$ so that $\rho(K)$ is a co-compact Fuchsian group. We have a natural one-to-one map

$$R(K) \to (\mathrm{PSL}(2,\mathbb{R}))^{2\gamma+r} : \rho \mapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_\gamma), \rho(b_\gamma), \rho(c_1), \ldots, \rho(c_r)), \tag{5}$$

which permits to see $R(K)$ as a subset of $(\mathrm{PSL}(2,\mathbb{R}))^{2\gamma+r}$ and, in particular, to give it a topological structure.

The *Teichmüller space* of K is defined as the quotient space $T(K)$ obtained by the following equivalence relation: $\rho_1 \sim \rho_2$ if and only there is some $A \in \mathrm{PSL}(2,\mathbb{R})$ so that $\rho_2(x) = A\rho_1(x)A^{-1}$, for every $x \in K$. One provides to $T(K)$ with the quotient topology. It is known [13] that $T(K)$ is in fact a simply-connected manifold of complex dimension $3\gamma - 3 + r$.

Now, if $\widehat{K}$ is another co-compact Fuchsian group that contains K as a finite index d subgroup, then there is a natural embedding $T(\widehat{K}) \subset T(K)$; so $\dim(T(\widehat{K})) \leq \dim(T(K))$ [15]. For most of the situations, it happens that $\dim(T(\widehat{K})) < \dim(T(K))$. The exceptional cases occur when $\dim(T(\widehat{K})) = \dim(T(K))$ and the list of these pairs $(K, \widehat{K})$ is provided in [31]. Below we recall these lists from [31].

Table 1. normal inclusions

K	$\widehat{K}$	$[\widehat{K} : K]$
$(2; -)$	$(0; 2, 2, 2, 2, 2, 2)$	2
$(1; t, t)$	$(0; 2, 2, 2, 2, t)$	2
$(1; t)$	$(0; 2, 2, 2, 2t)$	2
$(0; t, t, t, t)$	$(0; 2, 2, 2, t)$	4
$(0; t_1, t_1, t_2, t_2)$	$(0; 2, 2, t_1, t_2)$	2
$(0; t, t, t)$	$(0; 3, 3, t)$	3
$(0; t, t, t)$	$(0; 2, 3, 2t)$	6
$(0; t_1, t_1, t_2)$	$(0; 2, t_1, 2t_2)$	2

Table 2. non-normal inclusions

K	$\widehat{K}$	$[\widehat{K} : K]$
$(0; 7, 7, 7)$	$(0; 2, 3, 7)$	24
$(0; 2, 7, 7)$	$(0; 2, 3, 7)$	9
$(0; 3, 3, 7)$	$(0; 2, 3, 7)$	8
$(0; 4, 8, 8)$	$(0; 2, 3, 8)$	12
$(0; 3, 8, 8)$	$(0; 2, 3, 8)$	10
$(0; 9, 9, 9)$	$(0; 2, 3, 9)$	12
$(0; 4, 4, 5)$	$(0; 2, 4, 5)$	6
$(0; n, 4n, 4n)$	$(0; 2, 3, 4n)$	6
$(0; n, 2n, 2n)$	$(0; 2, 4, 2n)$	4
$(0; 3, n, 3n)$	$(0; 2, 3, 3n)$	4
$(0; 2, n, 2n)$	$(0; 2, 3, 2n)$	3

2.2. **Automorphisms in terms of Fuchsian groups.** Let Γ be a torsion free co-compact Fuchsian group and $\mathcal{X} = \mathbb{H}/\Gamma$ be its uniformized closed Riemann surface. A finite group G acts faithfully as a group of conformal automorphisms of $\mathcal{X}$ if there is some co-compact Fuchsian group K and an epimorphism $\theta : K \to G$ whose kernel is Γ.

3. Cyclic n-gonal Riemann surfaces

Let us consider a cyclic n-gonal Riemann surface $\mathcal{X}$. By the definition there is an order n conformal automorphism $\tau \in \mathrm{Aut}(\mathcal{X})$ and a regular branched cover $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$ whose deck group is $H = \langle \tau \rangle \cong C_n$. Let us assume the branch values of π are given by the points $a_1, \ldots, a_s \in \widehat{\mathbb{C}}$. Let us denote the branch order of π at a_j by $n_j \geq 2$ (which is a divisor of n).

3.1. **Fuchsian description of $\mathcal{X}$.** Let Γ be a torsion free co-compact Fuchsian group so that $\mathcal{X} = \mathbb{H}/\Gamma$. In this case, there is co-compact Fuchsian group K with signature $(0; n_1, \ldots, n_s)$, so it has a presentation as in (3) with $\gamma = 0$ and $r = s \geq 3$, and there is some epimorphism $\rho : K \to C_n = \langle \tau \rangle$ with torsion free kernel Γ. The following fact, due to Harvey, asserts that the divisors n_j must satisfy some constrains.

Theorem 1 (Harvey's criterion [16])**.** *Let K be a Fuchsian group with signature $(0; n_1, \ldots, n_s)$, where each $n_j \geq 2$ is a divisor of n and $s \geq 3$. Then there exists a epimorphism $\rho : K \to C_n$ with torsion free kernel if and only if*

(a) $n = \mathrm{lcm}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_s)$ for all j;
(b) if n is even, then $\#\{j \in \{1, \ldots, s\} : n/n_j \text{ is odd}\}$ is even.

Let $\rho(c_j) = \tau^{l_j}$, where c_j as in (3), for $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$. The condition $c_1 \cdots c_s = 1$ is equivalent to have $l_1 + \cdots + l_s \equiv 0 \mod (n)$. The condition for $\Gamma = \ker(\rho)$ to be torsion free is then equivalent to have that $\gcd(n, l_j) = n/n_j$, for $j = 1, \ldots, s$. The surjectivity of ρ is equivalent to have $\gcd(n, l_1, \ldots, l_s) = 1$, which in our case is equivalent to condition (a). Condition (b) is then equivalent to say that for n even the number of l_j being odd is even, which trivially hold.

As a consequence of the above, the following holds.

Corollary 1. *The cyclic n-gonal Riemann surface $\mathcal{X}$ of genus $g \geq 2$ can be described as $\mathcal{X} = \mathbb{H}/\Gamma$, where Γ is a co-compact Fuchsian group being the kernel of an homomorphism*

$$\rho : K = \langle c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = c_1 \cdots c_s = 1 \rangle \to C_n = \langle \tau \rangle,$$

such that $\rho(c_j) = \tau^{l_j}$, $s \geq 3$, and

(1) $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$,
(2) $l_1 + \cdots + l_s \equiv 0 \mod (n)$,
(3) $n_j = n/\gcd(n, l_j)$, for all j,
(4) $\gcd(n, l_1, \ldots, l_s) = 1$.

Remark 1. Now, if G is a group of conformal automorphisms of the cyclic n-gonal Riemann surface $\mathcal{X} = \mathbb{H}/\Gamma$, containing the cyclic group $H = \langle \tau \rangle$, then there is a co-compact Fuchsian group N containing the group K so that Γ is a normal subgroup of N. If the signature of K does not belong to the list given in [31] (see above), then $N = K$ for the generic situation; so $G = H$ and τ will be central. This means that, generically, a cyclic n-gonal Riemann surface is a generalized superelliptic curve.

3.2. **Algebraic description of $\mathcal{X}$ and τ.** As a consequence of Corollary 1, the following must hold.

(1) If $a_1, \ldots, a_s \in \mathbb{C}$, $s \geq 3$, then $\mathcal{X}$ can be described by an equation of the form

$$\mathcal{X} : \quad y^n = \prod_{j=1}^{s} (x - a_j)^{l_j},$$

where

(a) $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$, $\gcd(n, l_j) = n/n_j$,
(b) $l_1 + \cdots + l_s \equiv 0 \mod (n)$,
(c) $\gcd(n, l_1, \ldots, l_s) = 1$.

(2) If one of the branched values, say $a_s=\infty$, then
$$\mathcal{X}: \quad y^n=\prod_{j=1}^{s-1}(x-a_j)^{l_j},$$
where
(a) $l_1,\ldots,l_{s-1}\in\{1,\ldots,n-1\}$, $\gcd(n,l_j)=n/n_j$ and $\gcd(n,l_1+\cdots+l_{s-1})=n/n_s$,
(b) $l_1+\cdots+l_{s-1}\not\equiv 0 \mod (n)$,
(c) $\gcd(n,l_1,\ldots,l_{s-1})=1$.

In any of the above situations, $\tau(x,y)=(x,\omega_n y)$, where $\omega_n=e^{2\pi i/n}$, and $\pi(x,y)=x$. The branch order of π at a_j is $n_j=n/\gcd(n,l_j)$ and, by the Riemann-Hurwitz formula, the genus g of $\mathcal{X}$ is given by
$$g=1+\frac{1}{2}\left((s-2)n-\sum_{j=1}^{s}\gcd(n,l_j)\right),$$
where in the case (2) $l_s\in\{1,\ldots,n-1\}$ is so that $l_1+\cdots+l_{s-1}+l_s\equiv 0 \mod (n)$.

3.3. **Conditions for τ to be central.** As seen in Section 3.2, we may consider a curve representation of $\mathcal{X}$ of the form (where $r=s$ if $\infty\notin\{a_1,\ldots,a_s\}$ and $r=s-1$ otherwise) $y^n=\prod_{j=1}^{r}(x-a_j)^{l_j}$, (with the corresponding constrains (a)-(c) on the exponents l_j and n, depending on the case if ∞ is or not a branch value of π). In this model, $\tau(x,y)=(x,\omega_n y)$ and $\pi(x,y)=x$.

Let N be the normalizer of $H=\langle\tau\rangle$ in $\mathrm{Aut}(\mathcal{X})$. As H is normal subgroup of N, the reduced group $\overline{N}=N/H$ is a finite group of Möbius transformations keeping invariant the set of branch values of π. Let us denote by $\theta: N\to\overline{N}$ the canonical projection map. If $\eta\in N$, then $\theta(\eta)$ is a Möbius transformation keeping invariant the set $\{a_1,\ldots,a_r\}$ if ∞ is not a branch value of π; otherwise, it keeps invariant the set $\{\infty,a_1,\ldots,a_r\}$.

The following fact provides condition for τ to be central in N by asking some extra constrains on the exponents $l_1,\ldots,l_r$.

Lemma 1. *The automorphism τ is central in N if and only if for every $\eta\in G$ and a_j and a_i in the same $\theta(\eta)$-orbits one has that $l_j=l_i$.*

Proof. Let $\eta\in N$ and assume $\theta(\eta)$ has order $m\geq 2$. As there is suitable Möbius transformation M so that $M\theta(\eta)M^{-1}$ is just the rotation $x\mapsto\omega_m x$, by post-composing π with M, we may assume that $\theta(\eta)(x)=\omega_m x$. So the cyclic n-gonal curve can be written as
$$(*)\quad y^n=x^s\prod_{j=1}^{r}(x-a_j)^{l_{j,1}}(x-a_j\omega_m)^{l_{j,2}}\cdots(x-a_j\omega_m^{m-1})^{l_{j,m}},$$
in which case $\tau(x,y)=(x,\omega_n y)$ and $\eta(x,y)=(\omega_m x,F(x,y))$, where $F(x,y)$ is a suitable rational map.

As $\eta(\tau(x,y))=(\omega_m x,F(x,\omega_n y))$, $\tau(\eta(x,y))=(\omega_m x,\omega_n F(x,y))$, the automorphism τ commutes with η when $F(x,y)=R(x)y$, for a suitable rational map $R(x)\in\mathbb{C}(x)$. As $\theta(\eta)^m=1$, it follows that $\eta^m\in\langle\tau\rangle$, from which we must have that $\left(\prod_{j=0}^{m-1}R(\omega_m^j x)\right)^n=1$.

Now, by applying $\theta(\eta)$ to the right part of $(*)$, we obtain

$$\omega_m^q x^s \prod_{j=1}^{r} \frac{(x-a_j)^{l_{j,1}}(x-a_j\omega_m)^{l_{j,2}}\cdots(x-a_j\omega_m^{m-1})^{l_{j,m}}}{(x-a_j)^{l_{j,1}-l_{j,2}}(x-a_j\omega_m)^{l_{j,2}-l_{j,3}}\cdots(x-a_j\omega_m^{m-1})^{l_{j,m}-l_{j,1}}}.$$

As $\eta(x,y)=(\omega_m x, R(x)y)$, it must be that

$$l_{j,1}-l_{j,2}=l_{j,2}-l_{j,3}=\cdots=l_{j,m-1}-l_{j,m}=l_{j,m}-l_{j,1}=\alpha.$$

But this asserts that $l_j=\alpha+l_{j+1}$, for $j=1,\ldots,m-1$ and $l_m=\alpha+l_1$, in particular, $l_1=m\alpha+l_1$, that is, $m\alpha=0$. Since $m\geq 2$, $\alpha=0$. $\square$

Remark 2. In the case that $N=\mathrm{Aut}(\mathcal{X})$ (for instance, if $n=p$ is a prime integer so that either $g>(p-1)^2$ or $g<(p-1)(p-5)/10$), Lemma 1 states the conditions for $\mathcal{X}$ to be a generalized superelliptic Riemann surface.

4. Generalized superelliptic Riemann surfaces and their automorphisms

Let $\mathcal{X}$ be a generalized superelliptic Riemann surface of level n, let $\tau\in\mathrm{Aut}(\mathcal{X})$ be a generalized superelliptic automorphism of level n and $H=\langle\tau\rangle$. Let us consider a regular branched cover $\pi:\mathcal{X}\to\widehat{\mathbb{C}}$ with H as its deck group. Let $a_1,\ldots,a_r\in\mathbb{C}$ be its finite branch values (it might be that ∞ is also a branch value of π).

We know from Section 3.2, that there is curve representation of the form $y^n=\prod_{j=1}^{r}(x-a_j)^{l_j}$, with the corresponding constrains (a)-(c) on the exponents l_j and n, depending on the case if ∞ is or not a branch value of π, and in that representation we have that $\tau(x,y)=(x,\omega_n y)$ and $\pi(x,y)=x$.

4.1. **Description of generalized superelliptic Riemann in terms of their automorphism group.** The group $G=\mathrm{Aut}(\mathcal{X})$ descends by π to obtain the reduced group $\overline{G}=G/H$, which is a finite group of Möbius transformations keeping invariant the set of branch values of π. It is well known that $\overline{G}\in\{C_m,D_m,A_4,S_4,A_5\}$.

As a consequence of Lemma 1, all branch values of π belonging to the same $\overline{G}$-orbit must have the same exponent. This fact permits to imitate the arguments in Horiuchi in [21], done for the hyperelliptic situation, to write down the above equation in terms of $\overline{G}$. In the following theorem $n_j=\gcd(n,l_j)$, for $i=0,\ldots,r+1$ and symbols $\frac{n}{n_i}$ or $\frac{2n}{n_i}$ not appearing in the signature if $l_i=0$.

Theorem 2. *Let $\mathcal{X}$ be a generalized superelliptic Riemann surface of level n, $\tau\in G=\mathrm{Aut}(\mathcal{X})$ its generalized superelliptic automorphisms, and $H=\langle\tau\rangle$. Then, up to isomorphisms, $\mathcal{X}$ and G are described as indicated below in terms of the reduced group $\overline{G}=G/H$.*

(1) $\overline{G}=\langle a(x)=\omega_m x\rangle\cong C_m$:

$$\mathcal{X}:\quad y^n=x^{l_0}(x^m-1)^{l_1}\prod_{j=2}^{r}(x^m-a_j^m)^{l_j},$$

such that $a_1,\ldots,a_r\in\mathbb{C}-\{0,1\}$, $a_i^m\neq a_j^m$, where $\gcd(n,l_0,l_1,\ldots,l_r)=1$ and if $l_0=0$, then $m(l_1+\cdots+l_r)\equiv 0 \mod (n)$. The group of automorphisms is

$$G=\langle\tau,A:\tau^n=1,A^m=\tau^{l_0},\tau A=A\tau\rangle,$$

where

$$A(x,y)=(\omega_m x, \omega_m^{l_0/n} y).$$

Moreover, if we set $n_j=\gcd(n,l_j)$*, then the signature of* $\mathcal{X}\to\mathcal{X}/H$ *is*

$$\begin{cases} \left(0; \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0=0,\\ \left(0; \frac{n}{n_0}, \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0\neq 0, l_0+m\sum_{j=1}^r l_j\equiv 0 \mod (n),\\ \left(0; \frac{n}{n_0}, \frac{n}{n_{r+1}}, \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0\neq 0, l_0+m\sum_{j=1}^r l_j\not\equiv 0 \mod (n), \end{cases}$$

the signature of $\mathcal{X}\to\mathcal{X}/G$ *is*

$$\begin{cases} \left(0; m, m, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0=0,\\ \left(0; m, \frac{mn}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0\neq 0,\ l_0+m\sum_{j=1}^r l_j\equiv 0 \mod (n),\\ \left(0; \frac{mn}{n_0}, \frac{mn}{n_{r+1}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0\neq 0,\ l_0+m\sum_{j=1}^r l_j\not\equiv 0 \mod (n), \end{cases}$$

and the genus of $\mathcal{X}$ *is*

$$\begin{cases} 1+\frac{1}{2}\left((rm-2)n-m\sum_{j=1}^r n_j\right), & \text{if } l_0=0,\\ 1+\frac{1}{2}\left((rm-1)n-m\sum_{j=1}^r n_j\right), & \text{if } l_0\neq 0,\ l_0+m\sum_{j=1}^r l_j\equiv 0 \mod (n),\\ 1+\frac{1}{2}\left(rmn-m\sum_{j=1}^r n_j\right), & \text{if } l_0\neq 0,\ l_0+m\sum_{j=1}^r l_j\not\equiv 0 \mod (n). \end{cases}$$

(2) $\overline{G}=\left\langle a(x)=\omega_m x, b(x)=\frac{1}{x}\right\rangle\cong D_m$*:*

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m-1)^{l_{r+1}}(x^m+1)^{l_{r+2}}\prod_{j=1}^r (x^m-a_j^m)^{l_j}(x^m-a_j^{-m})^{l_j},$$

such that $a_i^{\pm m}\neq a_j^{\pm m}\neq 0,\pm 1$*, where the following hold:*
a) $2l_0+m(l_{r+1}+l_{r+2})+2m(l_1+\cdots+l_r)\equiv 0 \mod (n)$*,*
b) $\gcd(n,l_0,l_1,\dots,l_{r+2})=1$*.*

The group of automorphisms is

$$G=\langle \tau, A, B : \tau^n=1, A^m=\tau^{l_0},\ B^2=\tau^{l_{r+1}},\ \tau A=A\tau,\ \tau B=B\tau\rangle,$$

where

$$A(x,y)=(\omega_m x, \omega_m^{l_0/n} y), \quad B(x,y)=\left(\frac{1}{x}, \frac{(-1)^{l_{r+1}/n} y}{x^{(2l_0+m(l_{r+1}+l_{r+2}+2(l_1+\cdots+l_r)))/n}}\right).$$

Let $n_j=\gcd(n,l_j)$*, then the signature of* $\mathcal{X}\to\mathcal{X}/H$ *is*

$$\left(0; \frac{n}{n_0}, \frac{n}{n_0}, \frac{n}{n_{r+1}}, \overset{m}{\dots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{m}{\dots}, \frac{n}{n_{r+2}}, \frac{n}{n_1}, \overset{2m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{2m}{\dots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X}\to\mathcal{X}/G$ *is*

$$\left(0; \frac{mn}{n_0}, \frac{2n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ *is*

$$g=1+\frac{1}{2}\left(2m(r+1)n-2n_0-m\left(n_{r+1}+n_{r+2}+2\sum_{j=1}^r n_j\right)\right).$$

(3) $\overline{G} = \left\langle a(x) = -x, b(x) = \frac{i-x}{i+x} \right\rangle \cong A_4$*:*

$$\mathcal{X}: \quad y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+1}} R_3(x)^{l_{r+2}} \prod_{j=1}^{r} (R_1(x)^3 + 12 f(a_j)\sqrt{3} i R_3(x)^2)^{l_j},$$

where

(a) $R_1(x) = x^4 - 2\sqrt{3}ix^2 + 1,\ R_2(x) = x^4 + 2\sqrt{3}ix^2 + 1,\ R_3(x) = x(x^4-1)$*,*

(b) $f(a_j) \neq f(a_i) \neq 0, 1, \infty$*,*

$$f(x) = \frac{R_1(x)^3}{-12\sqrt{3}iR_3(x)^2},$$

(c) $8l_{r+1} + 6l_{r+2} + 12(l_1 + \cdots + l_r) \equiv 0 \mod (n)$*,*

(d) $\gcd(n, l_1, \ldots, l_{r+2}) = 1$*.*

The group of automorphisms is

$$G = \langle \tau, A, B : \tau^n = 1, A^2 = \tau^{l_{r+2}},\ B^3 = \tau^{-(5l_{r+1}+3l_{r+2}+6(l_1+\cdots+l_r))},$$
$$(AB)^3 = \tau^{-3(l_{r+1}+(l_1+\cdots+l_4)},\ \tau A = A\tau,\ \tau B = B\tau \rangle,$$

where

$$A(x,y) = (-x, (-1)^{l_{r+2}/n} y), \quad B(x,y) = (b(x), F(x)y),$$

and

$$F(x) = \frac{2^{(4l_{r+1}+3l_{r+2}+6(l_1+\cdots+l_r))/n} i^{(l_{r+2}+2(l_1+\cdots+l_r))/n}}{(x+i)^{(8l_{r+1}+6l_{r+2}+12(l_1+\cdots+l_r))/n}}.$$

Let $n_j = \gcd(n, l_j)$*, then the signature of* $\mathcal{X} \to \mathcal{X}/H$ *is*

$$\left(0; \frac{n}{n_{r+1}}, \overset{8}{\ldots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{6}{\ldots}, \frac{n}{n_{r+2}}, \frac{n}{n_1}, \overset{12}{\ldots}, \frac{n}{n_1}, \ldots, \frac{n}{n_r}, \overset{12}{\ldots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X} \to \mathcal{X}/G$ *is*

$$\left(0; \frac{3n}{n_{r+1}}, \frac{3n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ *is*

$$g = 1 + 6(r+1)n - 4n_{r+1} - 3n_{r+2} - 6\sum_{j=1}^{r} n_j.$$

(4) $\overline{G} = \left\langle a(x) = ix, b(x) = \frac{i-x}{i+x} \right\rangle \cong S_4$*:*

$$\mathcal{X}: \quad y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+2}} R_3(x)^{l_{r+3}} \prod_{j=1}^{r} (R_1(x)^3 - 108 f(a_j) R_3(x)^4)^{l_j},$$

where

(a) $R_1(x) = x^8 + 14x^4 + 1,\ R_2(x) = x^{12} - 33x^8 - 33x^4 + 1,\ R_3(x) = x(x^4-1)$*,*

(b) $f(a_j) \neq f(a_i) \neq 0, 1, \infty$*,*

$$f(x) = \frac{R_1(x)^3}{108R_3(x)^4},$$

(c) $8l_{r+1} + 12l_{r+2} + 6l_{r+3} + 24(l_1 + \cdots + l_r) \equiv 0 \mod (n)$*,*

(d) $\gcd(n, l_1, \ldots, l_{r+3}) = 1$*.*

The group of automorphisms is

$$G=\langle \tau, A, B; \tau^n=1,\ A^4=\tau^{l_{r+3}},\ B^3=\tau^{-(5l_{r+1}+6l_{r+2}+3l_{r+3}+15(l_1+\cdots+l_r))},$$
$$(AB)^2=\tau^{-(4l_{r+1}+5l_{r+2}+2l_{r+3}+12(l_1+\cdots+l_r))},\ \tau A=A\tau,\ \tau B=B\tau\rangle,$$

where $A(x,y)=(ix, i^{l_{r+3}/n}y)$, $B(x,y)=(b(x),F(x)y)$, *and*

$$F(x)=\frac{(-1)^{l_{r+2}/n} i^{l_{r+3}/n} 2^{(4l_{r+1}+6l_{r+2}+3l_{r+3}+12(l_1+\cdots+l_r))/n}}{(x+i)^{(8l_{r+1}+12l_{r+2}+6l_{r+3}+24(l_1+\cdots+l_r))/n}}.$$

Let $n_j=\gcd(n,l_j)$, *then the signature of* $\mathcal{X}\to\mathcal{X}/H$ *is*

$$\left(0; \frac{n}{n_{r+1}}, \overset{8}{\ldots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{12}{\ldots}, \frac{n}{n_{r+2}}, \frac{n}{n_{r+3}}, \overset{6}{\ldots}, \frac{n}{n_{r+3}}, \frac{n}{n_1}, \overset{24}{\ldots}, \frac{n}{n_1}, \ldots, \frac{n}{n_r}, \overset{24}{\ldots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X}\to\mathcal{X}/G$ *is*

$$\left(0; \frac{3n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{4n}{n_{r+3}}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ *is*

$$g=1+12(r+1)n-4n_{r+1}-6n_{r+2}-3n_{r+3}-12\sum_{j=1}^{r} n_j.$$

(5) $\overline{G}=\left\langle a(x)=\omega_5 x, b(x)=\frac{(1-\omega_5^4)x+(\omega_5^4-\omega_5)}{(\omega_5-\omega_5^3)x+(\omega_5^2-\omega_5^3)}\right\rangle \cong A_5$:

$$\mathcal{X}: \quad y^n=R_1(x)^{l_{r+1}}R_2(x)^{l_{r+2}}R_3(x)^{l_{r+3}}\prod_{j=1}^{r}(R_1(x)^3-1728f(a_j)R_3(x)^5)^{l_j},$$

where

(a) $R_1(x)=-x^{20}-1+228x^5(x^{10}-1)-494x^{10}$, $R_2(x)=x^{30}+1+522x^5(x^{20}-1)-10005x^{10}(x^{10}+1)$, $\quad R_3(x)=x(x^{10}+11x^5-1)$,

(b) $f(a_j)\neq f(a_i)\neq 0,1,\infty$,

$$f(x)=\frac{R_1(x)^3}{1728R_3(x)^5},$$

(c) $20l_{r+1}+30l_{r+2}+12l_{r+3}+60(l_1+\cdots+l_r)\equiv 0 \mod (n)$,

(d) $\gcd(n,l_1,\ldots,l_{r+3})=1$.

The elements $a(x)$ *and* $b(x)$ *induce the automorphisms*

$$A(x,y)=(a(x),\omega_5^{s_3/n}y),\ B(x,y)=(b(x),L(x)y),$$

where $L(x)$ *is a rational map satisfying*

$$L(b^2(x))L(b(x))L(x)=\omega_5^l,$$

for a suitable $l\in\{0,1,2,3,4\}$, *and*

$$L(x)^n=T_1^{l_{r+1}+3(l_1+\cdots+l_r)}(x)T_2^{l_{r+2}}(x)T_3^{l_{r+3}}(x),$$

where $T_j(x)=R_j(b(x))/R_j(x)$, *for* $j=1,2,3$.

The group of automorphisms is

$$G=\langle \tau, A, B:\ A^5=\tau^{l_{r+3}},\ B^3=\tau^l\rangle$$

Let $n_j=\gcd(n,l_j)$,*and the signature of* $\mathcal{X}\to\mathcal{X}/H$ *is*

$$(0; \frac{n}{n_{r+1}}, \overset{20}{\ldots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{30}{\ldots}, \frac{n}{n_{r+2}}, \frac{n}{n_{r+3}}, \overset{12}{\ldots}, \frac{n}{n_{r+3}}, \frac{n}{n_1}, \overset{60}{\ldots}, \frac{n}{n_1}, \ldots, \frac{n}{n_r}, \overset{60}{\ldots}, \frac{n}{n_r}),$$

the signature of $\mathcal{X} \to \mathcal{X}/G$ *is*

$$(0; \frac{3n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{5n}{n_{r+3}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}),$$

and the genus of $\mathcal{X}$ *is*

$$g = 1 + 30(r+1)n - 10n_{r+1} - 15n_{r+2} - 6n_{r+3} - 30\sum_{j=1}^{r} n_j.$$

Proof. Let $\mathcal{X}$ be a generalized superelliptic curve of level n and let $H = \langle \tau \rangle \cong C_n$ be a generalized superelliptic group of level n for $\mathcal{X}$. We set $G = \mathrm{Aut}(\mathcal{X})$ and $\overline{G} = G/H$. Let us assume that $\overline{G}$ is not the trivial group, so $\overline{G} \in \{C_m, D_m, A_4, S_4, A_5\}$, where $m \geq 2$. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a regular branched cover with $\overline{G}$ as its deck group; we have that f is a degree $|\overline{G}|$-rational map. Let us write $f(x) = P(x)/Q(x)$, for suitable relatively prime polynomials $P(x), Q(x)$.

We already know that $\mathcal{X}$ is represented by a cyclic n-gonal curve of the form $y^n = \prod_{j=1}^{r}(x-b_j)^{d_j}$, where $d_1, \dots, d_r \in \{1, \dots, n-1\}$ satisfy Harvey's conditions in Corollary 1, and either the collection $\{b_1, \dots, b_r\}$ is $\overline{G}$-invariant if $d_1 + \cdots + d_r \equiv 0 \mod (n)$ or the collection $\{\infty, b_1, \dots, b_r\}$ is $\overline{G}$-invariant if $d_1 + \cdots + d_r \not\equiv 0 \mod (n)$. Lemma 1 asserts that for b_i, b_j so that there is some $T \in \overline{G}$ with $T(b_i) = b_j$, then $d_i = d_j$. If the disjoint $\overline{G}$-orbits (eliminating ∞ from its orbit if it is a branch value of π) are given by

$$\{a_{1,1}, \dots, a_{1,r_1}\}, \dots, \{a_{q,1}, \dots, a_{q,r_q}\},$$

then our curve can be written as follows

$$y^n = \prod_{j=1}^{q}\left(\prod_{i=1}^{r_j}(x - a_{j,i})\right)^{l_j}.$$

The $\overline{G}$-invariance of these sets (if $l_1 + \cdots + l_r \not\equiv 0 \mod (n)$, then one of these orbits must be enlarged by adding ∞) asserts that we might write, for the case $r_j = |\overline{G}|$,

$$\prod_{i=1}^{r_j}(x - a_{j,i}) = P(x) - f(a_{j,1})Q(x),$$

and for the case that the r_j is a proper divisor of $|\overline{G}|$ a similar equality holds but we will need to take care of multiplicities.

(1) In the case that $\overline{G} = C_m$, up to a Möbius transformation, we may assume that $\overline{G} = \langle a(x) = \omega_m x \rangle \cong C_m$, $m \geq 2$, and $f(x) = x^m$. In this case the possible $\overline{G}$-orbits are given by (up to conjugation by $T(x) = \lambda x$, for a suitable $\lambda \neq 0$, orbits of length one $\{\infty\}$ and/or $\{0\}$, the m-roots of unity, and m-roots of other complex numbers, that is,

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m - 1)^{l_1}\prod_{j=2}^{r}(x^m - a_j^m)^{l_j},$$

$$a_2, \dots, a_r \in \mathbb{C} - \{0, 1\},\ a_i^m \neq a_j^m,$$

where the following Harvey's constrains are as indicated in the theorem. In this case, a induces the automorphism

$$A(x, y) = (\omega_m x, \omega_m^{l_0/n} y),$$

and it can be seen that

$$G = \langle \tau, A : \tau^n = 1, A^m = \tau^{l_0}, \tau A = A\tau \rangle.$$

The signatures of $\mathcal{X}/H$ and $\mathcal{X}/G$ are easily obtained from the curve above (and Harvey's constrains) and, the formula of the genus of $\mathcal{X}$ is obtained from the signature of $\mathcal{X}/H$.

(2) In the case that $\overline{G} = D_m$, up to a Möbius transformation, we may assume that $\overline{G} = \left\langle a(x) = \omega_m x, b(x) = \frac{1}{x} \right\rangle \cong D_m$ and $f(x) = x^m + x^{-m}$. In this case, the $\overline{G}$-orbits are $\{0, \infty\}$, the m-roots or unity, the m-roots of -1 and some other orbits of length $2m$, that is,

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m - 1)^{l_{r+1}}(x^m + 1)^{l_{r+2}} \prod_{j=1}^{r} (x^m - a_j^m)^{l_j}(x^m - a_j^{-m})^{l_j},$$

$$a_i^{\pm m} \neq a_j^{\pm m} \neq 0, \pm 1,$$

where Harvey's conditions now read as
(a) $2l_0 + m(l_{r+1} + l_{r+2}) + 2m(l_1 + \cdots + l_r) \equiv 0 \mod (n)$,
(b) $\gcd(n, l_0, l_1, \ldots, l_{r+2}) = 1$.

In this case, the elements $a(x)$ and $b(x)$ induce the automorphisms

$$A(x, y) = (\omega_m x, \omega_m^{l_0/n} y),$$

$$B(x, y) = \left(\frac{1}{x}, \frac{(-1)^{l_{r+1}/n} y}{x^{(2l_0 + m(l_{r+1} + l_{r+2} + 2(l_1 + \cdots + l_r)))/n}} \right),$$

and

$$G = \langle \tau, A, B : \tau^n = 1, A^m = \tau^{l_0},\ B^2 = \tau^{l_{r+1}},\ \tau A = A\tau,\ \tau B = B\tau \rangle,$$

The signatures of $\mathcal{X}/H$ and $\mathcal{X}/G$ are easily obtained from the curve above (and Harvey's constrains) and, the formula of the genus of $\mathcal{X}$ is obtained from the signature of $\mathcal{X}/H$.

(3) In the case that $\overline{G} = A_4$, up to a Möbius transformation, we may assume that $\overline{G} = \left\langle a(x) = -x, b(x) = \frac{i-x}{i+x} \right\rangle \cong A_4$. In this case,

$$f(x) = \frac{R_1(x)^3}{-12\sqrt{3}iR_3(x)^2},$$

where

$$R_1(x) = x^4 - 2\sqrt{3}ix^2 + 1,\ R_2(x) = x^4 + 2\sqrt{3}ix^2 + 1,\ R_3(x) = x(x^4 - 1),$$

and the curve we obtain is of the form

$$\mathcal{X}: \quad y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+1}} R_3(x)^{l_{r+2}} \prod_{j=1}^{r} (R_1(x)^3 + 12f(a_j)\sqrt{3}iR_3(x)^2)^{l_j},$$

$$f(a_j) \neq f(a_i) \neq 0, 1, \infty,$$

where Harvey's condition now read as
(a) $8l_{r+1} + 6l_{r+2} + 12(l_1 + \cdots + l_r) \equiv 0 \mod (n)$,
(b) $\gcd(n, l_1, \ldots, l_{r+2}) = 1$.

Since,

$$R_1(b(x)) = \frac{2(1-\sqrt{3}i)}{(x+i)^4} R_1(x), \quad R_1(a(x)) = R_1(x),$$

$$R_2(b(x)) = \frac{2(1+\sqrt{3}i)}{(x+i)^4} R_2(x), \quad R_2(a(x)) = R_2(x),$$

$$R_3(b(x)) = \frac{8i}{(x+i)^6} R_3(x), \quad R_3(a(x)) = -R_3(x),$$

we see that $a(x)$ and $b(x)$ induce the automorphisms

$$A(x,y) = (-x, (-1)^{l_{r+2}/n} y), \ B(x,y) = (b(x), F(x)y),$$

where

$$F(x) = \frac{2^{(4l_{r+1}+3l_{r+2}+6(l_1+\cdots+l_r))/n} i^{(l_{r+2}+2(l_1+\cdots+l_r))/n}}{(x+i)^{(8l_{r+1}+6l_{r+2}+12(l_1+\cdots+l_r))/n}}.$$

and we obtain that

$$G = \langle \tau, A, B : \tau^n = 1, A^2 = \tau^{l_{r+2}}, \ B^3 = \tau^{-(5l_{r+1}+3l_{r+2}+6(l_1+\cdots+l_r))},$$
$$(AB)^3 = \tau^{-3(l_{r+1}+(l_1+\cdots+l_4)}, \ \tau A = A\tau, \ \tau B = B\tau \rangle.$$

The signatures of $\mathcal{X}/H$ and $\mathcal{X}/G$ are easily obtained from the curve above (and Harvey's constrains) and, the formula of the genus of $\mathcal{X}$ is obtained from the signature of $\mathcal{X}/H$.

(4) In the case that $\overline{G} = S_4$, up to a Möbius transformation, we may assume that $\overline{G} = \left\langle a(x) = ix, b(x) = \frac{i-x}{i+x} \right\rangle \cong S_4$. In this case,

$$f(x) = \frac{R_1(x)^3}{108 R_3(x)^4},$$

where

$$R_1(x) = x^8 + 14x^4 + 1, \ R_2(x) = x^{12} - 33x^8 - 33x^4 + 1, \ R_3(x) = x(x^4-1),$$

and the curve has the form

$$\mathcal{X}: \quad y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+2}} R_3(x)^{l_{r+3}} \prod_{j=1}^{r} (R_1(x)^3 - 108 f(a_j) R_3(x)^4)^{l_j},$$

$$f(a_j) \neq f(a_i) \neq 0, 1, \infty,$$

where Harvey's condition now read as

(a) $8l_{r+1} + 12l_{r+2} + 6l_{r+3} + 24(l_1 + \cdots + l_r) \equiv 0 \mod (n)$,
(b) $\gcd(n, l_1, \ldots, l_{r+3}) = 1$.

Since,

$$R_1(a(x)) = R_1(x), \quad R_1(b(x)) = \frac{16}{(x+i)^8} R_1(x),$$

$$R_2(a(x)) = R_2(x), \quad R_2(b(x)) = \frac{-64}{(x+i)^{12}} R_2(x),$$

$$R_3(a(x)) = iR_3(x), \quad R_3(b(x)) = \frac{8i}{(x+i)^6} R_3(x),$$

the elements $a(x)$ and $b(x)$ induce the automorphisms

$$A(x,y) = (ix, i^{l_{r+3}/n} y), \ B(x,y) = (b(x), F(x)y)$$

$$F(x)=\frac{(-1)^{l_{r+2}/n}i^{l_{r+3}/n}2^{(4l_{r+1}+6l_{r+2}+3l_{r+3}+12(l_1+\cdots+l_r))/n}}{(x+i)^{(8l_{r+1}+12l_{r+2}+6l_{r+3}+24(l_1+\cdots+l_r))/n}},$$

and one obtains that

$$G=\langle \tau,A,B;\tau^n=1,\ A^4=\tau^{l_{r+3}},\ B^3=\tau^{-(5l_{r+1}+6l_{r+2}+3l_{r+3}+15(l_1+\cdots+l_r))},$$
$$(AB)^2=\tau^{-(4l_{r+1}+5l_{r+2}+2l_{r+3}+12(l_1+\cdots+l_r))},\ \tau A=A\tau,\ \tau B=B\tau\rangle.$$

The signatures of $\mathcal{X}/H$ and $\mathcal{X}/G$ are easily obtained from the curve above (and Harvey's constrains) and, the formula of the genus of $\mathcal{X}$ is obtained from the signature of $\mathcal{X}/H$.

(5) In the case that $\overline{G}=A_5$, up to a Möbius transformation, we may assume that $\overline{G}=\left\langle a(x)=\omega_5 x, b(x)=\frac{(1-\omega_5^4)x+(\omega_5^4-\omega_5)}{(\omega_5-\omega_5^3)x+(\omega_5^2-\omega_5^3)}\right\rangle\cong A_5$. In this case,

$$f(x)=\frac{R_1(x)^3}{1728R_3(x)^5},$$

where

$$R_1(x)=-x^{20}-1+228x^5(x^{10}-1)-494x^{10},\ R_2(x)=x^{30}+1+522x^5(x^{20}-1)-10005x^{10}(x^{10}+1),$$
$$R_3(x)=x(x^{10}+11x^5-1),$$

and the curve we obtain has the form

$$\mathcal{X}:\quad y^n=R_1(x)^{l_{r+1}}R_2(x)^{l_{r+2}}R_3(x)^{l_{r+3}}\prod_{j=1}^{r}(R_1(x)^3-1728f(a_j)R_3(x)^5)^{l_j},$$
$$f(a_j)\neq f(a_i)\neq 0,1,\infty,$$

and Harvey's conditions read in this case as

(a) $20l_{r+1}+30l_{r+2}+12l_{r+3}+60(l_1+\cdots+l_r)\equiv 0 \mod (n)$,
(b) $\gcd(n,l_1,\ldots,l_{r+3})=1$.

In this case,

$$R_1(a(x))=R_1(x),\ R_2(a(x))=R_2(x),\ R_3(a(x))=\omega_5R_3(x),$$

and let us consider the rational maps

$$T_j(x)=R_j(b(x))/R_j(x),\quad j=1,2,3.$$

It can be checked that $T_1^3=T_3^5$ and that there is rational map $L(x)$ so that

$$L(b^2(x))L(b(x))L(x)=\omega_5^l,$$

for a suitable $l\in\{0,1,2,3,4\}$ and

$$L^n=T_1^{l_{r+1}+3(l_1+\cdots l_r)}T_2^{l_{r+2}}T_3^{l_{r+3}}.$$

In this case,

$$G=\langle\tau,A,B\rangle,$$

where

$$A(x,y)=(a(x),\omega_5^{s_3/n}y),\ B(x,y)=(b(x),L(x)y),$$
$$(A^5=\tau^{l_{r+3}},\ B^3=\tau^l).$$

The signatures of $\mathcal{X}/H$ and $\mathcal{X}/G$ are easily obtained from the curve above (and Harvey's constrains) and, the formula of the genus of $\mathcal{X}$ is obtained from the signature of $\mathcal{X}/H$.

□

4.2. **On the uniqueness of the generalized superelliptic group.** In this section we study the uniqueness of the generalized superelliptic group $H = \langle\tau\rangle$ of level n. Let $\overline{G} = \mathrm{Aut}(\mathcal{X})/H$ the reduced group with respect to H, which we know to be either trivial, cyclic, dihedral or one of the Platonic groups A_4, S_4 or A_5. As the Platonic groups and the dihedral groups of order not divisible by 4 there is no non-trivial central element, we may observe the following fact.

Proposition 1. *Let $\mathcal{X}$ be a generalized superelliptic Riemann surface of level n and let H be a generalized superelliptic group of level n. If the reduced group $\overline{G}$ of automorphism with respect to H is either a dihedral group of order not divisible by 4 or A_4 or S_4 or A_5, then H is the unique generalized superelliptic group of level n for $\mathcal{X}$.*

Proof. Assume, by the contrary, that there is a generalized superelliptic automorphism η of level n with $\eta \notin H$. Then η induces a non-trivial central element of the reduced group $\overline{G}$, a contradiction. □

As a consequence of the above, the only possibility for $\mathcal{X}$ to admit another generalized superelliptic group of level n is when $\overline{G}$ is either a non-trivial cyclic group or a dihedral group of order $4m$. In the following, we observe that if $\mathcal{X}$ has at least two different generalized superelliptic groups of level n, then it belong to a certain family of "exceptional" generalized superelliptic Riemann surfaces.

Theorem 3. *If $\mathcal{X}$ is a generalized superelliptic Riemann surface of level n admitting at least two different generalized superelliptic groups of level n, then $n = 2d$, $d \geq 2$, and it can be represented by a cyclic n-gonal curve of the form*

$$\mathcal{X}: \quad y^{2d} = x^2\left(x^2-1\right)^{l_1}\left(x^2-a_1^2\right)^{l_2}\prod_{j=3}^{L}\left(x^2-a_j^2\right)^{2\widehat{l_j}},$$

where

$$l_1, l_2, 2\widehat{l_3}, \ldots, 2\widehat{l_L} \in \{1, \ldots, 2d-1\}, \quad l_1 \text{ is odd,}$$

and

(1) *for $l_2 = 2\widehat{l_2}$, $\gcd\left(d, l_1, \widehat{l_2}, \ldots, \widehat{l_L}\right) = 1$.*

(2) *for l_2 odd, then $l_1 + l_2 = 2d$ and $\gcd\left(d, l_1, l_2, \widehat{l_3}, \ldots, \widehat{l_L}\right) = 1$.*

In these cases, $\tau(x,y) = (x, \omega_{2d}y)$ and $\eta(x,y) = (-x, \omega_{2d}y)$ are generalized superelliptic automorphisms of level n so that $K = \langle\tau,\eta\rangle \cong C_{2d}\times C_2$. The quotient orbifold $\mathcal{X}/K$ has signature

$$\left(0; 2, 2d, \frac{2d}{\gcd(2d,l_1)}, \frac{2d}{\gcd(2d,l_2)}, \frac{d}{\gcd\left(d,\widehat{l_3}\right)}, \ldots, \frac{d}{\gcd\left(d,\widehat{l_L}\right)}\right),$$

in the case that $1 + l_1 + l_2 + 2\left(\widehat{l_3} + \cdots + \widehat{l_L}\right) \equiv 0 \mod (d)$, or

$$\left(0; 2, 2d, \frac{2d}{\gcd(2d,l_1)}, \frac{2d}{\gcd(2d,l_2)}, \frac{d}{\gcd\left(d,\widehat{l_3}\right)}, \ldots, \frac{d}{\gcd\left(d,\widehat{l_L}\right)}, \frac{d}{\gcd\left(d, 1+l_1+l_2+\widehat{l_3}+\cdots+\widehat{l_L}\right)}\right),$$

in the case that $1 + l_1 + l_2 + 2\left(\widehat{l_3} + \cdots + \widehat{l_L}\right) \not\equiv 0 \mod (d)$.

The genus of $\mathcal{X}$ is, in the first case, equal to

$$2d(1+L)-\gcd(2d,l_1)-\gcd(2d,l_2)-2\sum_{j=3}^{L}\gcd\left(d,\widehat{l_j}\right),$$

and, in the second case, equal to

$$d(3+2L)-\gcd(2d,l_1)-\gcd(2d,l_2)-2\sum_{j=3}^{L}\gcd\left(d,\widehat{l_j}\right)-\gcd\left(d,1+l_1+l_2+2\sum_{j=3}^{L}\widehat{l_j}\right).$$

Remark 3. Let us observe that the Riemann surfaces described by the cyclic $2d$-gonal curves in Theorem 3 are not all of them necessary generalized superelliptic; the theorem only asserts that the exceptional ones are some of them (for $L\geq 3$ they are generically generalized superelliptic of level $2d$). For example, the cyclic $2d$-gonal curve $y^{2d}=x^2(x^2-1)^{l_1}$, with $l_1=d-1$ and $d\geq 2$ even, admits the extra automorphism

$$\alpha(x,y)=\left(\frac{x(x^2-1)^{d/2}}{y^d},\frac{y^{l_1}}{(x^2-1)^{(l_1^2-1)/2d}}\right),$$

which does not commutes with τ.

Proof of Theorem 3. As $n=2$ corresponds to the hyperelliptic situation, which we already know to be unique, we must have $n\geq 3$. Let us assume $\mathcal{X}$ has two different generalized superelliptic groups of level n, say $H=\langle\tau\rangle$ and $\langle\eta\rangle$, where $\eta\notin H=\langle\tau\rangle$. Let us consider, as before, the canonical quotient homomorphism $\theta:G\to\overline{G}=G/H$, where $G=\mathrm{Aut}(\mathcal{X})$, and let $\pi:\mathcal{X}\to\widehat{\mathbb{C}}$ be a regular branched cover with deck group H.

As τ is central, $K=\langle\tau,\eta\rangle<G$ is an abelian group and $\overline{K}=K/H=\langle\theta(\eta)\rangle\cong C_m$, where $n=md$ and $m\geq 2$. Since $\theta(\eta)$ has order m, $\eta^m\in H$ and it has order d. So, replacing τ by a suitable power (still being a generator of H) we may assume that $\eta^m=\tau^m$.

Theorem 2 asserts that we may assume $\mathcal{X}$ to be represented by an cyclic n-gonal curve of the form

$$\mathcal{X}:\quad y^n=x^{l_0}(x^m-1)^{l_1}\prod_{j=2}^{L}(x^m-a_j^m)^{l_j},$$

where the following Harvey's conditions are satisfied:

(1) $l_0=0$, $m(l_1+\cdots+l_L)\equiv 0 \mod (n)$ and $\gcd(n,l_1,\ldots,l_L)=1$; or
(2) $l_0\neq 0$ and $\gcd(n,l_0,l_1,\ldots,l_L)=1$.

In this algebraic model, $\tau(x,y)=(x,\omega_n y)$, $\pi(x,y)=x$ and $\theta(\eta)(x)=\omega_m x$, where $\omega_r=e^{2\pi i/r}$. In this way, $\eta(x,y)=(\omega_m x,\omega_m^{l_0/n}y)$.

As we are assuming $\eta^m=\tau^m$ and η has order n, we may assume the following

$$\begin{cases} \text{if } l_0\neq 0: & \eta(x,y)=(\omega_m x,\omega_n y) \quad \text{and} \quad l_0=m,\\ \text{if } l_0=0: & \eta(x,y)=(\omega_m x,y) \quad \text{and} \quad n=m.\end{cases}$$

(I) Case $l_0=m$; so $\eta(x,y)=(\omega_m x,\omega_n y)$ and we are in case (2) above.

The η-invariant algebra $\mathbb{C}[x,y]^{\langle\eta\rangle}$ is generated by the monomials $u=x^m, v=y^n$ and those of the form $x^a y^b$, where $a\in\{0,1,\ldots,m-1\}$ and $b\in\{0,1,\ldots,n-1\}$ (the case $a=b=0$ not considered) satisfy that $a+b/d\equiv 0 \mod (m)$. In particular, $b=dr$ for $r\in\{0,1,\ldots,[(n-1)/d]\}$ so that $a+r\equiv 0 \mod (m)$. As $0\leq a+r\leq$

$(m-1)+[(n-1)/d] \leq (m-1)+[(md-1)/d] < 2m$, it follows that $a+r \in \{0,m\}$. As the case $a+r=0$ asserts that $a=b=0$, which is not considered, we must have $a+r=m$, from which we see that the other generators are given by $t_1,\ldots,t_m$, where $t_j = x^{m-j}y^{dj}$ (observe that $t_m = v$). As consequence of invariant theory, the quotient curve $\mathcal{X}/\langle \eta \rangle$ corresponds to the algebraic curve

$$\mathcal{Y}: \quad \left\{ \begin{array}{lcl} t_1^m & = & u^{m-1}v, \\ t_2^m & = & u^{m-2}v^2, \\ & \vdots & \\ t_{m-1}^m & = & uv^{m-1}, \\ v & = & u(u-1)^{l_1}\prod_{j=2}^{L}(u-a_j^m)^{l_j}. \end{array} \right.$$

The curve $\mathcal{Y}$ admits the automorphisms $T_1,\ldots,T_{m-1}$, where T_j is just amplification of the t_j-coordinate by ω_m and acts as the identity on all the other coordinates. The group generated by all of these automorphisms is

$$(*) \quad \mathcal{U} = \langle T_1,\ldots,T_{m-1}\rangle \cong C_m^{m-1}.$$

The regular branched cover map $\pi_{\mathcal{U}}: \mathcal{Y} \to \widehat{\mathbb{C}} : (u,v,t_1,\ldots,t_{m-1}) \mapsto u$ has $\mathcal{U}$ as its deck group. Let us observe that the values 0, $a_1^m,\ldots,a_L^m$ belongs to the branch set of $\pi_{\mathcal{U}}$.

Since $\mathcal{Y} = \mathcal{X}/\langle \eta \rangle$ has genus zero and the finite Abelian groups of automorphisms of the Riemann sphere are either the trivial group, a cyclic group or $V_4 = C_2^2$, the group $\mathcal{U}$ is either one of these three types. As $m \geq 2$, the group $\mathcal{U}$ cannot be the trivial group nor it can be isomorphic to V_4. It follows that $\mathcal{U}$ is a cyclic group; so $m=2$ and, in particular, $n=2d$, where $d \geq 2$, and

$$\mathcal{X}: \quad y^{2d} = x^2(x^2-1)^{l_1}\prod_{j=2}^{L}(x^2-a_j^2)^{l_j}.$$

Harvey's condition (a) is equivalent to have $\gcd(2d,2,l_1,\ldots,l_L)=1$, which is satisfied if some of the exponents l_j is odd. Without loss of generality, we may assume that l_1 is odd. In this case the curve $\mathcal{Y}$ is given by

$$\mathcal{Y}: \quad \left\{ \begin{array}{lcl} t_1^2 & = & uv, \\ v & = & u(u-1)^{l_1}\prod_{j=2}^{L}(u-a_j^2)^{l_j}, \end{array} \right.$$

which is isomorphic to the curve

$$w^2 = (u-1)^{l_1}\prod_{j=2}^{L}(u-a_j^2)^{l_j}.$$

As this curve must have genus zero, and l_1 is odd, the number of indices $j \in \{2,\ldots,L\}$ for which l_j is odd, must be at most one.

(i) If l_1 is the only odd exponent, then if we write $l_j = 2\widehat{l_j}$, for $j=2,\ldots,L$, the we must have $\gcd(2d,2,l_1,2\widehat{l_2},\ldots,2\widehat{l_L}) = 1$, which is equivalent to $\gcd(d,l_1,\widehat{l_2},\ldots,\widehat{l_L}) = 1$.

(ii) If there are exactly two of the exponents being odd, then we may assume, without loss of generality, that l_1 and l_2 are the only odd exponents. In this case, we must then have that $l_1+l_2 \equiv 0 \mod (2d)$, that is, $l_1+l_2=2d$. If we write $l_j = 2\widehat{l_j}$, for $j=3,\ldots,L$, then we must have

$\gcd(2d, 2, l_1, l_2, 2\widehat{l_3}, \ldots, 2\widehat{l_L}) = 1$, which is equivalent to $\gcd(d, l_1, l_2, \widehat{l_3}, \ldots, \widehat{l_L}) = 1$.

(II) Let us now consider the case $l_0 = 0$; so $m = n$ and $\eta(x, y) = (\omega_n x, y)$ and we are in case (1) above.

The η-invariants algebra $\mathbb{C}[x, y]^{\langle\eta\rangle}$ is generated by the monomials $u = x^n, v = y$. As consequence of invariant theory, the quotient curve $\mathcal{X}/\langle\eta\rangle$ corresponds to the algebraic curve

$$\mathcal{Y}: \quad \left\{ \begin{array}{rcl} v^n & = & (u-1)^{l_1} \prod_{j=2}^{L} (u - a_j^n)^{l_j}. \end{array} \right.$$

As $\mathcal{Y}$ must have genus zero and $n \geq 3$, we should have $L = 1, 2$ (and for $L = 2$ we must also have $l_1 + l_2 \equiv 0 \mod (n)$). So either

$$\mathcal{X}: \quad y^n = (x^n - 1)^{l_1}, \quad L = 1,$$

or

$$\mathcal{X}: \quad y^n = (x^n - 1)^{l_1}(x^n - a_2^n)^{l_2}, \; l_1 + l_2 \equiv 0 \mod (n), \quad L = 2.$$

Note that, for $L = 1$ we may assume $l_1 = 1$ (this is the classical Fermat curve of degree n). As the group of automorphisms of classical Fermat curve of degree n is $C_n^2 \rtimes S_3$, we may see that τ is not central; that is, it is not a generalized superelliptic Riemann surface of level n.

In the case $L = 2$, Harvey's conditions holds exactly when $\gcd(n, l_1, l_2) = 1$. As $l_1 + l_2 \equiv 0 \mod (n)$ and $l_1, l_2 \in \{1, \ldots, n-1\}$, we have that $l_1 + l_2 = n$. If we write $l_2 = n - l_1$, then

$$\left(\frac{x^n - 1}{x^n - a_2^n} \right)^{l_1} = \frac{y^n}{(x^n - a_2^n)^n},$$

and by writing $l_1 = n - l_2$ we also have that

$$\left(\frac{x^n - a_2^n}{x^n - 1} \right)^{l_2} = \frac{y^n}{(x^n - 1)^n}.$$

Then the Möbius transformation $M(x) = a/x$ induces the automorphism

$$\alpha(x, y) = \left(\omega_n \frac{a_2}{x}, \frac{-a_2^{l_2}(x^n - 1)(x^n - a_2^n)}{x^n y} \right),$$

which does not commute with $\eta(x, y) = (\omega_n x, y)$ since $n \geq 3$, a contradiction. □

Since for an exceptional generalized superelliptic Riemann surface of level n we must have n even, we may observe the following fact.

Corollary 2. *Let n be either equal to two or an odd integer. Then every generalized superelliptic curve of level n has a unique generalized superelliptic group of level n.*

If $\mathcal{X}$ is an exceptional generalized superelliptic Riemann surface of level n and H one of its generalized superelliptic groups of level n, then the quotient orbifold $\mathcal{X}/H$ has a cone point of order $n/2$. In particular, we observe the following.

Corollary 3. *Let $n \geq 4$ be even, let $\mathcal{X}$ be a generalized superelliptic Riemann surface of level n and let H be a generalized superelliptic group of level n. If $\mathcal{X}/H$ has no cone point of order $n/2$, then H is the unique generalized superelliptic group of $\mathcal{X}$ of level n. In particular, every superelliptic Riemann surface of level n admits a unique superelliptic group of level n.*

4.3. **A remark about a result due to Sanjeewa.** In [28] it was determined the groups G of conformal automorphisms of a cyclic n-gonal Riemann surface $\mathcal{X}$ for which the n-gonal group $H = \langle \tau \rangle$ is assumed to be normal subgroup and all cone points of S/H of order n (in particular, this contains the case of superelliptic Riemann surfaces). For the case $\overline{G} = C_m$ it was stated (see Theorems 3.2 and 4.1 in [28]) that either $G = C_{nm}$ or $G = \langle r, s : r^n = 1, s^m = 1, srs^{-1} = r^l \rangle$, where $(l, n) = 1$ and $l^m \equiv 1 \mod (n)$ (if $(m, n) = 1$, then $l = n - 1$). In the case that τ is central (in the superelliptic situation), the last situation only happens if $l = 1$, that is, either $G = C_{nm}$ or $G = C_n \times C_m$.

In the generalized superelliptic situation things changes as can be seen in the next example which considers a generalized superelliptic curve of genus seventeen for which the quotient has 16 cone points, two of them of order 2 and all others of order 4, whose reduced group is a cyclic group of order four. In this case $G \cong C_2 \times C_8$, which is neither C_{16} or C_4^2 as will be in the previous consideration for superelliptic curves of level four.

Example 1. Let us consider two values $\lambda, \mu \in \mathbb{C}$ so that $\lambda^4 \neq \mu^4$, $\lambda^4, \mu^4 \in \mathbb{C} - \{0, 1\}$, and the curve

$$\mathcal{X} : \quad y^4 = x^2(x^4 - 1)(x^4 - \lambda^4)(x^4 - \mu^4),$$

which has genus $g = 17$ and admits the automorphisms

$$\tau(x, y) = (x, iy), \quad \eta(x, y) = (ix, \sqrt{i}\, y).$$

For generic values of λ and μ, we have that $G = \langle \tau, \eta \rangle$ is the full group of automorphisms of $\mathcal{X}$ and that $G \cong C_2 \times C_8$ (the factor C_2 is generated by $\tau\eta^2$ and the factor C_8 is generated by η). In this case, the automorphisms τ is a generalized superelliptic automorphism of level $n = 4$ and $G/\langle \tau \rangle = C_4$.

Let us observe that another automorphism of order 4 is given by $\rho = \eta^2$, that is, $\rho(x, y) = (-x, iy)$. As $\mathbb{C}[x, y]^{\langle \rho \rangle}$ is generated by $u = x^2$, $v = xy^2$ and $w = y^4$, we may see that $\mathcal{X}/\langle \rho \rangle$ is isomorphic to

$$\widehat{v}^2 = (u - 1)(u - \lambda^2)(u - \mu^4),$$

$(\widehat{v}u = v)$ which has genus one.

5. Minimal fields of definition of generalized superelliptic curves

Let us consider a closed Riemann surface $\mathcal{X}$ of genus g, describe as a projective, irreducible, algebraic curve defined over $\mathbb{C}$, say given as the common zeroes of the polynomials $P_1, \ldots, P_r$, and let us denote by $G = \mathrm{Aut}\,(\mathcal{X})$ the full automorphism group of $\mathcal{X}$.

If $\sigma \in \mathrm{Gal}(\mathbb{C})$, then X^σ will denote the curve defined as the common zeroes of the polynomials $P_1^\sigma, \ldots, P_r^\sigma$, where P_j^σ is obtained from P_j by applying σ to its coefficients. The new algebraic curve $\mathcal{X}^\sigma$ is again a closed Riemann surface of the same genus g.

Let us observe that, if $\sigma, \tau \in \mathrm{Gal}(\mathbb{C})$, then $X^{\tau\sigma} = (X^\sigma)^\tau$.

5.1. **Field of definition.** A subfield k_0 of $\mathbb{C}$ is called a *field of definition* of $\mathcal{X}$ if there is a curve $\mathcal{Y}$, defined over k_0, which is isomorphic to $\mathcal{X}$ over $\mathbb{C}$. It is clear that every subfield of $\mathbb{C}$ containing k_0 is also a field of definition of it. In the other direction, a subfield of k_0 might not be a field of definition of $\mathcal{X}$.

Weil's descent theorem [32] provides sufficient conditions for a subfield k_0 of $\mathbb{C}$ to be a field of definition. Let us denote by $\mathrm{Gal}(\mathbb{C}/k_0)$ the group of field automorphisms of $\mathbb{C}$ acting as the identity on k_0.

Theorem 4 (Weil's descent theorem [32])**.** *Assume that $\mathcal{X}$ has genus $g \geq 2$. If for every $\sigma \in \mathrm{Gal}(\mathbb{C}/k_0)$ there is an isomorphism $f_\sigma : \mathcal{X} \to \mathcal{X}^\sigma$ satisfying the Weil's co-cycle condition*

$$f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau, \quad \forall \sigma, \tau \in \mathrm{Gal}(\mathbb{C}/k_0,$$

then there is a curve $\mathcal{Y}$, defined over k_0, and there is an isomorphism $R : \mathcal{X} \to \mathcal{Y}$, defined over a finite extension of k_0, so that $R = R^\sigma \circ f_\sigma$, for every $\sigma \in \mathrm{Gal}(\mathbb{C}/k_0)$.

Clearly, the sufficient conditions in Weil's descent theorem are trivially satisfied if $\mathcal{X}$ has no non-trivial automorphisms (a generic situation for $\mathcal{X}$ of genus at least three).

Corollary 4. *If $\mathcal{X}$ has trivial group of automorphisms and for every $\sigma \in \mathrm{Gal}(\mathbb{C}/k_0)$ there is an isomorphism $f_\sigma : \mathcal{X} \to \mathcal{X}^\sigma$, then $\mathcal{X}$ can be defined over k_0.*

5.2. **Field of moduli.** The notion of field of moduli was originally introduced by Shimura for the case of abelian varieties and later extended to more general algebraic varieties by Koizumi. If $G_{\mathcal{X}}$ is the subgroup of $\mathrm{Gal}(\mathbb{C})$ consisting of those σ so that $\mathcal{X}^\sigma$ is isomorphic to $\mathcal{X}$, then the fixed field $M_{\mathcal{X}}$ of $G_{\mathcal{X}}$ is called *the field of moduli* of $\mathcal{X}$.

A result due to Koizumi [23] asserts that the field of moduli of $\mathcal{X}$ coincides with the intersection of all its fields of definition and there is always a field of definition that is a finite extension of the field of moduli. This is the field of definition of the representing point $\mathfrak{p} = [\mathcal{X}]$ in the moduli space $\mathcal{M}_g$.

It is known that every curve of genus $g \leq 1$ can be defined over its field of moduli. If $g \geq 2$, to determine the field of moduli and to decide if it is a field of definition is difficult task and it is an active research topic. Examples of algebraic curves which cannot be defined over their field of moduli have been provided by Earle [12], Huggins [22] and Shimura [30] for the hyperelliptic situation and by the first author [17] and Kontogeorgis [24] in the non-hyperelliptic situation. In other words, $\mathcal{M}_g$ is not a *fine* moduli space.

Investigating the obstruction for the field of moduli to be a field of definition is part of descent theory for fields of definition and has many consequences in arithmetic geometry. Many works have been devoted to this problem, most notably by Weil [32], Shimura [30] and Grothendieck, among many others. Weil's criterion [32] assures that if a curve has no non-trivial automorphisms then its field of moduli is a field of definition. On the other extreme, if the curve $\mathcal{X}$ is quasiplatonic (that is, when the quotient orbifold $\mathcal{X}/\mathrm{Aut}(\mathcal{X})$ has genus zero and exactly three cone points), then Wolfart [33] proved that the field of moduli is also a field of definition. Hence, the real problem occurs when the curve has non-trivial automorphism group but the quotient orbifold $\mathcal{X}/\mathrm{Aut}(\mathcal{X})$ has non-trivial moduli.

It is known that a cyclic n-gonal Riemann surface is either definable over its field of moduli or over an degree two extension of it. In the particular case of superelliptic curves, with extra automorphisms, an equation over an at most quadratic extension of its field of moduli has been provided in [10] using the Shaska invariants. A direct consequence of Weil's descent theorem is the following.

Corollary 5. *Every curve with trivial group of automorphisms can be defined over its field of moduli.*

As a consequence of Belyi's theorem [4], every quasiplatonic curve $\mathcal{X}$ can be defined over $\overline{\mathbb{Q}}$ (so over a finite extension of $\mathbb{Q}$).

Theorem 5 (Wolfart [33])**.** *Every quasiplatonic curve can be defined over its field of moduli (which is a number field).*

5.3. **Two practical sufficient conditions.** When the curve $\mathcal{X}$ has a non-trivial group of automorphisms, then Weil's conditions (in Weil's descent theorem) are in general not easy to check. Next we consider certain cases for which it is possible to check for $\mathcal{X}$ to be definable over its field of moduli.

5.3.1. *Sufficient condition 1: unique subgroups.* Let H be a subgroup of $\mathrm{Aut}\,(\mathcal{X})$. In general it might be another different subgroup K which is isomorphic to H and with $\mathcal{X}/K$ and $\mathcal{X}/H$ having the same signature. For instance, the genus two curve $\mathcal{X}$ defined by $y^2 = x(x-1/2)(x-2)(x-1/3)(x-3)$ has two conformal involutions, τ_1 and τ_2, whose product is the hyperelliptic involution. The quotient $\mathcal{X}/\langle \tau_j \rangle$ has genus one and exactly two cone points (of order two).

We say that H is *unique* in $\mathrm{Aut}\,(\mathcal{X})$ if it is the unique subgroup of $\mathrm{Aut}\,(\mathcal{X})$ isomorphic to H and with quotient orbifold of same signature as $\mathcal{X}/H$. Typical examples are (i) $H = \mathrm{Aut}\,(\mathcal{X})$ and (ii) H being the cyclic group generated by the hyperelliptic involution for the case of hyperelliptic curves.

If H is unique in $\mathrm{Aut}\,(\mathcal{X})$, then it is a normal subgroup; so we may consider the reduced group $\overline{\mathrm{Aut}}\,(\mathcal{X}) = \mathrm{Aut}\,(\mathcal{X})/H$, which is a group of automorphisms of the quotient orbifold $\mathcal{X}/H$. In [19] the following sufficient condition for a curve to definable over its field of moduli was obtained.

Theorem 6 (Hidalgo and Quispe [19])**.** *Let $\mathcal{X}$ be a curve of genus $g \geq 2$ admitting a subgroup H, which is unique in $\mathrm{Aut}\,(\mathcal{X})$, and so that $\mathcal{X}/H$ has genus zero. If the reduced group of automorphisms $\overline{\mathrm{Aut}}\,(\mathcal{X}) = \mathrm{Aut}\,(\mathcal{X})/H$ is different from trivial or cyclic, then $\mathcal{X}$ is definable over its field of moduli.*

If $\mathcal{X}$ is a hyperelliptic curve, then a consequence of the above is the following result (originally due to Huggins [22]).

Corollary 6. *Let $\mathcal{X}$ be a hyperelliptic curve with extra automorphisms and reduced automorphism group $\overline{\mathrm{Aut}}\,(\mathcal{X})$ not isomorphic to a cyclic group. Then, the field of moduli of $\mathcal{X}$ is a field of definition.*

5.3.2. *Sufficient condition 2: Odd signature.* Another sufficient condition of a curve $\mathcal{X}$ to be definable over its field of moduli, which in particular contains the case of quasiplatonic curves, was provided in [3]. We say that $\mathcal{X}$ has *odd signature* if $\mathcal{X}/\mathrm{Aut}(\mathcal{X})$ has genus zero and in its signature one of the cone orders appears an odd number of times.

Theorem 7 (Artebani and Quispe [3])**.** *Let $\mathcal{X}$ be a curve of genus $g \geq 2$. If $\mathcal{X}$ has odd signature, then it can be defined over its field of moduli.*

5.4. **Most of generalized superelliptic curves are definable over their field of moduli.** The exceptional generalized superelliptic Riemann surfaces of level n are definable over their fields of moduli. As a consequence of Corollary 2 and Theorem 6, we obtain the following fact concerning the field of moduli of the non-exceptional generalized superelliptic curves.

Theorem 8. *Let $\mathcal{X}$ be a non-exceptional generalized superelliptic curve of genus $g \geq 2$ with generalized superelliptic group $H \cong C_n$. If the reduced group of automorphisms $\overline{\mathrm{Aut}}\,(\mathcal{X}) = \mathrm{Aut}\,(\mathcal{X})/H$ is different from trivial or cyclic, then $\mathcal{X}$ is definable over its field of moduli.*

As a consequence of the above, we only need to take care of the case when the reduced group $\overline{G} = G/H$ is either trivial or cyclic. As a consequence of Theorem 7 we have the following fact.

Theorem 9. *Let $\mathcal{X}$ be a generalized superelliptic curve of genus $g \geq 2$ with generalized superelliptic group $H \cong C_n$ so that $\overline{G} = G/H$ is either trivial or cyclic. If $\mathcal{X}$ has odd signature, then it can be defined over its field of moduli.*

As a consequence, the only cases were the generalized superelliptic Riemann surfaces cannot be defined over their field of moduli are those non-exceptional generalized superelliptic curves with reduced group $\overline{G} = G/H$ being either trivial or cyclic and with $\mathcal{X}/G$ having not an odd signature.

6. Appendix

In order to compute all the cyclic n-gonal curves of genus $g \geq 2$ one proceeds as follows. We consider the collection $\mathcal{F}_g$ of all tuples $(n, r; n_1, \ldots, n_r)$ satisfying the following properties (Harvey's conditions):

(1) $n \geq 2$, $r \geq 3$;
(2) $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq n$;
(3) n_j is a divisor of n, for each $j = 1, \ldots, r$;
(4) $\mathrm{lcm}\,(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r) = n$, for every $j = 1, \ldots, r$;
(5) if n is even, then $\#\{j \in \{1, \ldots, r\} : n/n_j \text{ is odd}\}$ is even;
(6) $2(g-1) = n\left(r - 2 - \sum_{j=1}^{r} n_j^{-1}\right)$.

For each tuple $(n, r; n_1, \ldots, n_r) \in \mathcal{F}_g$ we consider the collection $\mathcal{F}_g(n, r; n_1, \ldots, n_r)$ of tuples $(l_1, \ldots, l_r)$ so that

(1) $l_1, \ldots, l_r \in \{1, \ldots, n-1\}$;
(2) $\gcd(n, l_j) = n/n_j$, for each $j = 1, \ldots, r$.

Now, for each such tuple $(l_1, \ldots, l_r) \in \mathcal{F}_g(n, r; n_1, \ldots, n_r)$ we may consider the epimorphism

$$\theta : \Delta = \langle c_1, \ldots, c_r : c_1^{n_1} = \cdots = c_r^{n_r} = c_1 \cdots c_r = 1 \rangle \to C_n = \langle \tau \rangle : c_j \mapsto \tau^{l_j}.$$

Our assumptions above ensure that the kernel $\Gamma = \ker(\theta)$ is a torsion free normal co-compact Fuchsian subgroup of Δ with $\mathcal{X} = \mathbb{H}/\Gamma$ a closed Riemann surface of genus g admitting a cyclic group $H \cong C_n$ as a group of conformal automorphisms with quotient orbifold $\mathcal{X}/H = \mathbb{H}/\Delta$; a genus zero orbifold with exactly r cone points of respective orders $n_1, \ldots, n_r$. The surface $\mathcal{X}$ corresponds to a cyclic n-gonal curve

$$C(n, r; l_1, \ldots, l_r; a_1, \ldots, a_r) : \quad y^n = \prod_{j=1}^{r} (x - a_j)^{l_j},$$

for suitable pairwise different values $a_1, \ldots, a_r \in \mathbb{C}$, and H being generated by $\tau(x, y) = (x, \omega_n y)$.

We should note that there might be different tuples $(l_1, \ldots, l_r)$ and $(l_1', \ldots, l_r')$, necessarily belonging to the same $\mathcal{F}_g(n, r; n_1, \ldots, n_r)$, for which the pairs $(\mathcal{X}, H)$ and $(\mathcal{X}', H')$ are isomorphic (i.e., an isomorphism between the Riemann surfaces conjugating the cyclic groups). In general this is a difficult problem to determine such pairs defining same isomorphic pairs. But, in the (non-exceptional) generalized superelliptic situation, the uniqueness of the superelliptic cyclic group of level n (see Theorem 2) permits to see that $(\mathcal{X}, H)$ and $(\mathcal{X}', H')$ are isomorphic pairs if and only if the corresponding curves $C(n, r; l_1, \ldots, l_r; a_1, \ldots, a_r)$ and $C(n, r; l'_1, \ldots, l'_r; a'_1, \ldots, a'_r)$ are isomorphic, and this last being equivalent to the existence of

(i) Möbius transformation $A \in \mathrm{PSL}(2, \mathbb{C})$,
(ii) a permutation $\eta \in S_r$,
(iii) an element $u \in \{1, \ldots, n-1\}$ with $\gcd(u, n) = 1$,

so that

(iv) $l'_j \equiv u l_{\eta(j)} \mod (n)$, for $j = 1, \ldots, r$,
(v) $a'_j = M(a_j)$, for $j = 1, \ldots, r$.

All the above (together Lemma 1) permits to construct all the possible generalized superelliptic curves of lower genus in a similar fashion as done for the superelliptic case [27].

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Departamento de Matemática y Estadística, Universidad de La Frontera, Temuco, Chile.
E-mail address: `ruben.hidalgo@ufrontera.cl, saul.quispe@ufrontera.cl`

Department of Mathematics and Statistics, Oakland University, Rochester, MI, 48386.
E-mail address: `shaska@oakland.edu`